An Review on Realization Theory for Infinite-Dimensional Systems*

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1 Introduction

Already since the beginning of infinite-dimensional systems theory, there has been interest in the state-space realization problem. The state-space realization problem is the problem of finding for a given function a system in state-space form whose transfer function equals the given function. If we start with a rational function, then it is well-known, that one can always find a system with finite-dimensional state space whose transfer function is the given rational function. Moreover, it is always possible to find a controllable and observable realization and all controllable and observable realizations are equivalent, i.e., let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be two realizations that are controllable and observable, then there exists an invertible matrix S such that $A_1 = SA_2S^{-1}, B_1 = SB_2, C_1 = C_2S^{-1}$, and $D_1 = D_2$.

Since every finite-dimensional system has a rational transfer function, for a non-rational function we only can find a (state-space) realization with an infinite-dimensional state space. For functions that are analytic and bounded in some right-half plane, the realization problem was investigated by a number of people, e.g. Baras and Brockett [BB73], Fuhrmann [Fuh81], Helton [Hel76], Yamamoto [Yam81, Yam82], Salamon [Sal89] and Weiss [Wei89c, Wei97]. Here we present the realization theory in the language of well-posed linear systems.

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We end this section with some notation.

$$\begin{array}{rcl} \mathbb{C}_\delta^+ &:=& \{z \in \mathbb{C} \mid \operatorname{Re} z > \delta\}, \quad \delta \in \mathbb{R}, \\ \mathbb{C}_\delta^- &:=& \{z \in \mathbb{C} \mid \operatorname{Re} z < \delta\}, \quad \delta \in \mathbb{R}, \\ \boldsymbol{H}_\infty(\Omega) &:=& \{f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic and bounded}\}, \quad \Omega \subset \mathbb{C}. \end{array}$$

Furthermore, a holomorphic function $G : \mathbb{C}_0^+ \to \mathbb{C}$ is called *inner* if $|G(z)| \leq 1$ for $z \in \mathbb{C}_0^+$, and $|G(it)| = 1$ for almost every $t \in \mathbb{R}$.

We see that every inner function is an element of $\boldsymbol{H}_\infty(\mathbb{C}_0^+)$. Hence we may define $G(i\cdot)$ as the non-tangential limits of $G(z)$. This limit exists almost everywhere, and so the condition in the above definition makes sense.

2 Well-posed linear systems and realization theory

A quite large and well-studied class of infinite-dimensional linear systems is the class of well-posed linear systems introduced by Salamon [Sal89] and Weiss [Wei89a], [Wei89b].

Let $T(t)$ be a C_0-semigroup on the separable Hilbert space H, and let A its generator. We define the space H_{-1} to be the completion of H with respect to the norm

$$\|x\|_{-1} := \|(\beta I - A)^{-1} x\|$$

and the space H_1 to be $D(A)$ with the norm

$$\|x\|_1 := \|(\beta I - A)x\|,$$

where $\beta \in \rho(A)$, the resolvent set of A. It is easy to verify that H_{-1} and H_1 do not depend on $\beta \in \rho(A)$. Moreover, $\|\cdot\|_1$ is equivalent to the graph norm on $D(A)$, so H_1 is complete. In Weiss [Wei89a, Remark 3.4] it is shown that $T(t)$ has a restriction to a C_0-semigroup on H_1 whose generator is the restriction of A to $D(A)$, and $T(t)$ can be extended to a C_0-semigroup on H_{-1} whose generator is an extension of A with domain H. Therefore, we get

$$A \in \mathcal{L}(H_1, H) \quad \text{and} \quad A \in \mathcal{L}(H, H_{-1}).$$

H_{-1} equals the dual of $D(A^*)$, where we have equipped $D(A^*)$ with the graph norm (see [Wei89a]). Following [Wei89a] and [Wei89b] we introduce admissible control operators and observation operators for $T(t)$.

Definition 2.1 *1. Let $B \in \mathcal{L}(\mathbb{C}, H_{-1}) = H_{-1}$. For $t \geq 0$ we define the operator $\mathcal{B}_t : \boldsymbol{L}_2(0,\infty) \to H_{-1}$ by*

$$\mathcal{B}_t u := \int_0^t T(t-\rho) B u(\rho)\, d\rho.$$

Then B is called an admissible control operator for $T(t)$*, if for some (and hence any) $t > 0$, $\mathcal{B}_t \in \mathcal{L}(\boldsymbol{L}_2(0,\infty), H)$.*

2. *Let B be an admissible control operator for $T(t)$. B is called an* infinite-time admissible control operator for $T(t)$, *if $T(\cdot)Bu(\cdot) : [0,\infty) \to H_{-1}$ is integrable for every $u \in \boldsymbol{L}_2(0,\infty)$, and the operator $\mathcal{B}_\infty : \boldsymbol{L}_2(0,\infty) \to H_{-1}$, given by*

$$\mathcal{B}_\infty u := \int_0^\infty T(t)Bu(t)dt,$$

satisfies $\mathcal{B}_\infty \in \mathcal{L}(\boldsymbol{L}_2(0,\infty), H)$.

3. *Let $C \in \mathcal{L}(H_1, \mathbb{C})$. Then C is called an* admissible observation operator for $T(t)$, *if for some (and hence any) $t > 0$, there is some $K > 0$ such that*

$$\|CT(\cdot)x\|_{\boldsymbol{L}_2(0,t)} \le K\|x\|, \quad x \in D(A).$$

4. *Let C be an admissible observation operator for $T(t)$. We call C an* infinite-time admissible observation operator *if there is some $K > 0$ such that*

$$\|CT(\cdot)x\|_{\boldsymbol{L}_2(0,\infty)} \le K\|x\|, \quad x \in D(A).$$

By definition, every infinite-time admissible control operator for $T(t)$ is an admissible control operator for $T(t)$. If $T(t)$ is exponentially stable, then the two notion of admissibility coincide. Similar statements hold for admissible observation operators for $T(t)$. Moreover, B is an (infinite-time) admissible control operator for $T(t)$ if and only if B^* is an (infinite-time) admissible observation operator for $T^*(t)$.

Let C be an admissible observation operator for $T(t)$. Then for $t \ge 0$ the operator $\Psi_t \in \mathcal{L}(D(A), \boldsymbol{L}_2(0,t))$, given by $\Psi_t x := CT(\cdot)x$, has a unique extension to $\mathcal{L}(H, \boldsymbol{L}_2(0,t))$ (again denoted by Ψ_t). Similarly, if C is an infinite-time admissible observation, then we can extend the operator $\Psi_\infty \in \mathcal{L}(D(A), \boldsymbol{L}_2(0,\infty))$, given by $\Psi_\infty x := CT(\cdot)x$.

Definition 2.2 *Let B be an admissible control operator for $T(t)$, then*

1. (T,B) is exactly controllable in finite time *if there exists a time t_0 such that $\operatorname{Im} \mathcal{B}_{t_0} = H$.*

2. (T,B) is approximately controllable *if $\overline{\cup_{t\ge0} \operatorname{Im} \mathcal{B}_t} = H$.*

3. (T,B) is exactly controllable *if B be an infinite-time admissible control operator for $T(t)$ and $\operatorname{Im} \mathcal{B}_\infty = H$.*

Let C be an admissible observation operator for $T(t)$, then

1. (T,C) is exactly observable in finite time *if there exists a time t_0 such that $\|\Psi_{t_0} x\| \ge c\|x\|$ for some positive c.*

2. (T,C) is approximately observable *if $\cap_{t\ge0} \ker \Psi_t = \{0\}$.*

3. (T,C) is exactly observable *if $\Psi_\infty \in \mathcal{L}(H, \boldsymbol{L}_2(0,\infty))$ and $\|\Psi_\infty x\| \ge c\|x\|$ for some positive c.*

If B is infinite-time admissible, then it is easy to see that (T, B) is approximately controllable if and only if $\overline{\mathrm{Im}\mathcal{B}_\infty} = H$. A similar result holds for the output operator and approximate observability. It is easy to see, that controllability and observability are dual notions, i.e., (T, B) is approximately (exactly) controllable if and only if (T^*, B^*) is approximately (exactly) observable. We are now in the position to introduce well-posed linear systems.

Definition 2.3 *(T, B, C, G) is called* well-posed linear system *if the following holds*

1. *$T(t)$ is a C_0-semigroup,*
2. *B is an admissible control operator for $T(t)$,*
3. *C is an admissible observation operator for $T(t)$,*
4. *There exists a constant $\rho > 0$ such that $G \in \boldsymbol{H}_\infty(\mathbb{C}_\rho^+)$ and*

$$\frac{G(s) - G(z)}{z - s} = C(sI - A)^{-1}(zI - A)^{-1}B, \qquad s, z \in \mathbb{C}_\rho^+, s \neq z, \tag{1}$$

where A is the generator of $T(t)$.

Here G is called the *transfer function.* A transfer function is determined by (T, B, C) up to an additive constant operator. Note, that this definition of a well-posed linear system is not the standard one introduced by Weiss [Wei89a] [Wei89b], but it is equivalent to Weiss's definition, see Curtain and Weiss [CW89]. Moreover, this definition of a well-posed linear system is equivalent to Salamon's definition of a time-invariant, linear control system, see Salamon [Sal89] and Weiss [Wei94a]. Following, Salamon [Sal89], the system trajectory of a well-posed linear system (T, B, C, G) is given by

$$\begin{aligned} x(t, x_0, u) &:= T(t)x_0 + \int_0^t T(t-\rho)Bu(\rho)\, d\rho, \qquad t \geq 0, \\ y(t, x_0, u) &:= C(x(t, x_0, u) - (\mu I - A)^{-1}Bu(t)) + G(\mu)u(t), \qquad t \geq 0. \end{aligned}$$

Here $u \in \boldsymbol{L}_2(0, \infty)$ denotes the input of the system, $x_0 \in H$ denotes the initial state, $x(t, x_0, u)$ denotes the state of the system at time t, and $y(t, x_0, u)$ denotes the output at time t. Note, that the definition of $y(t, x_0, u)$ does not depend on $\mu \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. Moreover, if additionally (T, B, C, G) is a *regular system*, i.e., $D := \lim_{s\to+\infty, s\in\mathbb{R}} G(s)$ exists, then we are able to choose D as feedthrough term and $y(t, x_0, u)$ is given by

$$y(t, x_0, u) = Cx(t, x_0, u) + Du(t), \qquad t \geq 0.$$

Note, that in general this D does not exists.

Definition 2.4 *A well-posed linear system (T, B, C, G) is called* exponentially stable, *if T is an exponentially stable C_0-semigroup. The system is called* infinite-time admissible *if B is an infinite-time admissible control operator and C is an infinite-time admissible observation operator for $T(t)$.*

Furthermore, it will be called approximately controllable*,* exactly controllable*, or* exactly controllable in finite-time *if* (T, B) *has that property. Similar for the observability property.*

Definition 2.5 *Let* $G \in \boldsymbol{H}_\infty(\mathbb{C}_\delta^+)$ *for some* $\delta > 0$. *We say that* G *has a realization as a well-posed linear system if there exists a well-posed linear system* (T, B, C, G).

Salamon [Sal89] proved that every G as above has a realization as a well-posed linear system. His realization is the well-known shift realization. The shift realization has already been studied by many mathematicians, see for example Baras and Brockett [BB73], Helton [Hel76], Fuhrmann [Fuh81], Yamamoto [Yam81, Yam82], and Weiss [Wei97]. Before we state the proof, we would like to present the motivation behind this proof.

Assume that G is the Laplace transform of a continuous function $h \in \boldsymbol{L}_1(0, \infty) \cap \boldsymbol{L}_2(0, \infty)$. Then finding a state-space realization just means finding a triple $(T(t), B, C)$ such that

$$h(t) = CT(t)B, \qquad t \geq 0. \tag{2}$$

To establish this equality the idea is very simple. One takes as state space $\boldsymbol{L}_2(0, \infty)$, and define for $z \in \boldsymbol{L}_2(0, \infty)$.

$$\left[T(t)z\right](x) = z(t + x), \quad x \geq 0. \tag{3}$$

Furthermore, for the continuous functions in $\boldsymbol{L}_2(0, \infty)$ we define

$$Cz = z(0). \tag{4}$$

If we now take $B = h$, then by combining (3) and (4) we see that

$$CT(t)B = \left[T(t)h\right](0) = h(t), \quad t \geq 0.$$

This is precisely equality (2), and hence we have constructed a realization.

For transfer functions that do not have a nice inverse Laplace transform, we see that there are difficulties in defining $CT(t)B$. However, as we shall show, the realization is still possible with these choices of $T(t), B$ and C. In order to prove the realization, it is easier to work with the Laplace transforms of the above object. Thus $\boldsymbol{L}_2(0, \infty)$ becomes $\boldsymbol{H}_2(\mathbb{C}_0^+)$, and the $T(t), B$ and C become their equivalent counterpart on this space.

Theorem 2.6 *Every function* $G \in \boldsymbol{H}_\infty(\mathbb{C}_0^+)$ *has a realization.*

Proof: As state space H we choose

$$H := \boldsymbol{H}_2(\mathbb{C}_0^+).$$

Our C_0-semigroup is given by

$$T(t)x = P_{\boldsymbol{H}_2(\mathbb{C}_0^+)}[e^{t\cdot}x(\cdot)], \qquad x \in H,$$

where $P_{\boldsymbol{H}_2(\mathbb{C}_0^+)}$ is the orthogonal projection from $\boldsymbol{L}_2(i\mathbb{R})$ to $\boldsymbol{H}_2(\mathbb{C}_0^+)$. This semigroup has as infinitesimal generator

$$(Ax)(s) = sx(s) - \check{x}(0), \qquad x \in D(A),$$

with

$$D(A) = \{x \in H \mid s \mapsto sx(s) - \check{x}(0) \in \boldsymbol{H}_2(\mathbb{C}_0^+)\},$$

where $\check{x}$ denotes the inverse Laplace transform of x.

Furthermore, we define

$$B = G,$$

and

$$Cx = \check{x}(0) \qquad \text{for } x \in D(A).$$

Having made these choices, we still have to prove that they form a well-posed linear system with transfer function G. We begin by showing that $T(t)$ is a C_0-semigroup on H.

1. Let $S(t)$ be the right-shift semigroup on H, i.e.

$$S(t)x := e^{-t\cdot}x(\cdot), \quad x \in H.$$

   It is easy to see that this is a C_0 semigroup, and that the adjoint of this C_0-semigroup equals $T(t)$. Since we are working in a Hilbert space, we have that the adjoint of a C_0-semigroup is again a C_0-semigroup. Hence we have shown that $T(t)$ is a C_0-semigroup.

2. Now we shall derive a simple formula for the resolvent operator of A. Since the right-shift semigroup has growth bound zero, so has its adjoint, T, and thus we have that the open right-half is contained in the resolvent set of A. Take $\beta \in \mathbb{C}_0^+$, and $x, y \in H$, then

$$\begin{aligned}
\langle y, (\beta I - A)^{-1}x\rangle &= \int_0^\infty \langle y, e^{-\beta t}T(t)x\rangle dt \\
&= \int_0^\infty e^{-\overline{\beta}t}\langle T(t)^*y, x\rangle dt \\
&= \int_0^\infty e^{-\overline{\beta}t}\int_{-\infty}^\infty e^{-i\omega t}y(i\omega)\overline{x(i\omega)}d\omega dt \\
&= \int_{-\infty}^\infty \int_0^\infty e^{-(\overline{\beta}+i\omega)t}y(i\omega)\overline{x(i\omega)}dtd\omega \\
&= \int_{-\infty}^\infty \frac{1}{\overline{\beta}+i\omega}y(i\omega)\overline{x(i\omega)}d\omega \\
&= \int_{-\infty}^\infty y(i\omega)\overline{\left(\frac{x(i\omega)}{\beta - i\omega}\right)}d\omega \\
&= \int_{-\infty}^\infty y(i\omega)\overline{\left(\frac{x(i\omega)-x(\beta)}{\beta - i\omega}\right)}d\omega \\
&= \langle y, \frac{x(\cdot)-x(\beta)}{\beta - \cdot}\rangle.
\end{aligned}$$

Here we have used that $x(\beta)/(\beta - \cdot)$ is in $\boldsymbol{H}_2(\mathbb{C}_0^-) = \left[\boldsymbol{H}_2(\mathbb{C}_0^+)\right]^\perp$. Furthermore, is it easy to see that $\frac{x(\cdot)-x(\beta)}{\beta-\cdot}$ is an element of $\boldsymbol{H}_2(\mathbb{C}_0^+)$. So we conclude that

$$\left[(\beta I - A)^{-1}x\right](s) = \frac{x(s) - x(\beta)}{\beta - s} \qquad \text{for } s \neq \beta. \tag{5}$$

3. The inverse Laplace transform of the function given in (5) is given by

$$\check{z}(t) = \int_t^\infty \check{x}(\tau)e^{-\beta\tau}d\tau.$$

From this we see that $\check{z}$ is a continuous function on $[0, \infty)$, and the value in zero equals $x(\beta)$.

From (5) an easy calculation shows that

$$(Ax)(s) = sx(s) - \check{x}(0), \qquad x \in D(A), \tag{6}$$

with

$$D(A) = \{x \in H \mid s \mapsto sx(s) - \check{x}(0) \in \boldsymbol{H}_2(\mathbb{C}_0^+)\}. \tag{7}$$

4. Using (5) we can identify $H_{-1} = D(A^*)'$ with the space

$$\left\{f : \mathbb{C}_0^+ \to \mathbb{C} \mid s \mapsto \frac{f(s) - f(\beta)}{\beta - s} \in H \text{ for some } \beta \in \mathbb{C}_0^+\right\}. \tag{8}$$

Moreover, the sesquilinear form $\langle \cdot, \cdot \rangle_{D(A^*)\times D(A^*)'}$ is given by

$$\langle f, g\rangle_{D(A^*)\times D(A^*)'} = \int_{-\infty}^\infty f(i\omega)\overline{g(i\omega)}\, d\omega, \qquad f \in D(A^*), g \in D(A^*)'.$$

5. From (8) it follows directly that $\boldsymbol{H}_\infty(\mathbb{C}_0^+)$ can be seen as a subspace of H_{-1}, and so $B = G$ is an element of H_{-1}

Now we show that B is admissible. Take $y \in D(A^*)$

$$\begin{aligned}
\langle y, \int_0^\infty T(t)Bu(t)dt\rangle_{D(A^*)\times D(A^*)'} &= \int_0^\infty \langle y, T(t)Bu(t)\rangle_{D(A^*)\times D(A^*)'}dt \\
&= \int_0^\infty \langle T(t)^*y, Bu(t)\rangle_{D(A^*)\times D(A^*)'}dt \\
&= \int_0^\infty \int_{-\infty}^\infty e^{-i\omega t}y(i\omega)\overline{G(i\omega)u(t)}d\omega dt \\
&= \int_{-\infty}^\infty y(i\omega)\overline{G(i\omega)\int_0^\infty e^{i\omega t}u(t)dt}d\omega \\
&= \int_{-\infty}^\infty y(i\omega)\overline{G(i\omega)\hat{u}(-j\omega)}d\omega \\
&= \langle y, P_{\boldsymbol{H}_2}\left(G(\cdot)\hat{u}(-\cdot)\right)\rangle.
\end{aligned}$$

Since for every $u \in \boldsymbol{L}_2(0,\infty)$, we have that $\hat{u}(-\cdot) \in \boldsymbol{L}_2(i\mathbb{R})$, and since G is bounded on the imaginary axis, we have that $G(\cdot)\hat{u}(-\cdot) \in \boldsymbol{L}_2(i\mathbb{R})$. Thus we have that

$$\int_0^\infty T(t)Bu(t)dt = P_{\boldsymbol{H}_2(\mathbb{C}_0^+)}\left(G(\cdot)\hat{u}(-\cdot)\right) \tag{9}$$

is well-defined for every $u \in \boldsymbol{L}_2(0,\infty)$ with values in H. Thus B is an admissible operator for T.

6. From part 2 we know that for $x \in D(A)$, $\check{x}(0)$ is well-defined. This immediately implies that C is a well-defined operator on $D(A)$. From part 3 and equation (5), we see that

$$C(\beta I - A)^{-1}x = x(\beta). \tag{10}$$

Since $C(\cdot I - A)^{-1}x$ is the Laplace transform of $CT(\cdot)x$, we get from the equation above that

$$CT(t)x = \check{x}(t), \tag{11}$$

for every $x \in H = \boldsymbol{H}_2(\mathbb{C}_0^+)$. From Paley-Wiener theorem we get that the $\boldsymbol{L}_2(0,\infty)$-norm of $\check{x}$ equals the H-norm of x. In other words for every $x \in H$ we have that $\Psi_\infty x := CT(t)x \in \boldsymbol{L}_2(0,\infty)$. Hence C is an admissible operator for T.

Note that we even have

$$\|\Psi_\infty x\| = \|x\|. \tag{12}$$

Thus (T, C) is exactly observable.

7. We now show that G is a transfer function of (T, B, C). We have to show that (1) holds. Combining equation (10) with (5) gives that

$$\begin{aligned} C(sI - A)^{-1}(\beta I - A)^{-1}B &= [(\beta I - A)^{-1}B](s) = [(\beta I - A)^{-1}G](s) \\ &= \frac{G(s) - G(\beta)}{\beta - s}. \end{aligned}$$

Thus we have constructed a realization of the transfer function G. ■

For a $G \in \boldsymbol{H}_\infty(\mathbb{C}_0^+)$ the Hankel operator with symbol G is defined as the operator $H_G : \boldsymbol{L}_2(0,\infty) \mapsto \boldsymbol{L}_2(0,\infty)$ given by

$$\widehat{H_G u} := P_{\boldsymbol{H}_2(\mathbb{C}_0^+)}(G(\cdot)\hat{u}(-\cdot)), \qquad u \in \boldsymbol{L}_2(0,\infty), \tag{13}$$

where $\hat{}$ denotes the Laplace transform. If (T, B, C, G) is a realization of G, B is an infinite-time admissible control operator, and C is an infinite-time admissible observation operator for $T(t)$ we get

$$H_G = \Psi_\infty \mathcal{B}_\infty.$$

From the Hankel operator we can derive special results.

Lemma 2.7 *If the Hankel operator with symbol G has closed range, then all infinite-time admissible, approximately controllable and approximately observable realization are equivalent, i.e., if (T_1, B_1, C_1, G) with state space H_1, and (T_2, B_2, C_2, G) with state space H_2 are both infinite-time admissible, approximately controllable and approximately observable realizations, then there exists a bounded, invertible operator $S \in \mathcal{L}(H_1, H_2)$ such that*

$$T_2 = ST_1S^{-1}, \quad \mathcal{B}_{2,\infty} = S\mathcal{B}_{1,\infty}, \quad \Psi_{2,\infty} = \Psi_{1,\infty}S^{-1}.$$

Furthermore, if a realization of G is exactly controllable, and exactly observable, then H_G has closed range.

Proof The first part is shown in Proposition 6.2 of Ober and Wu [OW96].

Let us now assume that G has an exactly controllable and exactly observable realization. This implies, that G can be written in the form

$$H_G = \Psi_\infty \mathcal{B}_\infty,$$

where $\mathcal{B}_\infty \in \mathcal{L}(\boldsymbol{L}_2(0,\infty), H)$, $\Psi_\infty \in \mathcal{L}(H, \boldsymbol{L}_2(0,\infty))$, $\mathcal{B}_\infty$ is surjective and $\|\Psi_\infty x\| \geq c\|x\|$, for some positive c. The surjectivity of $\mathcal{B}_\infty$ implies that the range of H_G equals the range of Ψ_∞, and the open mapping theorem shows that the range of Ψ_∞ is closed. Thus H_G has closed range. ■

Let us remark that the above result is not true if $\mathcal{B}_\infty$ and/or Ψ_∞ are not bounded operators on $\boldsymbol{L}_2(0,\infty)$ and H, respectively. An example can be found at the end of this section.

Corollary 2.8 *Suppose H_G has closed range and there exists an exactly controllable and exactly observable realization. Then every infinite-time admissible, approximately controllable and approximately observable realization is exactly controllable and exactly observable.*

If the transfer function is inner, then there exists a realization which is exactly controllable and exactly observable.

Theorem 2.9 *Let $G \in \boldsymbol{H}_\infty(\mathbb{C}_0^+)$ be an inner function. Then there exists an exactly controllable and exactly observable well-posed linear system (T, B, C, G).*

Furthermore, G has an exactly controllable and exactly observable realization with the C_0-semigroup having the additional property that

1. *it is exponentially stable if and only if $\inf_{\operatorname{Re} z \in (0,\alpha)} |G(z)| > 0$ for some $\alpha > 0$.*

2. *it is a group if and only if $\inf_{\operatorname{Re} z > \rho} |G(z)| > 0$ for some $\rho > 0$.*

Proof From the Theorem 2.6 we have that there exists a realization of the G. We use the notation as introduced in the proof of Theorem 2.6. Let V denote the closed subspace of $\boldsymbol{H}_2(\mathbb{C}_0^+)$ defined as

$$V = [G\boldsymbol{H}_2(\mathbb{C}_0^+)]^\perp,$$

where the orthogonal complement is taken in $\boldsymbol{H}_2(\mathbb{C}_0^+)$. We shall show that the realization $(T|_V, G, C|_V)$ has the desired properties.

We begin by showing that V is $T(t)$-invariant.

1. Take an arbitrary $x \in H$ and $v \in V$, then

$$\begin{aligned} \langle Gx, T(t)v \rangle &= \langle T(t)^* Gx, v \rangle = \langle e^{-t\cdot} G(\cdot)x(\cdot), v(\cdot) \rangle \\ &= \langle G(\cdot) e^{-t\cdot} x(\cdot), v(\cdot) \rangle = 0, \end{aligned}$$

   since $e^{-t\cdot} x \in H$, and so $G(\cdot)e^{-t\cdot}x(\cdot) \in V^\perp$.

   The set of all w that can be written as Gx is dense in $V^\perp = \overline{G\boldsymbol{H}_2(\mathbb{C}_0^+)}$, thus we have that $T(t)V \subset V$.

   From this we see that $T_V(t)$ defined as the restriction to V of $T(t)$ is a C_0-semigroup on V.

2. Now we show that $B := G$ is an admissible control operator for $T_V(t)$.

   From (9) we see that

$$\mathcal{B}_\infty u = \int_0^\infty T(t)Bu(t)dt = P_{\boldsymbol{H}_2(\mathbb{C}_0^+)} \left(G(\cdot)\hat{u}(-\cdot) \right).$$

   If we can show that this expression maps into V, then we are done. Take an $x \in H$, and consider

$$\begin{aligned} \langle Gx, \mathcal{B}_\infty u \rangle &= \int_{-\infty}^\infty G(i\omega)x(i\omega)\overline{G(i\omega)\hat{u}(-i\omega)} d\omega \\ &= \int_{-\infty}^\infty x(i\omega)\overline{\hat{u}(-i\omega)} d\omega = 0 \end{aligned}$$

   where we have used $x \in H = \boldsymbol{H}_2(\mathbb{C}_0^+)$, and $\hat{u}(-\cdot) \in \boldsymbol{H}_2(\mathbb{C}_0^+)^\perp$.

   So we have shown that $\mathcal{B}_\infty u$ is orthogonal to any Gx with $x \in \boldsymbol{H}_2(\mathbb{C}_0^+)$. This proves that $\mathcal{B}_\infty$ maps into V.

3. Since C is admissible for $T(t)$ it is directly clear that $C|_V$ defined as

$$C_V x := \check{x}(0) = Cx, \quad x \in V \tag{14}$$

   is admissible for $T_V(t)$

4. Combing the above results we see that we found a second realization of G. We shall now prove that it is exactly controllable. Note that from equations (14) and (12) it follows that (T_V, C_V) is exactly observable.

5. We begin by showing that the range of $\mathcal{B}_\infty$ is closed. From (9) we see that its range equals

$$\operatorname{Im} \mathcal{B}_\infty = P_{\boldsymbol{H}_2(\mathbb{C}_0^+)} \left[G\boldsymbol{H}_2(\mathbb{C}_0^-) \right].$$

   Consider a sequence $\{f_n\}$ in $\boldsymbol{H}_2(\mathbb{C}_0^-)$ such that $\{Gf_n\}$ is converging in $\boldsymbol{L}_2(i\mathbb{R})$ to x. Since G has absolute value one on the imaginary axis, we see that the operator define as $f \mapsto Gf$ is an unitary operator on $\boldsymbol{L}_2(i\mathbb{R})$, and so f_n is converging in $\boldsymbol{L}_2(i\mathbb{R})$. Since $f_n \in \boldsymbol{H}_2(\mathbb{C}_0^-)$, and this space is a closed subspace of $\boldsymbol{L}_2(i\mathbb{R})$, we have that $G^{-1}x = \lim_{n\to\infty} f_n \in \boldsymbol{H}_2(\mathbb{C}_0^-)$.

This implies that $x \in \text{Im}\,\left[G\boldsymbol{H}_2(\mathbb{C}_0^-)\right]$. Thus the subspace $\text{Im}\,\left[G\boldsymbol{H}_2(\mathbb{C}_0^-)\right]$ is a closed subspace of $\boldsymbol{L}_2(i\mathbb{R})$. Since orthogonal projection maps closed subspaces onto closed subspaces we see that $\text{Im}\,\mathcal{B}_\infty$ is closed.

Hence we have that the range is closed. If we can show that the range is dense in V, then we have proved the assertion. Suppose that the range is not dense in V, then there exists a $v \in V$ such that $v \in [\text{Im}\,\mathcal{B}_\infty]^\perp$. So we have

$$\begin{aligned} 0 &= \langle v, \mathcal{B}_\infty u\rangle \\ &= \int_{-\infty}^{\infty} v(i\omega)\overline{G(i\omega)\hat{u}(-i\omega)}d\omega \\ &= \int_{-\infty}^{\infty} \overline{G(i\omega)}v(i\omega)\,\overline{\hat{u}(-i\omega)}d\omega. \end{aligned}$$

Since this holds for every $\hat{u}(-\cdot) \in \boldsymbol{H}_2(\mathbb{C}_0^-)$, we have that

$$x := \overline{G}v \in \boldsymbol{H}_2(\mathbb{C}_0^+)$$

Since G is inner, we get from the above equation that

$$v = Gx,$$

for the $x \in \boldsymbol{H}_2(\mathbb{C}_0^+)$. This means that $v \in V^\perp$, and since it is an element of V it must be zero. This proves that the range of $\mathcal{B}_\infty$ equals V.

6. We see that it remains to prove that the semigroup has the additional properties.

   Gearhart [Gea78, Theorem 2.1 and Theorem 2.2] or Moeller [Moe62, Theorem 3.1 and Theorem 3.2] shows $\sigma(T(1)) \subseteq \{z \in \mathbb{C} \mid |z| < e^{-\alpha}\}$ if and only if $\inf_{\text{Re}\,z\in(0,\alpha)} |G(z)| > 0$. Thus this proves part 1.

   Moreover, from Gearhart [Gea78, Theorem 3.4] we get that also part 2 of the theorem holds. ■

Next we give an example, which shows that Lemma 2.7 does not hold without the assumption of infinite-time admissibility.

Example 2.10 *Let A_0 be an infinitesimal generator on an infinite-dimensional Hilbert space H that satisfies $A_0 = -A_0^*$, and let $b \in H$. Define the operator $B_0 \in \mathcal{L}(\mathbb{C}, H)$ as $B_0 u = b \cdot u$. We assume that the system (A_0, B_0^*) is approximately observable. It is well-known that*

$$G(s) := 1 - B_0^*(sI - A_0 + \frac{1}{2}B_0B_0^*)^{-1}B_0$$

is an inner function and Lemma 2.7 together with Theorem 2.9 show that the Hankel operator H_G has closed range. Clearly, (T_0, B_0, B_0^*, G) is a well-posed linear system, where T_0 is the semigroup generated by $A_0 - \frac{1}{2}B_0B_0^*$. By the special structure of the system, we have that (T_0, B_0, B_0^*, G) is approximately *controllable and approximately observable. However, since B_0 is compact, we have that $(A_0 - \frac{1}{2}B_0B_0^*, B_0^*)$ is not exactly observable in finite time.*

We are now going to construct another realization. We begin by considering the realization (T, B, C, G) as constructed in Theorem 2.6 and Theorem 2.9. Note, that (T, B, C, G) is exactly controllable, exactly observable and the state space for the realization in Theorem 2.9 is given by

$$V = [G\boldsymbol{H}_2(\mathbb{C}_0^+)]^{\perp},$$

where the orthogonal complement in taken in $\boldsymbol{H}_2(\mathbb{C}_0^+)$. We define V_1 as the closure of V in the topology of $\boldsymbol{H}_2(\mathbb{C}_1^+)$. Note that this space is isometric isomorph (via Laplace transform) with the weighted $\boldsymbol{L}_2$-space

$$\{f \in \boldsymbol{L}_2^{\mathrm{loc}}(0,\infty) \mid \int_0^\infty |e^{-t}f(t)|^2 dt < \infty\}.$$

The semigroup on V is given by

$$T(t)v = P_{\boldsymbol{H}_2(\mathbb{C}_0^+}(e^{t\cdot}v),$$

It is easy to see that the inverse Laplace transform of $T(t)v$ is given by

$$(T(\check{t})v)(\tau) = \check{v}(\tau + t), \qquad \tau \geq 0.$$

Now we shall calculate the norm of $T(t)v$ in the new state space V_1.

$$\begin{aligned} \|T(t)v\|_{V_1}^2 &= \|(T(\check{t})v)\|_{V_1}^2 = \|\check{v}(\cdot + t)\|_{V_1}^2 = \int_0^\infty |e^{-\tau}\check{v}(\tau + t)\|^2 d\tau \\ &= e^{2t}\int_0^\infty |e^{-\tau}\check{v}(\tau)\|^2 d\tau = e^{2t}\|v\|_{V_1}^2. \end{aligned}$$

This implies that for every $t \geq 0$, we can extend the operator $T(t)$ to V_1. Since $T(t)$ is a C_0-semigroup on V, and since V is dense in V_1, we have that the extension is again a C_0-semigroup. We denote this new semigroup by $T_1(t)$.

Next we construct a well-posed realization of G with state space equal to V_1. As semigroup we take T_1, and as B_1 we take $B_1 = G$. From part 2. in Theorem 2.9 we see that the corresponding $\mathcal{B}_{1,\infty}$ satisfies

$$\mathcal{B}_{1,\infty} = \imath\mathcal{B}_\infty, \tag{15}$$

where $\imath$ denotes the inclusion of V into V_1. Thus B_1 is an infinite-time admissible control operator for $T_1(t)$. Furthermore, since the range of $\mathcal{B}_\infty$ equals V, and since V is dense in V_1, we have that (T_1, B_1) is approximately controllable.

As observation operator we take

$$C_1x := \check{x}(0), \quad x \in V_1.$$

Hence the extension of C to V_1. We have to show that this is admissible observation operator for $T_1(t)$. For $v \in V$, we have that $C_1T_1(t)v = CT(t)v = \check{v}(t)$, and thus

$$\int_0^1 |\check{v}(t)|^2 dt \leq e^2 \int_0^1 |e^{-t}\check{v}(t)|^2 dt \leq e^2 \int_0^\infty |e^{-t}\check{v}(t)|^2 dt = e^2\|v\|_{V_1}^2.$$

Since V is dense in V_1, we conclude that C_1 is an admissible observation operator for $T_1(t)$. From the definition of C_1 it is clear that (T_1, C_1) is approximately observable.

By (15), we see that

$$\imath(zI - A)^{-1}B = (zI - A)^{-1}B_1,$$

where A_1 is the infinitesimal generator of $T_1(t)$. Hence

$$\begin{aligned} C_1(sI - A_1)^{-1}(zI - A_1)^{-1}B_1 &= C_1(sI - A_1)^{-1}(zI - A)^{-1}B \\ &= C_1(sI - A)^{-1}(zI - A)^{-1}B \\ &= C(sI - A)^{-1}(zI - A)^{-1}B = \frac{G(s) - G(z)}{z - s}. \end{aligned}$$

Thus we have proved that (T_1, B_1, C_1, G) is realization of G as well. Furthermore, this realization is approximately controllable and observable. Thus the realizations (T_0, B_0, B_0^*, G), (T, B, C, G) and (T_1, B_1, C_1, G) are all approximately controllable and approximately observable.

We now assume that all approximately controllable and approximately observable realizations are equivalent. The equivalence of (T, B, C, G) and (T_1, B_1, C_1, G) implies that the topologies of V and V_1 would be equivalent. Since V_1 is the closure of V in the topology of V_1, this implies that $V = V_1$. In particular, there holds

$$\int_0^\infty e^{-2t}|f(t)|^2 dt \geq K \int_0^\infty |f(t)|^2 dt,$$

for all f whose Laplace transform lies in V_1, where K is independent of f. Using this we see that

$$\begin{aligned} K\int_0^\infty |f(t)|^2 dt &\leq \int_0^\infty e^{-2t}|f(t)|^2 dt \\ &= \int_0^{t_0} e^{-2t}|f(t)|^2 dt + \int_{t_0}^\infty e^{-2t}|f(t)|^2 dt \\ &\leq \int_0^{t_0} |f(t)|^2 dt + e^{-2t_0}\int_{t_0}^\infty |f(t)|^2 dt \\ &\leq \int_0^{t_0} |f(t)|^2 dt + e^{-2t_0}\int_0^\infty |f(t)|^2 dt. \end{aligned}$$

Hence for all t_0

$$[K - e^{-2t_0}]\int_0^\infty |f(t)|^2 dt \leq \int_0^{t_0} |f(t)|^2 dt$$

Thus for t_0 sufficiently large

$$\int_0^{t_0} |f(t)|^2 dt \geq K_1 \int_0^\infty |f(t)|^2 dt \tag{16}$$

for all f's whose Laplace transform lies in V_1. For the output y we have that

$$y(t) = C_1 T_1(t) v = \check{v}(t)$$

Hence with (16), we obtain that

$$\int_0^{t_0} |y(t)|^2 dt \geq K_1 \int_0^{\infty} |y(t)|^2 dt = K_1 \int_0^{\infty} |\check{v}(t)|^2 dt = K_1 \|v\|_V^2 \geq K_1 \|v\|_{V_1}^2.$$

Thus (C_1, T_1) *is exactly observable in finite-time. The equivalence of the systems* (T_0, B_0, B_0^*, G) *and* (T_1, B_1, C_1, G) *implies that the realization* (T_0, B_0, B_0^*, G) *is also exactly observable in finite time. However, this is not possible, since* B_0^* *is compact.*

Concluding we see that the realization (T_1, B_1, C_1, G) *cannot be equivalent with* (T, B, C, G) *whereas* (T, B, C, G) *is exactly controllable and exactly observable, and* (T_1, B_1, C_1, G) *is approximately controllable and approximately observable. Note that the realization* (T_1, B_1, C_1, G) *does not have an infinite-time admissible observation operator.*

References

[BB73] J. Baras and R. Brockett. H^2-functions and infinite-dimensional realization theory. *SIAM J. Control*, 13:221–241, 1973.

[CW89] R. F. Curtain and G. Weiss. Well–posedness of triples of operators (in the sense of linear systems theory). In W. Schappacher F. Kappel, K. Kunisch, editor, *"Distributed Parameter Systems", proc. of the conference in Vorau, Austria, July 1988*, pages 41–59, Birkhäuser Verlag Basel, 1989.

[Fuh81] P. A. Fuhrmann. *Linear Systems and Operators in Hilbert space.* McGraw-Hill, 1981.

[Gea78] L. Gearhart. Spectral theory for contraction semigroups on Hilbert space. *Transactions of the American Mathematical Society*, 236:385–394, 1978.

[Hel76] J. W. Helton. Systems with infinite-dimensional state-space: The Hilbert space approach. *Proc. IEEE*, 64:145–160, 1976.

[Moe62] J. W. Moeller. On the spectra of some translation spaces. *Journal of Mathematical Analysis and Applications*, 4:276–296, 1962.

[Ni86] N. K. Nikolskii. Treatise on the shift operator. Springer-Verlag, Berlin, 1986.

[OW96] R. J. Ober and Y. Wu. Infinite-dimensional continuous-time linear systems: Stability and structure analysis. *SIAM J. Control and Optimization*, 34(3):757–812, 1996.

[Sal89] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21:147–164, 1989.

[Wei89a] G. Weiss. Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, 27:527–545, 1989.

[Wei89b] G. Weiss. Admissible observation operators for linear semigroups. *Israel Journal of Mathematics*, 65:17–43, 1989.

[Wei89c] G. Weiss. The representation of regular linear systems on Hilbert spaces. In W. Schappacher F. Kappel, K. Kunisch, editor, *"Distributed Parameter Systems", proc. of the conference in Vorau, Austria, July 1988*, pages 401–416, Birkhäuser Verlag Basel, 1989.

[Wei94a] G. Weiss. Regular linear systems with feedback. *Math. Control Signals Systems*, 7:23–57, 1994.

[Wei94b] G. Weiss. Transfer functions of regular linear systems, part I: Characterizations of regularity. *Trans. American Mathematical Society*, 342:827–854, 1994.

[Wei97] G. Weiss. Latest HORROR stories from Exeter. personal correspondence, December 15, 1997.

[Yam81] Y. Yamamoto. Realization theory of infinite-dimensional linear systems. part I. *Math. Systems Theory*, 15:55–77, 1981.

[Yam82] Y. Yamamoto. Realization theory of infinite-dimensional linear systems. part II. *Math. Systems Theory*, 15:169–190, 1982.