Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain*

Hans Zwart†, Yann Le Gorrec, Bernhard Maschke‡§ and Javier Villegas¶

April 2, 2008

Abstract

We study hyperbolic partial differential equations on a one dimensional spatial domain with control and observation at the boundary. Using the idea of feedback we show these systems are well-posed in the sense of Weiss and Salamon if and only if the state operator generates a C_0-semigroup. Furthermore, we show that the corresponding transfer function is regular, i.e., has a limit for s going to infinity.

keywords: Infinite-dimensional systems, hyperbolic boundary control systems, C_0-semigroup, well-posedness.

*file:artikel/Dirac/well-posedness_4.tex

†H. Zwart, Department of Applied Mathematics, University of Twente, 7500 AE Enschede, The Netherlands. h.j.zwart@math.utwente.nl

‡Y. Le Gorrec and B. Maschke are with LAGEP, CNRS UMR 5007, CPE Lyon - Bâtiment 308 G, Université Lyon-1, Université de Lyon, 43, bd du 11 Novembre 1918, F-69622 Villeurbanne cedex, France. {legorrec,maschke}@lagep.univ-lyon1.fr

§The contribution of Y. Le Gorrec and B. Maschke has been done within the context of the European sponsored project GeoPlex with reference code IST-2001-34166 and French NRA sponsored project RECIPROC with reference code ANR-06-JCJC-0011. Further information is available at http://www.geoplex.cc. and http://www.agence-nationale-recherche.fr/Intl

¶J.A. Villegas is with Department of Engineering Science, University of Oxford, Parks Road, Oxford, OX1 3PJ, England. Javier.Villegas@eng.ox.ac.uk

1 Introduction

Consider the abstract linear differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{1}$$

$$y(t) = Cx(t) + Du(t). \tag{2}$$

where x is assumed to take values in the Hilbert space X, u in the Hilbert space U, and y is the Hilbert space Y. The operators A, B, C, and D are linear operators. Under the standard assumption that A generates a C_0-semigroup, we know that the homogeneous equation, i.e., $u \equiv 0$ in (1) has a unique (weak) solution. Assuming that B, C, and D are bounded operators, there exists a solution of (1)–(2) for every $u \in L^2((0, t_f); U)$. Existence of solutions for an arbitrary initial condition $x_0 \in X$ and input $u \in L^2((0, t_f); U)$, such that $y \in L^2((0, t_f); Y)$ is called *well-posedness*, see e.g. [13, 16]. Hence if B, C, and D are bounded linear operators, then the system (1)–(2) is well-posed if and only if A is the infinitesimal generator of a C_0-semigroup. For more details on this class of systems we refer to chapter 2 and 3 of [1].

As is known for a very long time not all linear partial differential equations (p.d.e.'s) can be written into the format (1)–(2) with a bounded B and C. Consider for instance, the simple shift equation on the interval $[0, 1]$ with scalar control and observation on the boundary

$$\frac{\partial w}{\partial t}(t, z) = \frac{\partial w}{\partial z}(t, z), \quad w(0, z) = w_0(z), \qquad z \in [0, 1] \tag{3}$$

$$u(t) = w(t, 1), \tag{4}$$

$$y(t) = w(t, 0). \tag{5}$$

Without going into too much detail, it is clear that the mapping $f \mapsto f(0)$ is not a bounded mapping on the state space $X = L^2(0, 1)$ to the output space $\mathbb{C}$. Hence C will not be a bounded operator. Similarly, B is not a bounded operator from $\mathbb{C}$ to X either. It is not hard to see that for any $w_0 \in X$ and $u \in L^2(0, \infty)$, the solution of (3)–(5) is given by $w(t, z) = f(t + z)$ and $y(t) = f(t)$, where $f(\xi) = w_0(\xi), \xi \in [0, 1]$ and $f(\xi + 1) = u(\xi), \xi > 0$. So if $w_0 \in X = L^2(0, 1)$, and $u \in L^2(0, \infty)$, then $f \in L^2(0, \infty)$ and so $w(t, \cdot) \in X$ for all $t > 0$, and $y \in L^2(0, \infty)$. Hence this system is well-posed, without having bounded B and C operators.

Over the last decades there has been a growing interest in well-posedness of p.d.e.'s with control and observation on the boundary. The main books on this subject are the books by Lasiecka and Triggiani [6, 7] and by Tucsnak and Weiss [14]. It is very hard to give an overview of all the articles on this subject, but the references in the books gives a good starting point for the interested reader. The examples treated in these books and papers include among others p.d.e.'s on a two or three dimensional spatial domain. In this paper we consider a particular class of systems, namely linear boundary port Hamiltonian systems [4, 15], defined on a one-dimensional spatial domain. The aim of this paper is to show that for this class we have a similar theorem as for the class of systems with bounded B and C, see (1) and (2). Namely, it is well-posed if and only if the homogeneous equation, i.e., $u \equiv 0$ generates a C_0-semigroup.

The set-up of the proof is relatively simple. First we consider a system that consists out of a collection of delay lines, e.g. (3)–(5). If these delay lines are non-connected, then it is very easy to show well-posedness. If they are connected, then we regard this connection as a feedback of the non-connected situation, see Figure 1. Using a result by G. Weiss [16], we know that this connected case will lead to a well-posed system, provided the feedback gives a bounded closed loop transfer function. Hence we study the closed loop transfer function. For the considered class of systems we show that if the feedback connection does not give a bounded transfer function, then the homogeneous system does not generate a C_0-semigroup.

The general case is shown by performing a state basis transformation such that the system becomes a set of simple delay lines. This is rather standard by using the characteristic or the Riemann coordinates for our p.d.e.

2 Class of p.d.e.'s and main result.

For the sake of simplicity, in this paper we use the elementary hyperbolic p.d.e., i.e. the wave equation, as illustrative example

$$\frac{\partial^2 w}{\partial t^2}(t,z) = c^2 \frac{\partial^2 w}{\partial z^2}(t,z). \tag{6}$$

For the sake of the discussion in the Section 5 we consider this as the model of a uniform vibrating string. In this case the variable $w(t,z)$ denotes the position with respect to equilibrium of the string at position z and the constant c denotes the celerity $c = \sqrt{\frac{T}{\rho}}$ with T Young's modulus and ρ the mass density. However, the physical formulation of the model of the string is given as a set of coupled conservation laws and induces the following choice of the state variables: the momentum $x_1(t,z) = \rho \frac{\partial w}{\partial t}(t,z)$ and the elastic strain $x_2(t,z) = \frac{\partial w}{\partial z}(t,z)$ [10, 12, 15]. It may be noticed that this formulation is more general than (6) as it remains valid in the case of a non-uniform string when Young's modulus T and the mass density ρ may depend on z. In these variables the model of the vibrating string is written as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t,z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial z}\begin{pmatrix} \frac{1}{\rho}x_1 \\ Tx_2 \end{pmatrix}(t,z)$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial z}\left[\begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right](t,z). \qquad (7)$$

We have given two examples, the first one corresponding to the delay line (3) consisting in one conservation law and the second one (7), the vibrating string, consisting in two coupled conservation laws. More examples of coupled systems of conservation laws may be found of hyperbolic type [10, 9, 15] but also of parabolic type [3, 15].

In this paper we consider a general class of conservation laws with flux variables depending linearly on the state and linear source term:

$$\frac{\partial x}{\partial t}(t,z) = P_1\frac{\partial}{\partial z}(\mathcal{L}x)(t,z) + P_0(z)x(t,z), \qquad z \in [a,b], \qquad (8)$$

where x is a function taking values in $\mathbb{R}^n$, $\mathcal{L}$ is a multiplication operator satisfying $0 < mI \leq \mathcal{L}(z) \leq MI$, $z \in [a,b]$, for some positive constants m and M, P_1 is a constant (real) matrix satisfying $P_1^T = P_1$ and P_0 is some matrix valued function.

With the operator $\mathcal{L}$ we introduce the Hilbert space X as being the function space $L^2((a,b);\mathbb{R}^n)$ with the inner product

$$\langle f,g\rangle_{\mathcal{L}} = \int_a^b f(z)^*\mathcal{L}(z)g(z)dz. \qquad (9)$$

The norm associated to this inner product, denoted by $\|x\|_{\mathcal{L}}$, has the physical meaning of energy. The operator $\mathcal{L}$ may be interpreted as the variational derivative of the energy $\mathcal{L}x = \frac{1}{2}\frac{\partial\|x\|^2_{\mathcal{L}}}{\partial x}$. Note that since $mI \leq \mathcal{L}(z) \leq MI$ this new norm is equivalent to the standard L^2-norm.

Assume for a moment that P_0 is zero, and that we have a classical solution of the p.d.e. (8), then

$$\begin{aligned}
\frac{d}{dt}\|x(t)\|_{\mathcal{L}}^2 &= \langle \frac{\partial x}{\partial t}, x\rangle_{\mathcal{L}} + \langle x, \frac{\partial x}{\partial t}\rangle_{\mathcal{L}} \\
&= \langle P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right), x\rangle_{\mathcal{L}} + \langle x, P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)\rangle_{\mathcal{L}} \\
&= \int_a^b \left(P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)(t,z)\right)^* \mathcal{L}(z)x(z)dz + \\
&\quad \int_a^b x(z)^* \mathcal{L}(z)(P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)(t,z)dz \\
&= \int_a^b \frac{\partial}{\partial z}\left[\left(\mathcal{L}x\right)^*(t,z) P_1 \left(\mathcal{L}x\right)(t,z)\right] dz \\
&= \left[\left(\mathcal{L}x\right)^*(t,z) P_1 \left(\mathcal{L}x\right)(t,z)\right]_a^b, \qquad (10)
\end{aligned}$$

where we have used the symmetry of P_1. Hence by putting enough boundary conditions to zero, this last expression becomes zero, and the energy remains constant along solutions. This reflects the hyperbolic nature of (8). From the calculation above, we can clearly see the advantage of writing a p.d.e. in the form (8). The symmetry of P_1 tells that the system is hyperbolic in nature, and the $\mathcal{L}$ tells us which energy function should be used. In many models the P_1 only contains ones and zeros, telling how different physical domains are coupled, and all the physical parameters are in $\mathcal{L}$ [10, 9, 3, 15]. One may also see P_1 as the matrix obtained by putting all physical constants to one. Standard p.d.e. theory gives that we will not have a unique solution of (8) if we don't impose boundary conditions. If we set all $2n$ boundary conditions to zero, then we will only have the zero solution, and so we cannot choose a non-zero initial condition. It is well-known that we must impose n boundary conditions, if we want to have existence and uniqueness of solutions. In [4] all boundary conditions are characterized for which we have that the energy stays constant along solutions.

We decompose the set of n boundary conditions into two sets, the first one contains those boundary conditions that are set to zero, whereas the other class contains the boundary conditions which are free to choose, i.e., the inputs. More precisely, we shall complete the p.d.e. (8) with a set of boundary conditions defined in terms of the variational derivative of the energy, $\mathcal{L}x$, and not the state variable x, as follows:

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)(t,z) + P_0(z)x(t,z), \quad x(0,z) = x_0(z) \tag{11}$$

$$0 = M_{11}\left(\mathcal{L}x\right)(t,b) + M_{12}\left(\mathcal{L}x\right)(t,a) \tag{12}$$

$$u(t) = M_{21}\left(\mathcal{L}x\right)(t,b) + M_{22}\left(\mathcal{L}x\right)(t,a), \tag{13}$$

where $\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right]$ is a $n \times 2n$ matrix with rank n. The maximality of the rank implies that (12) and (13) are not conflicting. Furthermore, we assume that we observe the system through its boundary in terms of the variables $\mathcal{L}x$

$$y(t) = C_1\left(\mathcal{L}x\right)(t,b) + C_2\left(\mathcal{L}x\right)(t,a). \tag{14}$$

In order that we are not observing an input or a zero boundary condition, we assume that rank $\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ C_1 & C_2 \end{smallmatrix}\right] = n + \text{rank}\left[\begin{smallmatrix} C_1 & C_2 \end{smallmatrix}\right]$, in other words we choose the observations at the boundary to be complementary to the boundary control (13). Notice again that both the boundary input and output variables are defined as linear combinations of the variational derivative of the energy $\mathcal{L}x$ restricted to the boundary, which appears naturally in the energy balance equation (10).

The equations (11)–(14) give a system with boundary control and observation. The aim of this paper is to study solutions of this system. For this we define the notion of well-posedness. Note that this is the adaptation of a more general notion to our class of p.d.e.'s, see [13] or [16]. By H^1 we denote the Sobolev space of square integrable functions, whose derivative is also square integrable.

Definition 2.1 *Consider the system (11)–(14) and let k be the dimension of u. This system is* well-posed *if there exists a $t_f > 0$ and m_f such that the following holds:*

1. *The operator A defined as $P_0 + P_1 \frac{\partial}{\partial z}\mathcal{L}$ with domain,*

$$D(A) = \{x_0 \in X \mid \mathcal{L}x_0 \in H^1((a,b);\mathbb{R}^n), \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} (\mathcal{L}x_0)(b) \\ (\mathcal{L}x_0)(a) \end{smallmatrix}\right] = 0\}$$

   is the infinitesimal generator of a C_0-semigroup on X.

2. *The following inequality holds for all $\mathcal{L}x_0 \in H^1((a,b);\mathbb{R}^n)$ and $u \in C^2([0,t_f);\mathbb{R}^k)$ with $u(0) = M_{21}\left(\mathcal{L}x_0\right)(b) + M_{22}\left(\mathcal{L}x_0\right)(a)$, and $0 = M_{11}\left(\mathcal{L}x_0\right)(b) + M_{12}\left(\mathcal{L}x_0\right)(a)$*

$$\|x(t_f)\|_{\mathcal{L}}^2 + \int_0^{t_f} \|y(t)\|^2 dt \le m_f \left[\|x_0\|_{\mathcal{L}}^2 + \int_0^{t_f} \|u(t)\|^2 dt\right]. \tag{15}$$

Since $\mathcal{L}^{-1}H^1((a,b);\mathbb{R}^n)$ and $C^2([0,t_f);\mathbb{R}^k)$ are dense linear subspaces of X and $L^2([0,t_f);\mathbb{R}^k)$, respectively, we find that if (15) holds for all $\mathcal{L}x_0 \in H^1((a,b);\mathbb{R}^n)$ and $u \in C^2([0,t_f),\mathbb{R}^k)$, then (15) holds for all $x_0 \in X$ and $u \in L^2([0,t_f);\mathbb{R}^k)$. Hence if the system is well-posed, then (11)–(14) has for every initial condition and every square integrable input a unique (mild) solution, and (15) still holds.

It may seem strange why we did not wrote down (15) for general initial conditions and general inputs. However, in many occasions it is easy to prove existence of solutions for these smooth data first, and next to prove properties of it. This is formalized in the following lemma, see Malinen [11].

Lemma 2.2 *Let t_f be a positive real number. Assume further that condition 1. of Definition 2.1 is satisfied. Then for every $\mathcal{L}x_0 \in H^1((a,b);\mathbb{R}^n)$ and every $u(\cdot) \in C^2([0,t_f);\mathbb{R}^k)$ with $u(0) = M_{21}(\mathcal{L}x_0)(b) + M_{22}(\mathcal{L}x_0)(a)$, and $0 = M_{11}(\mathcal{L}x_0)(b) + M_{12}(\mathcal{L}x_0)(a)$, there exists a unique classical solution of (11)–(13) on $[0,t_f]$. Furthermore, the output (14) is well-defined and $y(\cdot)$ is continuous.*

For a well-posed system one can define the transfer function. Basically, this means that for an input of the form $u_0 \exp(st)$, one finds a state trajectory and an output of the form $y(t) = G(s)u_0 \exp(st)$. The function $G(s)$ is the *transfer function*, see Zwart [17]. For a well-posed system the transfer function exists on some right-half plane.

Definition 2.3 *Let $G(s)$ be the transfer function of (11)–(14). The system (11)–(14) is* regular *when $\lim_{s\to\infty} G(s)$ exists. If the system (11)–(14) is regular, then the* feed-through *term D is defined as $D = \lim_{s\to\infty} G(s)$.*

In the sequel we state the main result of this paper which gives the conditions under which, for the class of system given in (11)–(14), the dissipation inequality (15) of the definition 2.1 is a necessarily satisfied if the condition 1. is satisfied. Furthermore, when the condition 1. holds, then the transfer function corresponding to the well-posed system is regular. We formulate this in the following theorem.

Theorem 2.4 *Consider the partial differential equation (11)–(14) on the spatial interval $[a,b]$, with $z(x,t)$ taking values in $\mathbb{R}^n$. Let X be the Hilbert space $L^2((a,b);\mathbb{R}^n)$ with inner product (9). Furthermore, assume that*

- *P_1 is real-valued, invertible, and symmetric, i.e., $P_1^T = P_1$,*
- *$\mathcal{L}$ is a (real) multiplication operator on $L^2((a,b);\mathbb{R}^n)$ satisfying $0 < mI \le \mathcal{L}(z) \le MI$, $z \in [a,b]$.*

- *The multiplication operator $P_1\mathcal{L}$ can be written as*

$$P_1\mathcal{L}(z) = S^{-1}(z)\Delta(z)S(z), \tag{16}$$

  with $\Delta(z)$ a diagonal multiplication operator, and both $\Delta(z)$ and $S(z)$ are continuously differentiable,

- $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ *is a $n \times 2n$ matrix with rank n,*

- $\operatorname{rank} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ C_1 & C_2 \end{bmatrix} = n + \operatorname{rank} \begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

If the homogeneous p.d.e., i.e., $u \equiv 0$, generates a C_0-semigroup on X, then the system (11)–(14) is well-posed, and the corresponding transfer function G is regular. Furthermore, we have that $\lim_{\mathrm{Re}(s)\to\infty} G(s) = \lim_{s\to\infty} G(s)$.

As mentioned before, the last condition in the theorem tells us that we will not be observing inputs or boundary conditions which are set to zero. In other words, we are really observing a part of the state.

The third condition tells us that $P_1\mathcal{L}$ is diagonizable via a continuously differentiable basis transformation. In Kato [5, chapter II], one can find conditions on $P_1\mathcal{L}(z)$ such that this is possible. For simplicity, we have assumed that $\Delta(z)$ is continuously differentiable. However, the above theorem also holds if this does not hold.

Since $\mathcal{L}(z) > mI$, we see that for every $z \in [a,b]$ the signature of the matrix $\mathcal{L}(z)^{\frac{1}{2}}P_1\mathcal{L}(z)^{\frac{1}{2}}$ equals the signature of P_1. This implies that the signature of $\mathcal{L}(z)^{\frac{1}{2}}P_1\mathcal{L}(z)^{\frac{1}{2}}$ is independent of z. Furthermore, a simple calculation gives that the eigenvalues of $\mathcal{L}(z)^{\frac{1}{2}}P_1\mathcal{L}(z)^{\frac{1}{2}}$ are equal to the eigenvalues of $P_1\mathcal{L}(z)$. Concluding, we see that for all $z \in [a,b]$ zero is not an eigenvalue of $P_1\mathcal{L}(z)$, and that the number of negative and positive eigenvalues of $P_1\mathcal{L}(z)$ is independent of z. Thus without loss of generality, we may assume that

$$\Delta(z) = \begin{pmatrix} \Lambda(z) & 0 \\ 0 & \Theta(z) \end{pmatrix}, \tag{17}$$

where $\Lambda(z)$ is a diagonal (real) matrix, with positive functions on the diagonal, and $\Theta(z)$ is a diagonal (real) matrix, with negative functions on the diagonal.

In [4] necessary and sufficient conditions were given such that the homogeneous p.d.e. (11)–(13) generates a contraction semigroup and also gives the dissipation inequalities associated with the choices of the boundary output variables (14). As a consequence of the Theorem 2.4, these systems are furthermore proven to be well-posed and regular.

We prove this theorem in two main steps. First, in Section 3, we assume that $P_1\mathcal{L}$ is diagonal and prove the theorem in this case. Secondly, in Section 4, we consider the general case and use the fact that Theorem 2.4 remains valid under a basis transformation.

From the proof of Theorem 2.4, we see that we obtain an equivalent matrix condition for condition 1., i.e., item 1. of Theorem 2.4 holds if and only if K is invertible, see Theorem 3.3. Since the matrix K is obtained after a basis transformation, and depends on the negative and positive eigenvalues of $P_1\mathcal{L}$, it is not easy to rewrite this condition in a condition for M_{ij}.

A semigroup can be extended to a group, if the homogeneous p.d.e. has for every initial condition a solution for negative time. Using once more the proof of Theorem 3.3, we see that A in item 1. of Theorem 2.4 generates a group if and only if K and Q are invertible matrices.

3 The operator $P_1\mathcal{L}$ is diagonal.

In this section, we prove Theorem 2.4 if $P_1\mathcal{L}$ is diagonal. For this we need the following two lemma's.

Lemma 3.1 *Let $\lambda(z)$ be a positive continuous differentiable function on the interval $[a,b]$. With this function we define the scalar system*

$$\frac{\partial w}{\partial t}(t,z) = \frac{\partial}{\partial z}\left(\lambda(z)w(t,z)\right), \qquad w(0,z) = w_0(z) \qquad z \in [a,b] \tag{18}$$

The value at b we choose as input

$$u(t) = \lambda(b)w(t,b) \tag{19}$$

and as output we choose the value on the other end

$$y(t) = \lambda(a)w(t,a). \tag{20}$$

The system (18)–(20) is a well-posed system on the state space $L^2(a,b)$, and its solution is given as

$$w(t,z) = f(p(z)+t)\lambda(z)^{-1}, \tag{21}$$

where

$$p(z) = \int_a^z \lambda(\zeta)^{-1} d\zeta \tag{22}$$

$$f(p(z)) = \lambda(z)w_0(z), \qquad z \in [a,b] \tag{23}$$

$$f(p(b)+t) = u(t), \qquad t > 0. \tag{24}$$

The transfer function is given by

$$G(s) = e^{-p(b)s}. \tag{25}$$

This transfer function satisfies

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = 0. \tag{26}$$

Proof: It is easy to see that (21)–(24) is a classical solution of (18)–(19) provided w_0 is continuously differentiable and $\lambda(b)w_0(b) = u(0)$. Let $w(t,z)$ be this classical solution, then

$$\begin{aligned}
\frac{\partial}{\partial t}\int_a^b & w^*(t,z)\lambda(z)w(t,z)dz \\
&= \int_a^b \left[\frac{\partial}{\partial z}\left[\lambda(z)w(t,z)\right]^* \lambda(z)w(t,z) + w(t,z)^*\lambda(z)\frac{\partial}{\partial z}\left[\lambda(z)w(t,z)\right]\right] dz \\
&= \left[\lambda(z)w(t,z)\right]^* \lambda(z)w(t,z) \mid_a^b \\
&= |u(t)|^2 - |y(t)|^2.
\end{aligned}$$

Thus for all $t > 0$ we have that

$$\begin{aligned}
\int_a^b w^*(t,z)\lambda(z)w(t,z)dz - & \int_a^b w^*(0,z)\lambda(z)w(0,z)dz \\
&= \int_0^t |u(\tau)|^2 d\tau - \int_0^t |y(\tau)|^2 d\tau.
\end{aligned} \tag{27}$$

Since λ is strictly positive, we have that the norm $\int_a^b w^*(t,z)\lambda(z)w(t,z)dz$ is equivalent to the $L^2(a,b)$-norm, and so on a dense set an inequality like (15) is satisfied. Thus the system is well-posed, see also the remark following Definition 2.1.

The transfer function $G(s)$ is constructed by finding for $s \in \mathbb{C}$ and for all u_0 a triple $(u_s(t), w_s(t,z), y(t)) = (u_0 e^{st}, w_0(z)e^{st}, y_0 e^{st})$ satisfying (18)–(20). If such a triple can be constructed and y_0 is uniquely depending on u_0, then $G(s)$ is defined as $G(s)u_0 = y_0$, see [17]. Substituting a triple of this form in the p.d.e., gives

$$sw_0(z) = \frac{\partial}{\partial z}\left(\lambda(z)w_0(z)\right), \quad u_0 = \lambda(b)w_0(b), \quad y_0 = \lambda(a)w_0(a).$$

Thus $w_0(z) = u_0\lambda(z)^{-1}\exp(s(p(z) - p(b)))$, and $y_0 = u_0\exp(-sp(b))$. This proves (25). The property (26) follows directly from (25) and the fact that $p(b) > 0$. ■

For a p.d.e. with negative coefficient, we obtain a similar result.

Lemma 3.2 *Let $\theta(z)$ be a negative continuous function on the interval $[a,b]$. With this function we define the scalar system*

$$\frac{\partial w}{\partial t}(t,z) = \frac{\partial}{\partial z}\left(\theta(z)w(t,z)\right), \qquad w(0,z) = w_0(z) \qquad z \in [a,b] \tag{28}$$

The value at a we choose as input

$$u(t) = \theta(a)w(t,a) \tag{29}$$

and as output we choose the value on the other end

$$y(t) = \theta(b)w(t,b). \tag{30}$$

The system (28)–(30) is a well-posed system on the state space $L^2(a,b)$, and its solution is given as

$$w(t,z) = f(n(z)+t)\theta(z)^{-1}, \tag{31}$$

where

$$n(z) = \int_a^z \theta(\zeta)^{-1} d\zeta \tag{32}$$

$$f(n(z)) = \theta(z)w_0(z), \qquad z \in [a,b] \tag{33}$$

$$f(t) = u(t), \qquad t > 0. \tag{34}$$

The transfer function is given by

$$G(s) = e^{n(b)s}. \tag{35}$$

This transfer function satisfies

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = 0. \tag{36}$$

We use these two lemmas to prove Theorem 2.4 when $P_1\mathcal{L}$ is diagonal and the input space has dimension n.

Consider the following diagonal hyperbolic system on the spatial interval $z \in [a,b]$

$$\frac{\partial}{\partial t}\begin{pmatrix} x_+(t,z) \\ x_-(t,z) \end{pmatrix} = \frac{\partial}{\partial z}\left[\begin{pmatrix} \Lambda(z) & 0 \\ 0 & \Theta(z) \end{pmatrix}\begin{pmatrix} x_+(t,z) \\ x_-(t,z) \end{pmatrix}\right] \tag{37}$$

where $\Lambda(z)$ is a diagonal (real) matrix, with positive functions on the diagonal, and $\Theta(z)$ is a diagonal (real) matrix, with negative functions on the diagonal. Furthermore, we assume that Λ and Θ are continuously differentiable.

With this p.d.e. we associate the following boundary control and observation

$$u_s(t) := \begin{pmatrix} \Lambda(b)x_+(t,b) \\ \Theta(a)x_-(t,a) \end{pmatrix}, \tag{38}$$

$$y_s(t) := \begin{pmatrix} \Lambda(a)x_+(t,a) \\ \Theta(b)x_-(t,b) \end{pmatrix}. \tag{39}$$

Theorem 3.3 *Consider the p.d.e. (37) with u_s and y_s as defined in (38) and (39), respectively.*

- *The system defined by (37)–(39) is well-posed and regular. Furthermore, its transfer function converges to zero for $\mathrm{Re}(s) \to \infty$.*
- *To the p.d.e. (37) we define a new set of inputs and outputs. The new input $u(t)$ is written as*

$$u(t) = Ku_s(t) + Qy_s(t), \tag{40}$$

where K and Q are two square matrices, with $[K, Q]$ of rank n. The new output is written as

$$y(t) = O_1u_s(t) + O_2y_s(t). \tag{41}$$

where O_1 and O_2 are some matrices. For the system (37) with input $u(t)$ and output $y(t)$, we have the following possibilities:

1. *If K is invertible, then the system (37)–(41) is well-posed and regular. Furthermore, its transfer function converges to O_1K^{-1} for $\mathrm{Re}(s) \to \infty$*
2. *If K is not invertible, then the operator A_K defined as*

$$A_K \begin{pmatrix} g_+(z) \\ g_-(z) \end{pmatrix} = \frac{\partial}{\partial z}\left[\begin{pmatrix} \Lambda(z) & 0 \\ 0 & \Theta(z) \end{pmatrix}\begin{pmatrix} g_+(z) \\ g_-(z) \end{pmatrix}\right] \tag{42}$$

with domain

$$D(A_K) = \left\{ \begin{pmatrix} g_+(z) \\ g_-(z) \end{pmatrix} \in H^1((a,b),\mathbb{R}^n) \mid K\begin{pmatrix} \Lambda(b)g_+(b) \\ \Theta(a)g_-(a) \end{pmatrix} + Q\begin{pmatrix} \Lambda(a)g_+(a) \\ \Theta(b)g_-(b) \end{pmatrix} = 0 \right\} \tag{43}$$

does not generate a C_0-semigroup on $L^2((a,b);\mathbb{R}^n)$.

Note that the last item implies that the homogenous p.d.e. does not have a well-defined solution, when K is not invertible.

Proof: The first item is a direct consequence of Lemma 3.1 and 3.2 by noticing that the system (37)–(39) is built out of copies of the system (18)–(20) and the system (28)–(30). Furthermore, these sub-systems do not interact with each other.

For the proof of the first part of the second assertion, with K invertible, we rewrite the new input, as $u_s(t) = K^{-1}u(t) - K^{-1}Qy_s(t)$. This can be seen as a feedback interconnection on the system (37)–(39), as is depictured in Figure 1. The system contains one feedback loop with gain matrix

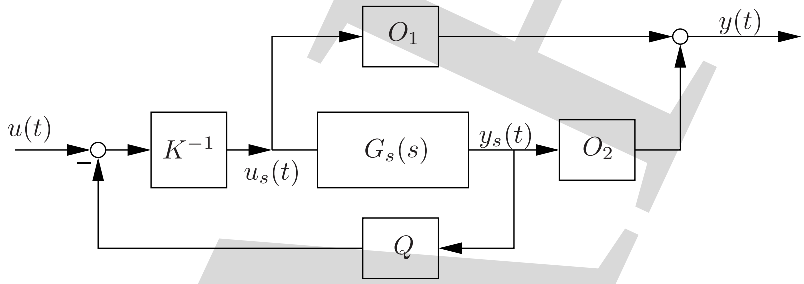

Figure 1: The system (37) with input (40) and output (41)

$K^{-1}Q$. By Weiss [16], see also Staffans [13, Chapter 7], we have that if $I + G_s(s)K^{-1}Q$ is invertible on some right-half plane and if this inverse is bounded on a right-half plane, then the closed loop system is well-posed. Since $\lim_{\mathrm{Re}(s)\to\infty} G_s(s) = 0$, we see that this holds for every K^{-1} and Q. So under the assumption that K is invertible, we find that (37) with input and output given by (40) and (41) is well-posed. The regularity follows easily. By regarding the loops in Figure 1 we see that the feed-though term is O_1K^{-1}.

So it remains to show that there is no C_0-semigroup when K is non-invertible. Since K is singular, there exists a non-zero $v \in \mathbb{R}^n$ such that $v^TK = 0$. Since $[K, Q]$ has full rank, we know that $q^T := v^TQ \neq 0$. So at least one of the components of q is unequal to zero. For the sake of the argument, we assume that this holds for the first one.

If A_K would be the infinitesimal generator of a C_0-semigroup, then for all $x_0 \in D(A_K)$ the abstract differential equation

$$\dot{x}(t) = A_Kx(t), \qquad x(0) = x_0 \tag{44}$$

would have classical solution, i.e., for all $t > 0$, $x(t)$ is differentiable, it is an element of $D(A_K)$, and it satisfies (44). Hence by (43), we have that

$x(t)$ is an element of H^1. Since we are working in a one dimensional spatial domain, we have that functions in H^1 are continuous. So we have that for every t, $x(t)$ is a continuous function of z satisfying the boundary conditions in (43).

So if A_K would generate a C_0-semigroup, then for every $x_0 \in D(A_K)$ there would be a function $x(t,z) := \begin{pmatrix} x_+(t,z) \\ x_-(t,z) \end{pmatrix}$ which is a (mild) solution to the p.d.e. (37), and satisfies for all $t > 0$ the boundary condition

$$K \begin{pmatrix} \Lambda(b)x_+(t,b) \\ \Theta(a)x_-(t,a) \end{pmatrix} + Q \begin{pmatrix} \Lambda(a)x_+(t,a) \\ \Theta(b)x_-(t,b) \end{pmatrix} = 0.$$

Using the vectors v and q, we see that this $x(t,z)$ must satisfy

$$0 = q^T \begin{pmatrix} \Lambda(a)x_+(t,a) \\ \Theta(b)x_-(t,b) \end{pmatrix}, \quad t > 0. \tag{45}$$

Now we construct an initial condition in $D(A_K)$, for which this equality does not hold. Note that we have chosen the first component of q unequal to zero.

The initial condition x_0 is chosen to have all components zero except for the first one. For this first component we choose an arbitrary function in $H^1(a,b)$ which is zero at a and b, but nonzero everywhere on the open set (a,b). It is clear that this initial condition is in the domain of A_K. Now we solve (37).

Standard p.d.e. theory gives that the solution of (37) can be written as

$$x_{+,m}(t,z) = f_{+,m}(p_m(z) + t)\lambda_m(z)^{-1},$$
$$x_{-,l}(t,z) = f_{-,l}(n_l(z) + t)\theta_l(z)^{-1},$$

where λ_m and θ_l are the m-th and the l-th diagonal element of Λ and Θ, respectively. Furthermore, $p_m(z) = \int_a^z \lambda_m(\zeta)^{-1} d\zeta$, $n_l(z) = \int_a^z \theta_l(\zeta)^{-1} d\zeta$, see also Lemmas 3.1 and 3.2. The functions f_+, f_- need to be determined from the boundary and initial conditions.

Using the initial condition we have that $f_{+,m}(p_m(z)) = \lambda_m(z)x_{0,+,m}(z)$ and $f_{-,l}(n_l(z)) = \theta_l(z)x_{0,-,l}(z)$. Since $p_m > 0$, and $n_l < 0$, we see that the initial condition determines f_+ on a (small) positive interval, and f_- on a small negative interval. By our choice of the initial condition, we find that

$$\begin{aligned} f_{+,1}(\xi) &= \lambda_1(\xi)x_{0,+,1}(\xi) && \xi \in [0, p_1(b)), \\ f_{+,m}(\xi) &= 0 && \xi \in [0, p_m(b)), \quad m \geq 2, \\ f_{-,l}(\xi) &= 0 && \xi \in [n_l(b), 0), \quad l \geq 1. \end{aligned} \tag{46}$$

The solution $x(t,z)$ must also satisfy (45), thus for all $t > 0$ we have that

$$0 = q^T \begin{pmatrix} f_+(t) \\ f_-(n(b)+t) \end{pmatrix} \tag{47}$$

Combining this with (46), we find

$$0 = q_1 f_{+,1}(p_1(z)) = q_1 x_{0,+,1}(z)\lambda_1^{-1}(z)$$

on some interval $[a, \beta]$. Since q_1 and λ_1 are unequal to zero, we find that x_0 must be zero on some interval. This is in contradiction with our choice of the initial condition. Thus A_K cannot be the infinitesimal generator of a C_0-semigroup. ∎

4 Proof of Theorem 2.4.

In this section we use the results of the previous section to prove Theorem 2.4. We begin with a useful lemma, its proof can be found in Chapter 7 of Staffans [13].

Lemma 4.1 *The system (11)–(14) is well-posed if and only if the system*

$$\frac{\partial x}{\partial t}(t,x) = P_1 \frac{\partial}{\partial z}(\mathcal{L}x)(t,z), \tag{48}$$

with inputs, outputs given by (12)–(14) is well-posed.

Let $G(s)$ denote the transfer function of (11)–(14) and $G_0(s)$ the transfer function of (48) with (12)–(14). Then

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty} G_0(s), \tag{49}$$

and

$$\lim_{\mathrm{Re}(s)\to\infty} G(s) = \lim_{\mathrm{Re}(s)\to\infty} G_0(s), \tag{50}$$

This lemma tells us that we may ignore any bounded linear term involving x.

By the third assumption of Theorem 2.4, the matrices P_1 and $\mathcal{L}$ satisfy the equation (16):

$$P_1\mathcal{L}(z) = S^{-1}(z)\Delta(z)S(z).$$

With this we introduce the new state vector

$$\tilde{x}(t,z) = S(z)x(t,z), \qquad z \in [a,b] \tag{51}$$

Under this basis transformation, the p.d.e. (11) becomes

$$\frac{\partial \tilde{x}}{\partial t}(t,z) = \frac{\partial}{\partial z}\left(\Delta \tilde{x}\right)(t,z) + S(z)\frac{dS^{-1}(z)}{dz}\Delta(z)\tilde{x}(t,z) + S(z)P_0(z)S(z)^{-1}\tilde{x}(t,z), \qquad x(0,z) = S(z)x_0(z) = \tilde{x}_0(z). \tag{52}$$

The relations (12)–(14) become

$$\begin{aligned} 0 =& M_{11}P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(t,b) + M_{12}P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(t,a) \\ =& \tilde{M}_{11}\Delta(b)\tilde{x}(t,b) + \tilde{M}_{12}\Delta(a)\tilde{x}(t,a) && (53) \\ u(t) =& M_{21}P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(t,b) + M_{22}P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(t,a) \\ =& \tilde{M}_{21}\Delta(b)\tilde{x}(t,b) + \tilde{M}_{22}\Delta(a)\tilde{x}(t,a) && (54) \\ y(t) =& C_1P_1^{-1}S^{-1}(b)\Delta(b)\tilde{x}(t,b) + C_2P_1^{-1}S^{-1}(a)\Delta(a)\tilde{x}(t,a) \\ =& \tilde{C}_1\Delta(b)\tilde{x}(t,b) + \tilde{C}_2\Delta(a)\tilde{x}(t,a). && (55) \end{aligned}$$

We introduce $\tilde{M} = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix}$ with

$$\begin{pmatrix} \tilde{M}_{j1} & \tilde{M}_{j2} \end{pmatrix} = \begin{pmatrix} M_{j1} & M_{j2} \end{pmatrix} \begin{pmatrix} P_1^{-1}S(b)^{-1} & 0 \\ 0 & P_1^{-1}S(a)^{-1} \end{pmatrix}, \quad j = 1,2.$$

and

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} P_1^{-1}S(b)^{-1} & 0 \\ 0 & P_1^{-1}S(a)^{-1} \end{pmatrix}.$$

Since the matrix $\begin{pmatrix} P_1^{-1}S(b)^{-1} & 0 \\ 0 & P_1^{-1}S(a)^{-1} \end{pmatrix}$ has full rank, we see that the rank conditions in Theorem 2.4 imply similar rank conditions for $\tilde{M}$ and $\tilde{C}$.

Using Lemma 4.1 we see that we only have to prove the result for the p.d.e.

$$\frac{\partial \tilde{x}}{\partial t}(t,z) = \frac{\partial}{\partial z}\left(\Delta \tilde{x}\right)(t,z). \tag{56}$$

with boundary conditions, inputs, and outputs as described in (53)–(55).

It is clear that if condition (53) is not present, then Theorem 3.3 gives that the above system is well-posed and regular if and only if the homogeneous p.d.e. generates a C_0-semigroup on $L^2((a,b);\mathbb{R}^n)$. Since the state transformation (51) defines a bounded mapping on $L^2((a,b);\mathbb{R}^n)$, we have proved Theorem 2.4 provided there is no condition (12).

Thus it remains to prove Theorem 2.4 if we have put part of the boundary conditions to zero. Or equivalently, to prove that the system (53)–(56) is well-posed and regular if and only if the homogeneous p.d.e. generates a C_0-semigroup.

We replace (53) by

$$v(t) = \tilde{M}_{11}\Delta(b)\tilde{x}(t,b) + \tilde{M}_{12}\Delta(a)\tilde{x}(t,a), \tag{57}$$

where we regard v as a new input. Hence we have the system (56) with the new extended input

$$\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \tilde{M}_{11} \\ \tilde{M}_{21} \end{pmatrix} \Delta(b)\tilde{x}(t,b) + \begin{pmatrix} \tilde{M}_{12} \\ \tilde{M}_{22} \end{pmatrix} \Delta(a)\tilde{x}(t,a). \tag{58}$$

and the output (55). Hence we have obtained a system without a condition (53). For this system we know that it is well-posed and regular if and only if the homogeneous equation generates a C_0-semigroup.

Assume that the system (56), (58) and (55) is well-posed, then we may choose any (locally) square input. In particular, we may choose $v \equiv 0$. Thus the system (52)–(56) is well-posed and regular as well.

Assume next that the p.d.e. with the extended input in (58) set to zero, does not generate a C_0-semigroup. Since this gives the same homogeneous p.d.e. as (56) with (53) and u in (54) set to zero, we know that this p.d.e. does not generate a C_0-semigroup. This finally proves Theorem 2.4.

5 Example of the vibrating string.

Consider again the example of the vibrating string written as the system of conservation laws (7).

This system belongs to the class of systems defined in the equation (8) with zero source term (i.e. $P_0 = 0$) and

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}.$$

The operator $P_1\mathcal{L}$ of this system is diagonalizable:

$$P_1\mathcal{L} = S(z)^{-1}\Delta(z)S(z) = \begin{pmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ \frac{-1}{2\gamma} & \frac{\rho}{2} \end{pmatrix}, \tag{59}$$

where $\gamma^2 = c = \frac{T}{\rho}$. We take γ to be positive.

Hence the state transformation under which the p.d.e. becomes diagonal is

$$\tilde{x} = \frac{1}{2} \begin{pmatrix} \frac{1}{\gamma} & \rho \\ \frac{-1}{\gamma} & \rho \end{pmatrix} x.$$

Since we assumed that $\gamma > 0$, we see that $\tilde{x}_1$, $\tilde{x}_2$ correspond to x_+, and x_- in equation (37), respectively and Λ, Θ to γ and $-\gamma$ respectively. Hence we have that the input and output u_s and y_s defined for the diagonal system (37) by the equations (38)–(39) are expressed in the original coordinates by

$$u_s(t) = \frac{1}{2}\begin{pmatrix} x_1(t,b) + \gamma\rho x_2(t,b) \\ x_1(t,a) - \gamma\rho x_2(t,a) \end{pmatrix}, \tag{60}$$

$$y_s(t) = \frac{1}{2}\begin{pmatrix} x_1(t,a) + \gamma\rho x_2(t,a) \\ x_1(t,b) - \gamma\rho x_2(t,b) \end{pmatrix}. \tag{61}$$

This pair of boundary input and output variables consist in complementary linear combinations of the momentum x_1 and the the strain x_2 at the boundaries: however they lack an obvious physical interpretation. One could consider another choice of boundary input and outputs, for instance the velocity and the strain at the boundary points and choose as input $u_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(t,b) \\ x_2(t,a) \end{pmatrix}$ and as output $y_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(t,a) \\ x_2(t,b)) \end{pmatrix}$. We may apply Theorem 3.3 to check whether this system is well-posed, and to find the feed-through. Expressing the input-output pair (u_1, y_1) in (u_s, y_s) gives

$$u_1(t) = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix} u_s(t) + \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix} y_s(t), \tag{62}$$

$$y_1(t) = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix} y_s(t). \tag{63}$$

Hence

$$K = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix} \quad Q = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix}, \tag{64}$$

$$O_1 = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \frac{1}{\sqrt{T\rho}} & 0 \end{pmatrix}, \qquad O_2 = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{-1}{\sqrt{T\rho}} \end{pmatrix}. \tag{65}$$

Since K is invertible, the system with the input-output pair (u_1, y_1) is well-posed and regular, and the feed-through term is given by $O_1 K^{-1} = \begin{pmatrix} 0 & -\gamma \\ \frac{1}{\gamma} & 0 \end{pmatrix}$.

Let us now interprete the boundary inputs u_1 and outputs y_1 in terms of the position w in the second order p.d.e. (6) for which the change of coordinates with respect to the state variable of the system of conservation laws is $x_1 = \rho\frac{\partial w}{\partial t}$ and $x_2 = \frac{\partial w}{\partial z}$. One obtains then the expression of the boundary inputs and outputs: $u_1(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,b) \\ \frac{\partial w}{\partial z}(t,a) \end{pmatrix}$ and $y_1(t) = \begin{pmatrix} \frac{\partial w}{\partial t}(t,a) \\ \frac{\partial w}{\partial z}(t,b) \end{pmatrix}$, respectively. However considering the p.d.e. (6), one could consider as one of the boundary inputs or output the position and not the velocity at the boundary points. This leads to consider:

$$u_2(t) = \begin{pmatrix} w(t,b) \\ \frac{\partial w}{\partial z}(t,a) \end{pmatrix} \tag{66}$$

$$y_2(t) = \begin{pmatrix} w(t,a) \\ \frac{\partial w}{\partial z}(t,b) \end{pmatrix}. \tag{67}$$

These inputs and outputs a related to the previously defined input u_1 and output y_1 by integration and derivation with respect to time and this can be de-pictured as

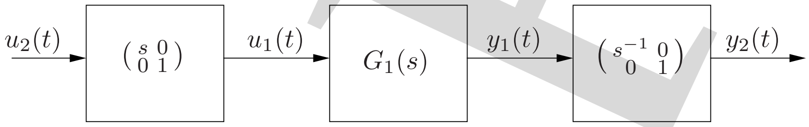

Figure 2: The system (6) with input and output (66) and (67)

From this it is clear that the transfer function, $G_2(s)$ of the system with input u_2 and output y_2 is given by

$$G_2(s) = \begin{pmatrix} s^{-1} & 0 \\ 0 & 1 \end{pmatrix} G_1(s) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}.$$

Since for large real s the transfer function $G_1(s)$ is approximately equal to $\begin{pmatrix} 0 & -\gamma \\ \gamma^{-1} & 0 \end{pmatrix}$, we see that $G_2(s)$ grows like s for large s. Any well-posed system has a transfer function which is bounded in some right-half plane. Thus the system (6) with input (66) and output (67) is not well-posed.

Putting the input u_2 to zero, is the same as putting the input u_1 to zero. Hence the homogenous p.d.e. will have a unique solution. Furthermore, for any initial condition the output corresponding to the homogeneous p.d.e. will stay square integrable. Hence the output is admissible, see Weiss [16]. On the other hand, since the transfer function grows to infinity for $s \to \infty$ the input will not be admissible.

Note that $u_3(t) = \frac{\partial w}{\partial z}(t,a)$, $w(t,b) = 0$, and $y_3(t) = y_2(t)$ will give a well-posed system.

6 A semigroup example

In this section, we show that it is possible to construct a closed operator A and a coercive operator $\mathcal{L}$ such that A with domain $D(A)$ is not the infinitesimal generator of a C_0-semigroup on the Hilbert space X, but the operator $A_{\mathcal{L}}$ defined as $A_{\mathcal{L}} := A\mathcal{L}$ with domain all $x \in X$ such that $\mathcal{L}x \in D(A)$ does generate a C_0-semigroup on X. The example is based on the results of the previous section.

We begin with defining A, $\mathcal{L}$, and the state space. As state space we choose $L^2((0,1);\mathbb{R}^2)$. The coercive operator $\mathcal{L}$ is just

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}, \tag{68}$$

where $\gamma > 0$. The operator A is defined as

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\partial}{\partial z} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \tag{69}$$

with domain

$$D(A) = \{x \in H^1((0,1);\mathbb{R}^2) \mid x_1(0) + x_2(0) = 0, x_2(1) = 0\} \tag{70}$$

Consider the p.d.e. associated to $A_{\mathcal{L}} = A\mathcal{L}$

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t,z) = \frac{\partial}{\partial z} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t,z) \right] \tag{71}$$

with boundary conditions

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t,0) + \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t,1). \end{aligned} \tag{72}$$

So we have the p.d.e. (7) with $\rho = 1$ and $T = \gamma^2$. We write the boundary condition (72) in terms of u_s and y_s, see (60) and (61). This gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1-\gamma \\ \gamma & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} 1+\gamma & 0 \\ 0 & -\gamma \end{pmatrix} y_s(t). \tag{73}$$

From Theorem 3.3 we know that for $\gamma = 1$ this does not generate a C_0-semigroup, and this is the only positive γ with this property. Hence we have that A does not generate a C_0-semigroup, whereas $A\mathcal{L}$ with $\gamma \in (0,1)\cup(1,\infty)$ does. Note that for every $\gamma > 0$ the operator $\mathcal{L}^{-1}A\mathcal{L}$ with domain equal to the domain of $A_{\mathcal{L}}$ does not generate a C_0-semigroup. This follows from the fact that $\mathcal{L}^{-1}A\mathcal{L}$ is similar to A, see exercise 2.5 of [1].

Finally, we remark, that it is easy to show that if A generates a contraction semigroup on a Hilbert space, then $A\,\mathcal{L}$ is the infinitesimal generator of a C_0-semigroup.

7 Well-posedness on L^q-spaces.

The theorem as formulated in Section 2 can also be formulated in L^q spaces. Thus

Theorem 7.1 *Let $q \geq 1$. Under the conditions of Theorem 2.4 we have that the homogeneous p.d.e. generates a C_0-semigroup on $L^q(a,b)$ if and only if the system is well-posed for L^q inputs and outputs, that is*

$$\int_0^{t_f} \|y(t)\|^q dt + \int_a^b \|x(t_f,z)\|^q dz \leq m_f \left[\int_0^{t_f} \|u(t)\|^q dt + \int_a^b \|x_0(z)\|^q dz \right]. \tag{74}$$

For the proof, one first proves the corresponding relation for the diagonal system, then uses a results of Staffans, chapter 7, to show that the feedback is allowed if K is invertible. The proof of the non-existences of the semigroup uses smooth functions, and so they are in L^q for all q and so this proof easily carries over. The only essential part in the proof is to show Lemma 3.1, in particular equation (27) for L^q-functions.

Consider the p.d.e. (18)–(20). Its solution is given by (21)–(24). Now define $Q(t)$ as

$$Q(t) = \int_a^b |w(t,z)|^q \lambda^{q-1}(z) dz. \tag{75}$$

Using (21) we see that $Q(t)$ is equal to

$$Q(t) = \int_a^b |f(p(z)+t)|^q \lambda^{-1}(z) dz$$

Performing the change of variables $\xi = p(z) + t$ in this integral, and using that $\dot{p} = \lambda^{-1}$, see (22), we find that

$$Q(t) = \int_{p(a)+t}^{p(b)+t} |f(\xi)|^q d\xi. \tag{76}$$

So we find that

$$\begin{aligned} Q(t_f) - Q(0) &= \int_0^{t_f} \dot{Q}(\tau) d\tau \\ &= \int_0^{t_f} \left[|f(p(b)+\tau)|^q - |f(p(a)+\tau)|^q \right] d\tau \\ &= \int_0^{t_f} |u(\tau)|^q d\tau - \int_0^{t_f} |y(\tau)|^q d\tau, \end{aligned} \tag{77}$$

where we have used (20) and (24). Combining (75) and (77) gives he desired estimate.

We conclude this session by remarking that the equality (77) does not hold for $q = \infty$, i.e., for the sup norm. Simply take $f(\xi) = \xi$ and $\lambda = 1$.

8 Conclusion

In this paper, we use physical port Hamiltonian modeling to study a class of hyperbolic partial differential equations on a one dimensional spatial domain with control and observation at the boundary. Even if for this class of systems the input mapping operator is unbounded, we show that the system is well-posed if and only if the state operator generates a C_0-semigroup. Furthermore, in this case the associated transfer function is regular and it is possible to give the expression of its direct transmission term. To prove this result we consider a basis state transformation to represent the system as a coupled set of delay lines. We regard this coupling as a feedback interconnection. The well posedness of the global system is given checking a condition on this feedback according to a result proposed by Weiss [16]. Furthermore, from the input and output definition in the diagonal case we characterize all the possible choices of boundary input and output that permit to obtain a well posed system and give, from his parameterization, the associated direct transmission term. With this work we showed that appropriate physical modeling is of great interest to prove properties of solutions associated to boundary control systems. Finally, we can remark that the class of systems treated in this paper is large enough to deal with well known examples from the literature as coupled wave equation (see [8, chapter 6] and [2]), Timoshenko beam equation, convection equation. Currently, we try to generalize these results to systems with a dissipation term.

References

[1] R.F. Curtain and H.J. Zwart, *An introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[2] R. Dáger and E. Zuazua, *Wave Propagation, Observation and Control in 1-d Flexible Multi-structures*, Mathématiques & Apllications, 50, Springer Verlag, 2006.

[3] Y. Le Gorrec, B.M. Maschke, H. Zwart and J.A. Villegas, Dissipative boundary control systems with application to distributed parameters

reactors, *Proc. IEEE International Conference on Control Applications*, pp. 668–673 Munich, Germany, Oct. 4-6, 2006.

[4] Y. Le Gorrec, H. Zwart, and B. Maschke, Dirac structures and boundary control systems associated with skew-symmetric differential operators, *SIAM J. Control and Optim.*, vol. 44(5), 2005, pp. 1864–1892; also available at `www.math.utwente.nl/publications`, Memorandum No. 1730, 2004.

[5] T. Kato, *Perturbation Theory for Linear Operators*, Corrected printing of the second edition, Springer Verlag, Berlin, 1980.

[6] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations I*, Encyclopedia of Mathematics and its Applications, 74, Cambridge University Press, 2000.

[7] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations II*, Encyclopedia of Mathematics and its Applications, 75, Cambridge University Press, 2000.

[8] Z-H. Luo, B-Z Guo, and O. Morgul, *Stability and Stabilization of Infinite-Dimensional Systems with Applications*, Springer Verlag, 1999.

[9] A. Macchelli and C. Melchiorri, Modeling and control of the Timoshenko beam. The distributed port Hamiltonian approach, *SIAM Journal On Control and Optimization*, **43**, 2, 743-767, 2004.

[10] B. Maschke and A.J. van der Schaft. *Advanced Topics in Control Systems Theory. Lecture Notes from FAP 2004*, chapter Compositional modelling of distributed-parameter systems, pp. 115–154. Lecture Notes on Control and Information Sciences. Springer, 2005.

[11] J. Malinen, Conservativity of time-flow invertible and boundary control systems, *Helsinki University of Technology, Institute of Mathematics, Research Report A479*, 2004, see also Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC'05).

[12] A.J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed parameter systems with boundary energy flow. *J. of Geometry and Physics* **42**, 166–174, 2002.

[13] O. Staffans, *Well-posed Linear Systems*, Encyclopedia of Mathematics and its Applications, 103, Cambridge University Press, 2005.

[14] M. Tucsnak and G. Weiss, *Passive and Conservative Linear Systems*, available at http://www.ee.imperial.ac.uk/gweiss/personal/

[15] J.A. Villegas, *A Port-Hamiltonian Approach to Distributed Parameter Systems*, Ph.D. thesis, University of Twente, May, 2007. Availible at http://doc.utwente.nl

[16] G. Weiss, Regular linear systems with feedback, *MCSS*, vol. 7, pp. 23–57, 1994.

[17] H. Zwart, Transfer functions for infinite-dimensional systems, *Systems & Control Letters*, vol. 52, pp. 247–255, 2004.

[18] H. Zwart, Y. Le Gorrec, B.M.J. Maschke and J.A. Villegas, Well-posedness and regularity for a class of hyperbolic boundary control systems, *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, Japan, pp. 1379–1883, 2006.