Left invertible semigroups on Hilbert spaces.*

Hans Zwart
Department of Applied Mathematics,
Faculty of Electrical Engineering, Mathematics and Computer Science,
University of Twente,
P.O. Box 217, 7500 AE Enschede,
The Netherlands.
h.j.zwart@utwente.nl

August 30, 2012

Abstract

For strongly continuous semigroups on a Hilbert space, we present a short proof of the fact that the left-inverse of a left-invertible semigroup can be chosen to be a semigroup as well. Furthermore, we show that this semigroup need not to be unique.

1 Introduction

In this paper we study left-invertible semigroups. In their paper [4] of 1983 Louis and Wexler showed that if a strongly continuous semigroup on a Hilbert space is left invertible for one (or equivalently all) positive time instants, then there exists a left inverse which is also a strongly continuous semigroup. Their proof uses optimal control and Riccati equations. We present a shorter proof with uses Lyapunov equations. Furthermore, using this Lyapunov equation, we can show that any left-invertible semigroup is a bounded perturbation of an isometric semigroup, see Theorem 2.6. Our results complement those found in [2], [5] and [6], who mainly concentrate on the relation between left invertibility and exact observability.

We conclude this introduction by introducing some notation. Z will denote a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The domain of the operator A will be denoted by $D(A)$, and by $\rho(A)$ we denotes its resolvent set. The

*artikel/left-inv2012.tex

point spectrum of A is given by $\sigma_p(A)$. If A is a densely defined operator on Z, then A^* will denote the adjoint of this operator. Furthermore, if A is the infinitesimal generator of a C_0-semigroup, then we denote this semigroup by $(T(t))_{t\geq 0}$.

The bounded linear operators from Z to Z are denoted by $\mathcal{L}(Z)$, and the bounded linear operators from the Hilbert space Z_1 to the Hilbert space Z_2 are denoted by $\mathcal{L}(Z_1, Z_2)$.

2 Left-invertible semigroups

In this section we present a new proof for the fact that the left inverse of a left invertible semigroup on a Hilbert space can be chosen to be a C_0-semigroup. We begin with the definition of a left-invertible semigroup.

Definition 2.1. The C_0-semigroup $(T(t))_{t\geq 0}$ is left-invertible if there exists a function $t \mapsto m(t)$ such that $m(t) > 0$ and for all $z_0 \in Z$ there holds

$$m(t)\|z_0\| \leq \|T(t)z_0\|, \qquad t \geq 0. \tag{1}$$

□

Using the semigroup property, it is not hard to show that the following are equivalent, see e.g. [6].

- $(T(t))_{t\geq 0}$ is left-invertible,
- The exists a $t_0 > 0$ such that $T(t_0)$ is left-invertible, i.e., there exists a m_0 such that for all $z_0 \in Z$ there holds $m_0\|z_0\| \leq \|T(t_0)z_0\|$.

To formulate the characterization of left-invertible semigroups we need the definition of exact observability, see e.g [1].

Definition 2.2. Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z and let $C \in \mathcal{L}(Z, Y)$, where Y is a second Hilbert space. The system $\Sigma(A, -, C)$ is *exactly observable in finite-time* if there exist $t_0 > 0$ and $m > 0$ such that for all $z_0 \in Z$

$$m\|z_0\|^2 \leq \int_0^{t_0} \|CT(t)z_0\|^2 dt. \tag{2}$$

□

In the proof of Theorem 2.4 the following lemma is essential.

Lemma 2.3. *Let A_1, A_2 be the infinitesimal generators of the C_0-semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$, respectively. Then $X \in \mathcal{L}(Z)$ satisfies the Sylvester equation*

$$\langle A_1 z_1, X z_2\rangle + \langle z_1, X A_2 z_2\rangle = 0, \quad z_1 \in D(A_1), z_2 \in D(A_2) \tag{3}$$

if and only if

$$T_1^*(t) X T_2(t) = X, \qquad \text{for all } t \geq 0. \tag{4}$$

Moreover, if X is (boundedly) invertible, then

$$X^{-1} T_1^*(t) X T_2(t) = I, \qquad \text{for all } t \geq 0.$$

Thus $\left(X^{-1}T_1^*(t)X\right)_{t\geq 0}$ is the left-inverse of $(T_2(t))_{t\geq 0}$.

Proof: We see that (4) is equivalent to $\langle T_1(t)z_1, XT_2(t)z_2\rangle = \langle z_1, Xz_2\rangle$ for all $z_1, z_2 \in Z$. Differentiating this expression for $z_1 \in D(A_1)$ and $z_2 \in D(A_2)$ and evaluating it at $t = 0$ gives (3).

If (3) holds, then for $z_{10} \in D(A_1)$ and $z_{20} \in D(A_2)$

$$\begin{aligned}\frac{d}{dt}\langle T_1(t)z_{10}, & XT_2(t)z_{20}\rangle \\ &= \langle A_1T_1(t)z_{10}, XT_2(t)z_2\rangle + \langle T_1(t)z_1, XA_2T_2(t)z_2\rangle = 0.\end{aligned}$$

Thus $\langle T_1(t)z_{10}, XT_2(t)z_{20}\rangle = \langle z_{10}, Xz_{20}\rangle$ for all $z_{10} \in D(A_1)$ and $z_{20} \in D(A_2)$ Since these domains are dense we conclude that (4) holds. □

Now we can formulate and prove the main result of this section.

Theorem 2.4. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z with infinitesimal generator A. Then the following are equivalent.*

1. *$(T(t))_{t\geq 0}$ is left-invertible.*
2. *The system $\Sigma(A, -, I)$ is exactly observable in finite-time.*
3. *There exists a Hilbert space Y and an operator $C \in \mathcal{L}(Z, Y)$ such that $\Sigma(A, -, C)$ is exactly observable in finite-time.*
4. *There exists an $\omega \in \mathbb{R}$, a Hilbert space Y, an operator $C \in \mathcal{L}(Z, Y)$, and a positive operator $X \in \mathcal{L}(Z)$ such that X is (boundedly) invertible and satisfies the Lyapunov equation*

   $$\langle (A - \omega I)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = -\langle Cz_1, Cz_2\rangle, \tag{5}$$

   for all $z_1, z_2 \in D(A)$.

5. *There exists a C_0-semigroup $(S(t))_{t\geq 0}$ such that $S(t)T(t) = I$ for all $t \geq 0$.*

PROOF: It is clear that 2. implies 3. and that 5. implies 1. So we concentrate on the other implications.

1. $\Rightarrow$ 2. Since (1) holds, we see that for all $t_0 > 0$ there holds

$$\int_0^{t_0} m(t)^2 dt \|z\|^2 = \int_0^{t_0} m(t)^2 \|z\|^2 dt \leq \int_0^{t_0} \|T(t)z\|^2 dt. \tag{6}$$

Thus $\Sigma(A, -, I)$ is exactly observable.

3. $\Rightarrow$ 4. Choose $\omega \in \mathbb{R}$ larger than the growth bound of $(T(t))_{t\geq 0}$. Then we have that the semigroup generated by $A - \omega I$ is exponentially stable. Now for $z \in Z$

$$\begin{aligned} m\|z\|^2 = \int_0^{t_0} \|CT(t)z\|^2 dt &\leq m_1 \int_0^{t_0} e^{-2\omega t} \|CT(t)z\|^2 dt \\ &\leq m_1 \int_0^{\infty} e^{-2\omega t} \|CT(t)z\|^2 dt \leq m_1 M \|z\|^2, \end{aligned} \tag{7}$$

where we have used that C is bounded and that $\left(e^{-\omega t}T(t)\right)_{t\geq 0}$ is exponentially stable. Define $\langle z_1, Xz_2\rangle = \int_0^\infty e^{-2\omega t}\langle CT(t)z_1, CT(t)z_2\rangle dt$. Then X is symmetric, and by the above relation $\frac{m}{m_1} I \leq X \leq MI$. In particular, X is a bounded operator with a bounded inverse. Similar as the proof of Lemma 2.3 we can show that X is a solution to the Lyapunov equation

$$\langle (A - \omega I)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = -\langle Cz_1, Cz_2\rangle,$$

for $z_1, z_2 \in D(A)$.

4. $\Rightarrow$ 5. We rewrite the Lyapunov equation (5) to the Sylvester equation

$$\langle (A - \omega I + X^{-1}C^*C)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = 0$$

for $z_1, z_2 \in D(A)$. Since this can be seen as (3) with $A_1 = A - \omega I + X^{-1}C^*C$ and $A_2 = A - \omega I$, we obtain by Lemma 2.3 that the semigroup generated by $A - \omega I$ is left invertible, and

$$X^{-1}T_1(t)^* XT(t)e^{-\omega t} = I,$$

where $(T_1(t))_{t\geq 0}$ is the semigroup generated by $A - \omega I + X^{-1}C^*C$. Thus $S(t) := X^{-1}T_1(t)^* X e^{-\omega t}$ is the left-inverse of $T(t)$. □

We remark that the theorem also holds for $C \in \mathcal{L}(Z,Y)$ for which $\Sigma(A,-,C)$ is exactly observable in infinite-time provided we have for all $z_0 \in Z$ that $\int_0^\infty \|CT(t)z_0\|^2 dt < \infty$. Just define X in part 4 as $\langle z_1, Xz_2\rangle = \int_0^\infty \langle CT(t)z_0, CT(t)z_2\rangle dt$ and take $\omega = 0$.

It is well-known that a C_0-semigroup can be extended to a group if and only if minus its generator is also a generator. For a left invertible semigroup a similar result holds.

Theorem 2.5. *Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space Z. Then $(T(t))_{t\geq 0}$ is left-invertible if and only if $-A$ can be extended to an infinitesimal generator of a C_0-semigroup.*

Proof: Assume that $(T(t))_{t\geq 0}$ is left-invertible and let $(S(t))_{t\geq 0}$ be a C_0-semigroup such that $S(t)T(t) = I$. Let A_2 with domain $D(A_2)$ be the infinitesimal generator of $(S(t))_{t\geq 0}$. For $z \in Z$ we have that

$$S(t)z - z = S(t)z - S(t)T(t)z = S(t)\left(z - T(t)z\right).$$

For $z \in D(A)$ we obtain, also using the strong continuity of $(S(t))_{t\geq 0}$,

$$-Az = \lim_{t\downarrow 0} \frac{S(t)\left(z - T(t)z\right)}{t} = \lim_{t\downarrow 0} \frac{S(t)z - z}{t}.$$

Thus $z \in D(A_2)$ and $A_2 z = -Az$. In other words A_2 is an extension of $-A$.

Assume next that A_2 is the infinitesimal generator of the C_0-semigroup $(S(t))_{t\geq 0}$ and that A_2 is an extension of $-A$. Then $-A^*$ is an extension of A_2^*. For $z_1 \in D(A_2^*)$ and $z_2 \in D(A)$ there holds

$$\begin{aligned}\langle A_2^* z_1, z_2\rangle + \langle z_1, Az_2\rangle &= \langle A_2^* z_1, z_2\rangle + \langle A^* z_1, z_2\rangle \\ &= \langle A_2^* z_1, z_2\rangle + \langle -A_2^* z_1, z_2\rangle = 0,\end{aligned}$$

where we first used that $D(A_2^*) \subset D(A^*)$ and next that $A^* z_1 = -A_2^* z_1$ for $z_1 \in D(A_2^*)$. Using now Lemma 2.3, we conclude that $S(t)T(t) = I$. □

In [3], Haase showed that the infinitesimal generator of a group is a bounded perturbation of the infinitesimal generator of a unitary group. Using Theorem 2.4, we show that a similar result holds for left-invertible semigroups.

Theorem 2.6. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z with generator A. Then the following are equivalent:*

1. *$(T(t))_{t\geq 0}$ is left-invertible;*

2. There exists a bounded operator Q and an equivalent inner product such that $A+Q$ generates an isometric semigroup in the new norm.

PROOF: By Theorem 2.4 we have the existence of an $\omega \in \mathbb{R}$ and a $X \in \mathcal{L}(Z)$, boundedly invertible, such that

$$\langle (A-\omega I)z_1, Xz_2\rangle + \langle z_1, X(A-\omega I)z_2\rangle = -\langle z_1, z_2\rangle,$$

for $z_1, z_2 \in D(A)$. Now we write it as

$$\langle (A-\omega I + \frac{1}{2}X^{-1})z_1, Xz_2\rangle + \langle z_1, X(A-\omega I + \frac{1}{2}X^{-1})z_2\rangle = 0.$$

By defining $Q = -\omega I + \frac{1}{2}X^{-1}$, taking as new inner product $\langle z_1, z_2\rangle_{\text{new}} = \langle z_1, Xz_2\rangle$, and using Lemma 2.3, we obtain the desired result. □

In the remaining part of this section, we address two issues related to Theorem 2.4. The first one being the equivalence between item 1 and 3 for unbounded C's. The second one being the uniqueness of the semigroup $S(t)$ in item 5.

In the proof of the implication 3 ⇒ 4, we used that $e^{-\omega t}CT(t)z_0$ is square integrable on $[0,\infty)$. Even when $(T(t))_{t\geq 0}$ is exponentially stable, this does not imply that C is bounded. The class of operators $C : D(A) \mapsto Y$ for which this holds is called admissible. Hence it may seem that if item 3 holds for an admissible C, then item 1 will hold as well. The following example shows that this does not hold for general admissible output operators.

Example 2.7 Consider the left-shift semigroup on $L^2(0,1)$, i.e.

$$\left(T(t)f\right)(\eta) = \begin{cases} f(\eta+t) & \eta+t \in [0,1] \\ 0 & \eta+t \geq 1 \end{cases}$$

with the observation at $\eta = 0$, i.e.,

$$Cf = f(0).$$

Then $(T(t))_{t\geq 0}$ not left-invertible, but since

$$\int_0^1 |CT(t)f|^2 dt = \|f\|^2,$$

it is admissible and exactly observable in finite time. □

In [2, 5, 6] more can be found on the relation between left invertibility and admissibility. In [2] it is shown that the equivalence between 1, 2, and 3 in Theorem 2.4 also holds in a general Banach space.

It is well-known that the left inverse need not to be unique. However, since in item 5 the left inverse is a C_0-semigroup, it may be expected that this extra structure induces uniqueness. The following example shows that the left-inverse semigroup in item 5 of Theorem 2.4 need not to be unique.

Example 2.8 As Hilbert space Z we take $L^2(0,\infty)\oplus L^2(0,1)$. Furthermore, we take

$$A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\dot{f}_1 \\ -\dot{f}_2 \end{pmatrix}$$

with domain

$$D(A_1) = \{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in Z \mid f_1 \in H^1(0,\infty), f_2 \in H^1(0,1) \\ \text{and } f_1(0) = \alpha f_2(0), f_2(1) = \sqrt{2} f_2(0)\},$$

where $H^1(\Omega)$ denotes the Sobolev space of $L^2(\Omega)$-functions whose (distributional) derivative lies in $L^2(\Omega)$. The operator A_2 is defined similarly,

$$A_2\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -\dot{g}_1 \\ -\dot{g}_2 \end{pmatrix}$$

with domain

$$D(A_2) = \{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in Z \mid g_1 \in H^1(0,\infty), g_2 \in H^1(0,1) \\ \text{and } g_1(0) = 0, \sqrt{2} g_2(1) = g_2(0)\}.$$

It is not hard to show that these operators generate strongly continuous semigroups on Z. For $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in D(A_1)$ there holds

$$\begin{aligned}
&\left\langle A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\rangle + \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, A_1\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\rangle \\
&= \int_0^\infty -\dot{f}_1(x)\overline{f_1(x)}dx + \int_0^\infty f_1(x)(-\overline{\dot{f}_1(x)})dx + \\
&\quad \int_0^1 -\dot{f}_2(x)\overline{f_2(x)}dx + \int_0^1 f_2(x)(-\overline{\dot{f}_2(x)})dx \\
&= -\left[|f_1(x)|^2\right]_0^\infty - \left[|f_2(x)|^2\right]_0^1 \\
&= -0 + |\alpha f_2(0)|^2 - 2|f_2(0)|^2 + |f_2(0)|^2 \le 0,
\end{aligned}$$

for $0 \leq \alpha \leq \sqrt{2}$. Thus for these values of α, A_1 generates a contraction semigroup. Next we show that (3) is satisfied for $X = I$ for all α.

$$\begin{aligned}
&\left\langle A_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, A_2 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle \\
&= \int_0^\infty -\dot{f}_1(x)\overline{g_1(x)}dx + \int_0^\infty f_1(x)(-\overline{\dot{g}_1(x)})dx + \\
&\quad \int_0^1 -\dot{f}_2(x)\overline{g_2(x)}dx + \int_0^1 f_2(x)(-\overline{\dot{g}_2(x)})dx \\
&= -\left[f_1(x)\overline{g_1(x)}\right]_0^\infty - \left[f_2(x)\overline{g_2(x)}\right]_0^1 \\
&= -0 + 0 - f_2(1)\overline{g_2(1)} + f_2(0)\overline{g_2(0)} \\
&= -\sqrt{2}f_2(0)\frac{1}{\sqrt{2}}\overline{g_2(0)} + f_2(0)\overline{g_2(0)} = 0,
\end{aligned}$$

where we used the boundary conditions. Hence by Lemma 2.3 A_1^* generates the left inverse, but because of it dependence on α it is not unique. $\square$

3 Open problem

For left invertibility semigroups on Banach spaces it is unknown whether a left-inverse can always be chosen to be a C_0-semigroup, and the author would like to pose this as an open problem .

For a long time, he believed that the right shift semigroup on $X := \{f : [0,\infty) \mapsto \mathbb{C} \mid f \text{ is continuous and } f(0) = f(\infty) = 0\}$ with the maximum norm could serve as a counter example. However, it is not hard to show that for every $\alpha > 0$

$$(S(t)f)(\eta) = f(t+\eta) - f(t)e^{-\alpha\eta}, \qquad f \in X,$$

defines a C_0-semigroup on X which is a left-inverse of the right shift semigroup.

References

[1] R.F. Curtain and H.J. Zwart, *An introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[2] B.H. Haak and E.M. Ouhabaz, Exact observability, square functions and spectral theory, *J. Funct. Anal.*, 262(6), pp. 2903–2927, 2012.

[3] M. Haase, A decomposition theorem for generators of strongly continuous groups on Hilbert spaces, *J. Operator Theory*, 52(1), pp. 21–37, 2004.

[4] J.-C. Louis and D. Wexler, On exact controllability in Hilbert spaces, *Journal of Differential Equations*, 49, pp. 258–269, 1983.

[5] G.Q. Xu, C. Liu and S.P. Yung, Necessary conditions for the exact observability of systems on Hilbert spaces, *Systems & Control Letters*, 57(3), pp. 222–227, 2008.

[6] G.Q. Xu and Y.F. Shang, Characteristic of left invertible semigroups and admissibility of observation operators, *Systems & Control Letters*, 58(8), pp. 561–566, 2009.