Left invertibility and invertible solutions of the Lyapunov equation*

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Abstract

If the solution of a Lyapunov or Riccati equation corresponding to a finite-dimensional system is invertible, then this leads to additional properties of this system. For infinite-dimensional systems these results do not need to hold. We show that it is necessary to put some additional conditions on the spectrum of the system operator A. For instance, if this operator has compact resolvent, then the results known for finite-dimensional systems extend to infinite-dimensional ones. Using this result, we obtain more insight in Lyapunov and Riccati equations for well-posed linear systems.

1 Introduction

In this paper we study two subjects. Left-invertible semigroups and invertible solutions of Lyapunov equations. It is well-known that there is a strong relation between these two concepts. For A the infinitesimal generator of an exponentially stable semigroup on a Hilbert space, Louis and Wexler [7] LoWe83 showed that there exists an bounded and boundedly invertible solution X of the Lyapunov equation

$$\langle Az, Xz\rangle + \langle Xz, Az\rangle = -\|z\|^2, \qquad z \in D(A) \tag{1}$$ eq:5

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if and only if the strongly continuous semigroup generated by A is left-invertible. The proof as presented by Louis and Wexler uses optimal control and Riccati equations. We present a shorter proof with uses the Lyapunov equation directly. Furthermore, using this Lyapunov equation, we can show that any left-invertible semigroup is a bounded perturbation of an isometric semigroup, see ?? Our results complement those found in ?? and ??.

We conclude this introduction by introducing some notation. Z will denote a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The domain of an operator A will be denoted by $D(A)$, and by $\rho(A)$ we denotes its resolvent set. The point spectrum of A is given by $\sigma_p(A)$. If A is a densely defined operator on Z, then A^* will denote the adjoint of this operator. Furthermore, if A is the infinitesimal generator of a C_0-semigroup, then we denote this semigroup by $(T(t))_{t\geq 0}$.

The bounded linear operators from Z to Z are denoted by $\mathcal{L}(Z)$, and the bounded linear operators from the Hilbert space Z_1 to the Hilbert space Z_2 are denoted by $\mathcal{L}(Z_1, Z_2)$.

2 Useful lemmas

sec2

In this section we prove a useful lemma, and show some easy consequence of this lemma. We start with this useful lemma. The origin of this lemma dates back to the proof of Lemma 8.3.2 in CuZw95 [1], and was used in IfZC05 [2].

L1.1 **Lemma 2.1.** *Let A_1 and A_2 be two closed, densely defined, linear operators on the Hilbert space Z. If $X \in \mathcal{L}(Z)$ is a boundedly invertible solution of the Lyapunov equation*

$$\langle A_1 z_1, X z_2 \rangle + \langle X^* z_1, A_2 z_2 \rangle = 0, \qquad z_1 \in D(A_1), z_2 \in D(A_2) \qquad (2)$$ eq:1

and if $\{\mathbb{C} \setminus \sigma_p(-A_1^*)\} \cap \rho(A_2) \neq \emptyset$ or if $\{\mathbb{C} \setminus \sigma_p(-A_2^*)\} \cap \rho(A_1) \neq \emptyset$, then the following holds:

1. *$XD(A_2) = D(A_1^*)$;*
2. *$X\mathrm{ran}(A_2) = \mathrm{ran}(A_1^*)$;*
3. *$A_2 = -X^{-1}A_1^*X$ on $D(A_2)$ and $-XA_2X^{-1} = A_1^*$ on $D(A_1^*)$;*
4. *$\rho(A_1) = \rho(-A_2^*)$;*
5. *$X^*D(A_1) = D(A_2^*)$ and $-X^{-*}A_2^*X^* = A_1$ on $D(A_1)$.*

PROOF: The proof is divided into several steps. In step 1 we show that $XD(A_2) \subset D(A_1^*)$. Using the condition on the resolvent sets, we show in step 2 and 3 that $XD(A_2) = D(A_1^*)$. In step 4, we show that this equality implies item 2.–5.

Step 1: Recall that $z \in D(A_1^*)$ if there exists a $\tilde{z} \in Z$ such that

$$\langle A_1 z_1, z\rangle = \langle z_1, \tilde{z}\rangle \qquad \text{for all } z_1 \in D(A_1).$$

If this holds, then $\tilde{z} = A_1^* z$.

Defining $z = Xz_2$, where $z_2 \in D(A_2)$, we see that the equation (15) implies that $z \in D(A_1^*)$ and

$$XA_2 z_2 = -A_1^* X z_2. \tag{3}$$

Thus

$$XD(A_2) \subset D(A_1^*) \tag{4}$$

and (3) holds for all $z_2 \in D(A_2)$.

Step 2: In this step we want to show that for some $s \in \mathbb{C}$ the following equality holds

$$(sI + A_1^*)^{-1} X = X(sI - A_2)^{-1} \qquad \text{on } Z. \tag{5}$$

We have to distinguish between two cases. Namely, $\{\mathbb{C} \setminus \sigma_p(-A_1^*)\} \cap \rho(A_2) \neq \emptyset$ and $\{\mathbb{C} \setminus \sigma_p(-A_2^*)\} \cap \rho(A_1) \neq \emptyset$.

First we assume that $\{\mathbb{C} \setminus \sigma_p(-A_1^*)\} \cap \rho(A_2) \neq \emptyset$. So we can choose $s \in \rho(A_2)$ such that $s \notin \sigma_p(-A_1^*)$. For this s we write the equality (3) as

$$X(sI - A_2) = (sI + A_1^*)X \qquad \text{on } D(A_2). \tag{6}$$

Since $s \in \rho(A_2)$ and X is invertible, we conclude that

$$Z = \operatorname{ran}(X(sI - A_2)) = \operatorname{ran}\left((sI + A_1^*)X|_{D(A_2)}\right).$$

Hence the range of $sI + A_1^*$ is Z. Since $s \notin \sigma_p(-A_1^*)$, this operator is injective as well, and we conclude that it is invertible. Using the invertibility of $(sI - A_2)$ and $(sI + A_1^*)$, we obtain from (6) that (5) holds.

Now we assume that $\{\mathbb{C} \setminus \sigma_p(-A_2^*)\} \cap \rho(A_1) \neq \emptyset$. So we can choose an $-\overline{s} \in \rho(A_1)$ such that $-\overline{s} \notin \sigma_p(-A_2^*)$. Note that the first condition is equivalent with $-s \in \rho(A_1^*)$. For this s we write the equality (3) as (6). If $s \in \sigma_p(A_2)$, then this equality implies that $-s \in \sigma_p(A_1^*)$. Since $-s \in \rho(A_1^*)$, this is not possible, and hence $s \notin \sigma_p(A_2)$. In order words, $sI - A_2$ is injective. Next we show that the range of this operator is closed.

Let $x_n \text{ran}(sI - A_2)$ such that $x_n \to x \in Z$, as $n \to \infty$. From (6) we see that x_n can be written as

$$x_n = (sI - A_2)z_n = X^{-1}(sI + A_1^*)Xz_n, \text{ with } z_n \in D(A_2)$$

Since X and $sI + A_1^*$ are boundedly invertible, we conclude from the convergence of x_n that z_n is converging as well. Since A_2 is a closed operator, we find that $z = \lim_{n\to\infty} z_n \in D(A_2)$ and $x = (sI - A_2)z$. Thus the range of $sI - A_2$ is closed. It remains to show that the range is the whole space.

The condition that $-\overline{s} \notin \sigma_p(-A_2^*)$ is equivalent to the condition that the range of $sI - A_2$ is dense. Combining this with the results above, we see that $sI - A_2$ is injective and surjective, and so it is boundedly invertible. Using (6) gives directly that (5) holds.

Step 3: Consider the following list of equalities on $D(A_1^*)$:

$$\begin{aligned} X^{-1} &= X^{-1}(sI + A_1^*)^{-1}XX^{-1}(sI + A_1^*) \\ &= X^{-1}X(sI - A_2)^{-1}X^{-1}(sI + A_1^*) \\ &= (sI - A_2)^{-1}X^{-1}(sI + A_1^*). \end{aligned}$$

Thus we conclude that X^{-1} maps the domain of A_1^* into the domain of A_2. Since we already know that X maps the domain of A_2 into that of A_1^*, we have proved that $XD(A_2) = D(A_1^*)$.

Step 4: In this step we prove the other assertions. If $z \in X\text{ran}(A_2)$, then by (3) it is also an element of the range of A_1^*. On the other hand, if z lies in the range of A_1^*, then there exists an $x \in D(A_1^*)$ such that $z = A_1^*x$. By the first assertion we know that we can write x as $x = X\tilde{x}$, with $\tilde{x} \in D(A_2)$. Using (3) once more shows that $z \in X\text{ran}(A_2)$. Thus we have proved item 2.

The assertion of item 3 follows from item 1 and equation (3). Since A_2 and $-A_1^*$ are similar, assertion 4 and 5 follow easily. □

The first corollary of this useful lemma is a characterization of the generator of a group.

C2.2 **Corollary 2.2.** *Let A be a closed, densely defined linear operator on the complex Hilbert space Z. Assume that $\text{Re}\langle Az, z\rangle = 0$ for all $z \in D(A)$, and that $\{\mathbb{C} \setminus \sigma_p(-A^*)\} \cap \rho(A) \neq \emptyset$. Then A is the infinitesimal generator of a unitary group.* □

PROOF: The equation $\text{Re}\langle Az, z\rangle = 0$ for all $z \in D(A)$ is the same as

$$\langle Az, z\rangle + \langle z, Az\rangle = 0$$

for all $z \in D(A)$. Choosing $z = z_1 + \alpha z_2$, where α is an element of $\mathbb{C}$ and $z_1, z_2 \in D(A)$, gives that (15) holds for $A_1 = A_2 = A$ and $X = I$. By the assumption on the resolvent sets, we conclude from Lemma 2.1 that $D(A^*) = D(A)$ and $A^* = -A$. Thus A is skew-adjoint, and now standard semigroup theory implies that A generates a unitary group on Z, see e.g. [1, p. 89]. □

C2.3 **Corollary 2.3.** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z, and let A denote its infinitesimal generator. Assume further that $\{\mathbb{C} \setminus \sigma_p(-A^*)\} \cap \rho(A) \neq \emptyset$. If $\|T(t)z_0\| = \|z_0\|$ for all $z_0 \in Z$ and all $t \geq 0$, then $(T(t))_{t\geq 0}$ is unitary group.* □

PROOF: Since $\|T(t)z_0\|^2 = \|z_0\|^2$ for all $z_0 \in Z$, we have that $\mathrm{Re}\langle Az, z\rangle = 0$ for all $z \in D(A)$. Now the rest follows from the previous corollary. □

One might wonder whether the above results still hold when we remove the condition on the spectrum of A_1 and A_2. The following example shows that this is not true.

E1.2 **Example 2.4** Let A be the differentiation operator on $Z = L^2(0, \infty)$, i.e.,

$$Ah = \frac{dh}{dx} \tag{7}$$ eq:2

on its domain $D(A) = \{h \in L^2(0,\infty) \mid h$ is absolutely continuous, and $\frac{dh}{dx} \in L^2(0,\infty)\}$. It is easy to see that the dual operator is given by

$$A^* f = -\frac{df}{dx} \tag{8}$$ eq:3

on its domain $D(A^*) = \{f \in L^2(0,\infty) \mid f$ is absolutely continuous, $f(0) = 0$ and $\frac{df}{dx} \in L^2(0,\infty)\}$.

Choose $A_1 = A_2 = A^*$ and $X = I$, then from equation (7) and (12) we conclude that equation (15) holds. In particular, $\mathrm{Re}\langle A_1 z, z\rangle = 0$ for all $z \in D(A_1)$. However,

$$XD(A_2) = D(A^*) \subsetneqq D(A) = D(A_1^*).$$

Thus we see that the conclusion of Lemma 2.1 does not hold. Furthermore, it is clear that A does not generate a group. Looking at the resolvent of A^*, we see that this equals the whole closed right-half plane. The point spectrum of A consist of all $\lambda \in \mathbb{C}$ with real part negative. Thus $\mathbb{C} \setminus \sigma_p(-A_1^*) \cap \rho(A_2) = \emptyset$. □

Next we present a second nice lemma, which shows that item 1 of Lemma 2.1 implies the condition on the spectrum.

Lemma 2.5. *Let A_1 and A_2 be two closed, linear operators on the Hilbert space Z. If $X \in \mathcal{L}(Z)$ is an invertible solution of (15) and $XD(A_2) = D(A_1^*)$, then $\rho(-A_1^*) = \rho(A_2)$. Furthermore,*

$$(sI + A_1^*)^{-1} = X(sI - A_2)^{-1}X^{-1} \quad \textit{for all } s \in \rho(A_2) \tag{9}$$

and the assertions 2–5 of Lemma 2.1 hold.

Proof: Since X is solution of the Lyapunov equation (15), we have that

$$XA_2 = -A_1^* X \qquad \text{on } D(A_2). \tag{10}$$

Now for $s \in \rho(A_2)$ define on $D(A_1^*)$ the operator

$$A_3 = X(sI - A_2)X^{-1}$$

Since $X^{-1}D(A_1^*) = D(A_2)$ this operator is well-defined. Furthermore, it is easy to see that it is invertible, with inverse

$$A_3^{-1} = X(sI - A_2)^{-1}X^{-1}.$$

Furthermore, from (10), we see that on $D(A_1^*)$

$$A_3 = sI - XA_2X^{-1} = sI + A_1^*$$

Hence, since A_3 is invertible, we conclude that $s \in \rho(-A_1^*)$ and (9) holds.

For $s \in \rho(-A_1^*)$ define on $D(A_2)$ the operator $A_4 = X^{-1}(sI + A_1^*)X$. This is a well-defined and invertible operator, and similar as above, we can show that $A_4 = sI - A_2$. Thus $\rho(-A_1^*) \subset \rho(A_2)$, which concludes the proof. The other assertions in Lemma 2.1 are now easy to show, see step 4 of the proof. □

Our third lemma relates the Sylvester equation (15) to a left-inverse of a semigroup.

Lemma 2.6. *Let A_1, A_2 be the infinitesimal generators of the C_0-semigroups $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$, respectively. Then $X \in \mathcal{L}(Z)$ satisfies the Sylvester equation*

$$\langle A_1 z_1, X z_2\rangle + \langle z_1, XA_2 z_2\rangle = 0, \quad z_1 \in D(A_1), z_2 \in D(A_2) \tag{11}$$

if and only if

$$T_1^*(t) X T_2(t) = X, \qquad \textit{for all } t \geq 0.$$

Moreover, if X is (boundedly) invertible, then

$$X^{-1} T_1^*(t) X T_2(t) = I, \qquad \textit{for all } t \geq 0.$$

Thus $\left(X^{-1} T_1^*(t) X\right)_{t \geq 0}$ is the left-inverse of $(T_2(t))_{t \geq 0}$.

Proof: Easy by differentiating (one direction) or (other implication) substituting $z_1 = T_1(t) z_{10}$ and $z_2 = T_2(t) z_{20}$, with $z_{10} \in D(A_1)$, $z_{20} \in D(A_2)$.□

3 Left-invertible semigroups

We begin with the definition of a left-invertible semigroup.

Definition 3.1. The C_0-semigroup $(T(t))_{t \geq 0}$ is left-invertible if there exists a function $t \mapsto m(t)$ such that $m(t) > 0$ and for all $z_0 \in Z$ there holds

$$m(t) \|z_0\| \leq \|T(t) z_0\|, \qquad t \geq 0. \tag{12}$$ eq:3

□

To formulate the characterization of left-invertible semigroups we need the definition of exact observability.

Definition 3.2. The system $\Sigma(A, -, C)$ is *exactly observable in finite-time* if there exist $t_0 > 0$ and $m > 0$ such that for all $z_0 \in Z$

$$m \|z_0\|^2 \leq \int_0^{t_0} \|C T(t) z_0\|^2 dt. \tag{13}$$ eq:6

□

T3.3 **Theorem 3.3.** *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup on the Hilbert space Z with infinitesimal generator A. Then the following are equivalent.*

1. *$(T(t))_{t \geq 0}$ is left-invertible.*
2. *The system $\Sigma(A, -, I)$ is exactly observable in finite-time.*
3. *There exists an operator $C \in \mathcal{L}(Z, Y)$ such that $\Sigma(A, -, C)$ is exactly observable in finite-time.*
4. *For all ω larger than the growth bound of $(T(t))_{t \geq 0}$ there exists a self-adjoint $X \in \mathcal{L}(Z)$ satisfying the Lyapunov equation*

$$\langle (A - \omega I) z_1, X z_2 \rangle + \langle z_1, X(A - \omega I) z_2 \rangle = -\langle C z_1, C z_2 \rangle, \tag{14}$$ eq:7

where $z_1, z_2 \in D(A)$.

5. *There exists a C_0-semigroup $(S(t))_{t\geq 0}$ such that $S(t)T(t) = I$ for all $t \geq 0$.*

PROOF: It is clear that 2. implies 3. and that 5. implies 1. So we concentrate on the other implications.

1. $\Rightarrow$ 2. Since (12) holds, we see that for all $t_0 > 0$ there holds

$$\int_0^{t_0} m(t)^2 \|z\|^2 \leq \int_0^{t_0} \|T(t)z\|^2 dt. \tag{15}$$

eq:1

Thus $\Sigma(A, -, I)$ is exactly observable.

3. $\Rightarrow$ 4. Choose $\omega \in \mathbb{R}$ larger than the growth bound of $(T(t))_{t\geq 0}$. Then we have that the semigroup generated by $A - \omega I$ is exponentially stable.

Now for $z \in Z$

$$\begin{aligned} m_0\|z\|^2 &= \int_0^\infty m(t)^2 e^{-2\omega t}\|z\|^2 dt \\ &\leq \int_0^\infty e^{-2\omega t}\|CT(t)z\|^2 dt \leq M\|z\|^2 \end{aligned}$$

Define $\langle z, Xz\rangle = \int_0^\infty e^{-2\omega t}\|CT(t)z\|^2 dt$. Then

- $m_0 I \leq X \leq MI$
- X is the solution to the Lyapunov equation

  $$\langle (A - \omega I)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = -\langle Cz_1, Cz_2\rangle,$$

  for $z_1, z_2 \in D(A)$.

4. $\Rightarrow$ 5. We rewrite the Lyapunov equation (14) to the Sylvester equation

$$\langle (A - \omega I + X^{-1}C^*C)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = 0.$$

The Sylvester equation

$$\langle (A - \omega I + X^{-1}C^*C)z_1, Xz_2\rangle + \langle z_1, X(A - \omega I)z_2\rangle = 0.$$

is a special case of our general Sylvester equation. Thus by the previous lemma we know that the semigroup generated by $A - \omega I$ is left invertible, and

$$X^{-1}T_1(t)^* XT(t)e^{-\omega t} = I$$

where $(T_1(t))_{t\geq 0}$ is the semigroup generated by $A - \omega I + X^{-1}C^*C$. Thus $S(t) := X^{-1}T_1(t)^* Xe^{-\omega t}$ is the left-inverse of $T(t)$. $\square$

We remark that the the theorem also holds for $C \in \mathcal{L}(Z, Y)$ for which $\Sigma(A, -, C)$ is exactly observable in infinite-time provided we have for all $z_0 \in Z$ that $\int_0^\infty \|CT(t)z_0\|^2 dt < \infty$. Just define X in part 4 as $\langle z_1, Xz_2 \rangle = \int_0^\infty \langle CT(t)z_0, CT(t)z_2 \rangle dt$.

In ?? Haase showed that that groups are bounded perturbation of unitary groups. Based on the proof in Theorem 3.3, we show that a similar result holds for left-invertible semigroups.

Theorem 3.4. *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup on the Hilbert space Z with generator A. Then the following are equivalent:*

1. *$(T(t))_{t \geq 0}$ is left-invertible;*
2. *There exists a bounded operator Q and an equivalent inner product such that $A + Q$ generates an isometric semigroup in the new norm.*

PROOF: By Theorem 3.3 we have the existence of a $X \in \mathcal{L}(Z)$, boundedly invertible, such that

$$\langle (A - \omega I)z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = -\langle z_1, z_2 \rangle,$$

for $z_1, z_2 \in D(A)$. Now we write it as

$$\langle (A - \omega I + \frac{1}{2}X^{-1})z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I + \frac{1}{2}X^{-1})z_2 \rangle = 0.$$

By defining $Q = -\omega I + \frac{1}{2}X^{-1}$, and taking as new inner product $\langle z_1, z_2 \rangle_{\text{new}} = \langle z_1, Xz_2 \rangle$, we obtain the desired result. □

Theorem 3.3 does not hold for general admissible output operators.

Example 3.5 Consider the left-shift semigroup on $L^2(0, 1)$, i.e.

$$(T(t)f)(\eta) = \begin{cases} f(\eta + t) & \eta + t \in [0, 1] \\ 0 & \eta + t \geq 1 \end{cases}$$

with the observation at $\eta = 0$, i.e.,

$$Cf = f(0).$$

Then $(T(t))_{t \geq 0}$ not left-invertible, but since

$$\int_0^1 |CT(t)f|^2 dt = \|f\|^2,$$

it is exactly observable in finite time. □

4 Banach spaces

In this section we define admissibility and exact observability in a much more general setting. To do this we first introduce Banach spaces $\mathcal{Y}_\tau$, $\tau > 0$ which satisfy the following

- $\mathcal{Y}_\tau \subset L((0,\tau);Y)$, where Y is the output space of the system.
- For every $t_0 < \tau$ there holds

$$\|f\|_{\mathcal{Y}_\tau} \leq \|\mathbb{1}_{[0,t_0]} f\|_{\mathcal{Y}_{t_0}} + \|\mathbb{1}_{[t_0,\tau]}(\cdot) f(t_0 + \cdot)\|_{\mathcal{Y}_{\tau - t_0}}. \tag{16}$$

eq:8

It is easy to see that for every $p \geq 1$, $L^p((0,\tau);Y)$ satisfy this property. Furthermore, it is satisfied by $C([0,\tau];Y)$.

We say that C is an *admissible output operator with respect to* $\mathcal{Y}_\tau$ if there exists a $m_\tau \geq 0$ and a dense set V of X such that

$$\|CT(\cdot)x_0\|_{\mathcal{Y}_\tau} \leq m_\tau \|x_0\| \qquad x_0 \in V. \tag{17}$$

eq:9

If this holds, then Φ_η defined by

$$\Phi_\tau x_0 = CT(\cdot)x_0, \qquad x_0 \in V.$$

extends to a bounded operator from X to $\mathcal{Y}_\tau$. We define

$$m_0 = \inf_{\eta > 0} \|\Phi_\eta\|$$

Furthermore, assume that the system is *exactly observable*, that is there exists a $\tau \in (0, \infty]$ and a $\gamma(\tau) > 0$ such that

$$\|\Phi_\tau x\| \geq \gamma(\tau)\|x\| \quad \text{for all } x.$$

Now we have for all $\eta < \tau$

$$\begin{aligned}
\gamma(\tau)\|x\| &\leq \|\Phi_\tau x\|_{\mathcal{Y}_\tau} \\
&\leq \|\mathbb{1}_{[0,\eta]}\Phi_\tau x\|_{\mathcal{Y}_\eta} + \|\mathbb{1}_{[\eta,\tau]}(\cdot)\Phi_\tau x(\eta + \cdot)\|_{\mathcal{Y}_{\tau-\eta}} \\
&= \|\Phi_\eta x\|_{\mathcal{Y}_\eta} + \|\Phi_{\tau-\eta} T(\eta)x\|_{\mathcal{Y}_{\tau-\eta}} \\
&\leq \|\Phi_\eta\|\|x\| + \|\Phi_{\tau-\eta}\|\|T(\eta)x\| \\
&\leq \|\Phi_\eta\|\|x\| + \|\Phi_\tau\|\|T(\eta)x\|.
\end{aligned}$$

Hence we have that

$$\|T(\eta)x\| \geq \frac{\gamma(\tau) - \|\Phi_\eta\|}{\|\Phi_\tau\|}\|x\|.$$

If for some τ and η, this is positive, then we have proved that the semigroup at time η is bounded from below. By standard semigroup theory, we get that the semigroup is bounded from below for all $\eta \in (0, \infty)$ and thus left-invertible. In particular, this holds when $\gamma(\tau) - m_0 > 0$ for some τ.

We first remark that in its most general form exact observability means that the norm of the initial state can be bounded from above by the norm of the output. Using this general definition Theorem 3.3 does no longer hold.

Example 4.1 As state space X we take the space of all continuous functions on the interval $[0, 1]$ which are zero at 1. This is a Banach space under the supremum norm. As C_0-semigroup we take the left shift, i.e.

$$(T(t)f)(\eta) = \begin{cases} f(t+\eta) & \eta + t \leq 1 \\ 0 & t + \eta > 1 \end{cases}$$

As observation we take $Cf = f(0)$. This is a bounded operator from X to $\mathbb{C}$. Furthermore, it is easy to see that

$$\sup_{t \in [0,1]} |CT(t)f| = \|f\|_X$$

and thus the system $\Sigma(A, -, C)$ is exactly observable for $\mathcal{Y}_\tau = C(0, \tau)$. Since the left-shift is not left-invertible, we see that Theorem 3.3 does not hold for general Banach spaces, with a general definition of exact observability. $\square$

Remark 4.2. It is important that other than the L^2 norm are considered in the study of exact observability and admissibility for semigroups on Banach spaces. Since if the system $\Sigma(A, -, C)$ is admissible and exactly observable on $\mathcal{Y}_\tau = L^2((0, \tau); Y)$ with Y a Hilbert space, then

$$c_1 \|x\| \leq \sqrt{\int_0^\tau \|CT(t)x_0\|^2 dt} \leq c_2 \|x\|.$$

Thus the (Banach space) norm on X is similar to a Hilbert space norm. This will never happen for $X = L^p$ with $p \neq 2$, and so any system $\Sigma(A, -, C)$ with Y a Hilbert space will never be admissible AND exactly observable for $\mathcal{Y}_\tau = L^2((0, \tau); Y)$. $\square$

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