Invertible solutions of the Lyapunov and algebraic Riccati equation

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Abstract

If the solution of a Lyapunov or Riccati equation corresponding to a finite-dimensional system is invertible, then this leads to additional properties of this system. For infinite-dimensional systems these results do not need to hold. We show that it is necessary to put some additional conditions on the spectrum of the system operator A. For instance, if this operator has compact resolvent, then the results known for finite-dimensional systems extend to infinite-dimensional ones. Using this result, we obtain more insight in Lyapunov and Riccati equations for well-posed linear systems.

1 Introduction

In many areas of systems theory, Lyapunov and Riccati equations play an important role. They appear in the study of stability, controllability, observability, and in the study of the quadratic optimal control problem, see e.g. [1, 4, 5, 6]. If the system is (exactly) controllable and/or observable, then the solution of the Lyapunov or Riccati equation will be strictly positive, and thus invertible, see e.g. [3]. This invertibility leads to nice relations between the open and the closed loop system. To illustrate this, we consider the finite-dimensional Lyapunov equation

$$A^*X + XA = -C^*C. \tag{1}$$

Since X is invertible, we easily obtain that this equation is equivalent to

$$A^* = -X\left(A + X^{-1}C^*C\right)X^{-1}. \tag{2}$$

Thus the system $\dot{z}(t) = \left(A + X^{-1}C^*C\right)z(t)$ is similar to the system $\dot{z}(t) = -A^*z(t)$. In particular, if A was completely unstable, then the closed loop system $A + X^{-1}C^*C$ is stable.

Using once more the invertibility of X, we multiply (1) by X^{-1} from the left and from the right, and we find that X^{-1} satisfies the Riccati equation

$$X^{-1}A^* + AX^{-1} + X^{-1}CC^*X^{-1}.$$

Hence the Lyapunov equation (1) transforms to a Riccati equation.

The relation between Riccati and Lyapunov equations is already known since the introduction of the Riccati equation. If X is the solution of the Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0, \tag{3}$$

then X also satisfies the Lyapunov equation

$$A^*X + X(A - BB^*X) = -C^*C. \tag{4}$$

So results which we obtain for Lyapunov equation, will be have an equivalent counterpart for algebraic Riccati equations.

Although, (1) is the best-known form of the Lyapunov equation, we study a more general form of the Lyapunov equation, namely,

$$A^*X + XA_1 = 0. \tag{5}$$

Under the assumption that X is invertible, it is not hard to write (1) into this form. One just defines $A_1 = A + X^{-1}C^*C$.

Again for the finite-dimensional case, equation (5) implies that $A^* = -XA_1X^{-1}$. For finite-dimensional systems the situation is slightly different. First of all, the Lyapunov equation (5) has to be interpreted as

$$\langle Az_1, Xz_1\rangle + \langle X^*z_1, A_1z_2\rangle, \qquad z_1 \in D(A), z_2 \in D(A_1), \tag{6}$$

where $D(A)$ and $D(A_1)$, denote the domain of the operator A and A_1, respectively. In contrast to the finite-dimensional case, the invertibility of X does not necessarily imply that A^* is similar to $-A_1$. Consider the following simple example. As state space we take $Z = L^2(0,1)$ and as A we take $A = \frac{d}{d\eta}$ with $D(A) = \{f \in L^2(0,1) \mid f \text{ is absolutely continuous and } f(0) = 0, f(1) = 0\}$, and we take A_1 to be equal to A, then we find

$$\begin{aligned} \langle Az_1, z_1\rangle + \langle z_1, A_1z_2\rangle &= \int_0^1 \frac{dz_1}{d\eta}(\eta)z_2(\eta)d\eta + \int_0^1 z_1(\eta)\frac{dz_1}{d\eta}(\eta)d\eta \\ &= z_1(\eta)z_2(\eta)|_0^1 = 0. \end{aligned}$$

Hence (6) is satisfied for $X = I$. However, $A^* = -\frac{d}{d\eta}$ with $D(A^*) = \{f \in L^2(0,1) \mid f \text{ is absolutely continuous}\}$, and this is not equal to $-A$, since the domains are different.

However, if we would take $Z = L^2(-\infty,\infty)$, and $A_1 = A$ with $A = \frac{d}{d\eta}$ with $D(A) = \{f \in L^2(-\infty,\infty) \mid f \text{ is absolutely continuous}\}$, then a similar calculation gives that (6) is satisfied. However, in this situation $A^* = -A_1$.

Concluding, we see that is not a priori clear when the finite-dimensional similarity result can be extended. In this paper we investigate this question, and we show that it is related to the resolvent of A and of A_1. After we have solved this question via the useful lemma in Section 2, we apply it to standard Lyapunov equations, and to Riccati equations.

As we have seen, invertible solutions of the Lyapunov equation relates the spectrum of A^* and $-A_1$. However, for infinite-dimensional systems there is a more important property. Within the theory of infinite-dimensional systems theory it is natural to assume that A generates a C_0-semigroup, i.e., the differential equation $\dot{x}(t) = Ax(t)$, $x(0) = x_0$, $t \geq 0$ is well-defined. If one assumes this for A_1 as well, then the similarity of $-A^*$ and A_1 gives that the differential equation $\dot{x}(t) = -Ax(t)$, $x(0) = x_0$, $t \geq 0$ is well-defined as well. This implies that A generates a group. Hence we see that invertible solutions of the Lyapunov equation (5) is related to the group property of the semigroup associated to A and A_1. Note that the invertibility of X is the essential property. Nowhere we use that X is positive.

We conclude this introduction by introducing some notation. Z will denote a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The domain of an operator A will be denoted by $D(A)$, and by $\rho(A)$ we denotes its resolvent set. The point spectrum of A is given by $\sigma_p(A)$. If A is a densely defined operator on Z, then A^* will denote the adjoint of this operator. The bounded linear operators from Z to Z are denoted by $\mathcal{L}(Z)$, and the bounded linear operators from the Hilbert space Z_1 to the Hilbert space Z_2 are denoted by $\mathcal{L}(Z_1, Z_2)$.

2 A useful lemma

In this section we prove a useful lemma, and show some easy consequence of this lemma. We start with this useful lemma. The origin of this lemma dates back to the proof of Lemma 8.3.2 in [1], and was used in [2].

Lemma 2.1. *Let A_1 and A_2 be two closed, densely defined, linear operators on the Hilbert space Z. If $X \in \mathcal{L}(Z)$ is a boundedly invertible solution of the Lyapunov equation*

$$\langle A_1 z_1, X z_2 \rangle + \langle X^* z_1, A_2 z_2 \rangle = 0, \qquad z_1 \in D(A_1), z_2 \in D(A_2) \tag{7}$$

and if $\{\mathbb{C} \setminus \sigma_p(-A_1^*)\} \cap \rho(A_2) \neq \emptyset$ or if $\{\mathbb{C} \setminus \sigma_p(-A_2^*)\} \cap \rho(A_1) \neq \emptyset$, then the following holds:

1. *$XD(A_2) = D(A_1^*)$;*
2. *$X\mathrm{ran}(A_2) = \mathrm{ran}(A_1^*)$;*
3. *$A_2 = -X^{-1}A_1^*X$ on $D(A_2)$ and $-XA_2X^{-1} = A_1^*$ on $D(A_1^*)$;*
4. *$\rho(A_1) = \rho(-A_2^*)$;*
5. *$X^*D(A_1) = D(A_2^*)$ and $-X^{-*}A_2^*X^* = A_1$ on $D(A_1)$.*

Proof. The proof is divided into several steps. In step 1 we show that $XD(A_2) \subset D(A_1^*)$. Using the condition on the resolvent sets, we show in step 2 and 3 that $XD(A_2) = D(A_1^*)$. In step 4, we show that this equality implies item 2.–5.

Step 1: Recall that $z \in D(A_1^*)$ if there exists a $\tilde{z} \in Z$ such that

$$\langle A_1 z_1, z \rangle = \langle z_1, \tilde{z} \rangle \qquad \text{for all } z_1 \in D(A_1).$$

If this holds, then $\tilde{z} = A_1^* z$.

Defining $z = Xz_2$, where $z_2 \in D(A_2)$, we see that the equation (7) implies that $z \in D(A_1^*)$ and

$$XA_2z_2 = -A_1^*Xz_2. \tag{8}$$

Thus

$$XD(A_2) \subset D(A_1^*) \tag{9}$$

and (8) holds for all $z_2 \in D(A_2)$.

Step 2: In this step we want to show that for some $s \in \mathbb{C}$ the following equality holds

$$(sI + A_1^*)^{-1} X = X(sI - A_2)^{-1} \qquad \text{on } Z. \tag{10}$$

We have to distinguish between two cases. Namely, $\{\mathbb{C} \setminus \sigma_p(-A_1^*)\} \cap \rho(A_2) \neq \emptyset$ and $\{\mathbb{C} \setminus \sigma_p(-A_2^*)\} \cap \rho(A_1) \neq \emptyset$.

First we assume that $\{\mathbb{C} \setminus \sigma_p(-A_1^*)\} \cap \rho(A_2) \neq \emptyset$. So we can choose $s \in \rho(A_2)$ such that $s \notin \sigma_p(-A_1^*)$. For this s we write the equality (8) as

$$X(sI - A_2) = (sI + A_1^*)X \qquad \text{on } D(A_2). \tag{11}$$

Since $s \in \rho(A_2)$ and X is invertible, we conclude that

$$Z = \operatorname{ran}\left(X(sI - A_2)\right) = \operatorname{ran}\left((sI + A_1^*)X|_{D(A_2)}\right).$$

Hence the range of $sI + A_1^*$ is Z. Since $s \notin \sigma_p(-A_1^*)$, this operator is injective as well, and we conclude that it is invertible. Using the invertibility of $(sI - A_2)$ and $(sI + A_1^*)$, we obtain from (11) that (10) holds.

Now we assume that $\{\mathbb{C} \setminus \sigma_p(-A_2^*)\} \cap \rho(A_1) \neq \emptyset$. So we can choose an $-\overline{s} \in \rho(A_1)$ such that $-\overline{s} \notin \sigma_p(-A_2^*)$. Note that the first condition is equivalent with $-s \in \rho(A_1^*)$. For this s we write the equality (8) as (11). If $s \in \sigma_p(A_2)$, then this equality implies that $-s \in \sigma_p(A_1^*)$. Since $-s \in \rho(A_1^*)$, this is not possible, and hence $s \notin \sigma_p(A_2)$. In order words, $sI - A_2$ is injective. Next we show that the range of this operator is closed.

Let $x_n \operatorname{ran}(sI - A_2)$ such that $x_n \to x \in Z$, as $n \to \infty$. From (11) we see that x_n can be written as

$$x_n = (sI - A_2)z_n = X^{-1}(sI + A_1^*)Xz_n, \text{ with } z_n \in D(A_2)$$

Since X and $sI + A_1^*$ are boundedly invertible, we conclude from the convergence of x_n that z_n is converging as well. Since A_2 is a closed operator, we find that $z = \lim_{n\to\infty} z_n \in D(A_2)$ and $x = (sI - A_2)z$. Thus the range of $sI - A_2$ is closed. It remains to show that the range is the whole space.

The condition that $-\overline{s} \notin \sigma_p(-A_2^*)$ is equivalent to the condition that the range of $sI - A_2$ is dense. Combining this with the results above, we see that $sI - A_2$ is injective and surjective, and so it is boundedly invertible. Using (11) gives directly that (10) holds.

Step 3: Consider the following list of equalities on $D(A_1^*)$:

$$\begin{aligned} X^{-1} &= X^{-1}(sI + A_1^*)^{-1} X X^{-1}(sI + A_1^*) \\ &= X^{-1}X(sI - A_2)^{-1}X^{-1}(sI + A_1^*) \\ &= (sI - A_2)^{-1}X^{-1}(sI + A_1^*). \end{aligned}$$

Thus we conclude that X^{-1} maps the domain of A_1^* into the domain of A_2. Since we already know that X maps the domain of A_2 into that of A_1^*, we have proved that $XD(A_2) = D(A_1^*)$.

Step 4: In this step we prove the other assertions. If $z \in X\operatorname{ran}(A_2)$, then by (8) it is also an element of the range of A_1^*. On the other hand, if z lies in the range of A_1^*, then there exists an $x \in D(A_1^*)$ such that $z = A_1^* x$. By the first assertion we know that we can write x as $x = X\tilde{x}$, with $\tilde{x} \in D(A_2)$. Using (8) once more shows that $z \in X\operatorname{ran}(A_2)$. Thus we have proved item 2.

The assertion of item 3 follows from item 1 and equation (8). Since A_2 and $-A_1^*$ are similar, assertion 4 and 5 follow easily. ■

The first corollary of this useful lemma is a characterization of the generator of a group.

Corollary 2.2. *Let A be a closed, densely defined linear operator on the complex Hilbert space Z. Assume that Re$\langle Az, z\rangle = 0$ for all $z \in D(A)$, and that $\{\mathbb{C} \setminus \sigma_p(-A^*)\} \cap \rho(A) \neq \emptyset$. Then A is the infinitesimal generator of a unitary group.*

Proof. The equation $\mathrm{Re}\langle Az, z\rangle = 0$ for all $z \in D(A)$ is the same as

$$\langle Az, z\rangle + \langle z, Az\rangle = 0$$

for all $z \in D(A)$. Choosing $z = z_1 + \alpha z_2$, where α is an element of $\mathbb{C}$ and $z_1, z_2 \in D(A)$, gives that (7) holds for $A_1 = A_2 = A$ and $X = I$. By the assumption on the resolvent sets, we conclude from Lemma 2.1 that $D(A^*) = D(A)$ and $A^* = -A$. Thus A is skew-adjoint, and now standard semigroup theory implies that A generates a unitary group on Z, see e.g. [1, p. 89]. ■

Corollary 2.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space Z, and let A denote its infinitesimal generator. Assume further that $\{\mathbb{C} \setminus \sigma_p(-A^*)\} \cap \rho(A) \neq \emptyset$. If $\|T(t)z_0\| = \|z_0\|$ for all $z_0 \in Z$ and all $t \geq 0$, then $(T(t))_{t\geq 0}$ is unitary group.*

Proof. Since $\|T(t)z_0\|^2 = \|z_0\|^2$ for all $z_0 \in Z$, we have that $\mathrm{Re}\langle Az, z\rangle = 0$ for all $z \in D(A)$. Now the rest follows from the previous corollary. ■

One might wonder whether the above results still hold when we remove the condition on the spectrum of A_1 and A_2. The following example shows that this is not true.

Example 2.4. *Let A be the differentiation operator on $Z = L^2(0, \infty)$, i.e.,*

$$Ah = \frac{dh}{dx} \tag{12}$$

on its domain $D(A) = \{h \in L^2(0, \infty) \mid h$ is absolutely continuous, and $\frac{dh}{dx} \in L^2(0, \infty)\}$. It is easy to see that the dual operator is given by

$$A^*f = -\frac{df}{dx} \tag{13}$$

on its domain $D(A^*) = \{f \in L^2(0, \infty) \mid f$ is absolutely continuous, $f(0) = 0$ and $\frac{df}{dx} \in L^2(0, \infty)\}$.

Choose $A_1 = A_2 = A^*$ and $X = I$, then from equation (12) and (13) we conclude that equation (7) holds. In particular, Re$\langle A_1 z, z\rangle = 0$ for all $z \in D(A_1)$. However,

$$XD(A_2) = D(A^*) \subsetneqq D(A) = D(A_1^*).$$

Thus we see that the conclusion of Lemma 2.1 does not hold. Furthermore, it is clear that A does not generate a group. Looking at the resolvent of A^*, we see that this equals the whole closed right-half plane. The point spectrum of A consist of all $\lambda \in \mathbb{C}$ with real part negative. Thus $\mathbb{C} \setminus \sigma_p(-A_1^*) \cap \rho(A_2) = \emptyset$. ■

Next we present a second nice lemma, which shows that item 1 of Lemma 2.1 implies the condition on the spectrum.

Lemma 2.5. *Let* A_1 *and* A_2 *be two closed, linear operators on the Hilbert space* Z*. If* $X \in \mathcal{L}(Z)$ *is an invertible solution of (7) and* $XD(A_2) = D(A_1^*)$*, then* $\rho(-A_1^*) = \rho(A_2)$*. Furthermore,*

$$(sI + A_1^*)^{-1} = X(sI - A_2)^{-1}X^{-1} \quad \text{for all } s \in \rho(A_2) \tag{14}$$

and the assertions 2–5 of Lemma 2.1 hold.

Proof. Since X is solution of the Lyapunov equation (7), we have that

$$XA_2 = -A_1^* X \qquad \text{on } D(A_2). \tag{15}$$

Now for $s \in \rho(A_2)$ define on $D(A_1^*)$ the operator

$$A_3 = X(sI - A_2)X^{-1}$$

Since $X^{-1}D(A_1^*) = D(A_2)$ this operator is well-defined. Furthermore, it is easy to see that it is invertible, with inverse

$$A_3^{-1} = X(sI - A_2)^{-1}X^{-1}.$$

Furthermore, from (15), we see that on $D(A_1^*)$

$$A_3 = sI - XA_2X^{-1} = sI + A_1^*$$

Hence, since A_3 is invertible, we conclude that $s \in \rho(-A_1^*)$ and (14) holds.

For $s \in \rho(-A_1^*)$ define on $D(A_2)$ the operator $A_4 = X^{-1}(sI + A_1^*)X$. This is a well-defined and invertible operator, and similar as above, we can show that $A_4 = sI - A_2$. Thus $\rho(-A_1^*) \subset \rho(A_2)$, which concludes the proof. The other assertions in Lemma 2.1 are now easy to show, see step 4 of the proof. ■

As always with useful lemma's, they can be used to prove other results. This will be done in the next two sections.

3 Invertible solutions of Lyapunov equations

We start with a simple application of Lemma 2.1 to the standard Lyapunov equation.

Lemma 3.1. *Let* A *be the infinitesimal generator of a* C_0*-semigroup on the Hilbert space* Z*, and let* Q *be a bounded, linear operator on* Z*. If* X *is an invertible solution of the Lyapunov equation*

$$\langle Az_1, Xz_2\rangle + \langle X^* z_1, Az_2\rangle = -\langle z_1, Qz_2\rangle, \qquad z_1, z_2 \in D(A), \tag{16}$$

and $\rho(A) \cap \mathbb{C} \setminus \sigma_p(-A^* - Q^* X^{-*}) \neq \emptyset$*, then the following holds*

1. $XD(A) = D(A^*)$*;*
2. A *and* $A + X^{-1}Q$ *are the infinitesimal generators of the strongly continuous group,* $(T(t))_{t\in\mathbb{R}}$ *and* $(T_Q(t))_{t\in\mathbb{R}}$*, respectively;*
3. *For all* $t \in \mathbb{R}$ *we have that* $T(t)^* = XT_Q(-t)X^{-1}$*.*

Proof. 1. If we choose $A_1 = A$ and $A_2 = A + X^{-1}Q$, then equation (16) is the same as (7). Hence we conclude from Lemma 2.1 that $XD(A) = XD(A_2) = D(A_1^*) = D(A^*)$.

2. Using the above definition of A_1 and A_2, we see that they both generate a C_0-semigroup. Since $-A_1^*$ is similar to A_2, we obtain that $-A_1^*$ is the infinitesimal generator of a C_0-semigroup as well. Since $-A_1^*$, and thus $-A_1$ is the infinitesimal generator of a C_0-semigroup, we obtain that $A_1 = A$ generates a group. Since A_2 is a bounded perturbation of A, it also generates a C_0-group.

3. This follows immediately from the similarity of the generators. ■

Remark 3.2. *Since the solution* X *of (16) may be hard to calculate, the condition* $\rho(A) \cap \mathbb{C} \setminus \sigma_p(-A^* - Q^*X^{-*}) \neq \emptyset$ *may be hard to check. However, when* A *generates a group,* A *and* $-A^* - Q^*X^{-*}$ *generates a* C_0*-semigroup, and hence the intersection of* $\rho(A)$ *with* $\rho(-A^* - Q^*X^{-*})$ *contains a right half-plane, and thus is nonempty. Another option would be when* A *has compact resolvent, then its spectrum consists out of only countable many points, and hence the condition* $\rho(A) \cap \mathbb{C} \setminus \sigma_p(-A^* - Q^*X^{-*}) \neq \emptyset$ *will be satisfied.*

From Lemma 2.5, one sees that the assertions as formulated in items 2. and 3. also hold, provided that the assertion of item 1. holds.

We now continue with the unbounded version of Lemma 3.1.

Let A be the infinitesimal generator of a C_0-semigroup. Following Weiss and Staffans, see [9] and [7, section3.6], we define $Z_{-1} = D(A^*)'$, where the dual is taken with respect of the pivot space Z. Similar, one can define $Z_{-1,d}$ as $D(A)'$. Since Z is the pivot space we have for $z_1 \in D(A)$ and $z \in Z$ that

$$\langle z_1, z\rangle_{D(A)\times Z_{-1,d}} = \langle z_1, z\rangle. \tag{17}$$

The operator A^* has a bounded extension to the operator A^*_{ext} from Z to $Z_{-1,d}$, via

$$\langle z_1, A^*_{\text{ext}} z_2\rangle_{D(A)\times Z_{-1,d}} = \langle Az_1, z\rangle. \tag{18}$$

If C is a bounded operator from $D(A)$ to the second Hilbert space Y, then the the dual is a bounded operator from Y to $Z_{-1,d}$ defined as

$$\langle z_1, C^* y\rangle_{D(A)\times Z_{-1,d}} = \langle Cz_1, y\rangle_Y. \tag{19}$$

For $C \in \mathcal{L}(D(A), Y)$ the *Lambda-extension* of C is defined as

$$C_\Lambda z = \lim_{s\to\infty} sC(sI - A)^{-1} z, \tag{20}$$

with its domain all $z \in Z$ for which the above limit exists.

Consider the system

$$\dot{z}(t) = Az(t), \quad z(0) = z_0, \quad y(t) = Cz(t).$$

It is well-known that there exist positive constants m and M such that for all $z_0 \in Z$,

$$m\|z_0\|^2 \leq \int_0^\infty \|y(t)\|^2 dt \leq M\|z_0\|^2$$

if and only if the Lyapunov equation

$$\langle Xz_1, Az_2\rangle + \langle Az_1, Xz_2\rangle = -\langle Cz_1, Cz_2\rangle_Y, \tag{21}$$

$z_1, z_2 \in D(A)$, has a bounded, and boundedly invertible, positive solution. Hence the system has a (unique) solution and is exactly observable if and only if (21) has a coercive and bounded solution.

First we prove a result concerning non-invertible solution of the Lyapunov equation (21). From [] we know that if the Lyapunov equation has a bounded (not necessarily invertible) solution, then the operators $\left(\begin{smallmatrix} -A^* & 0 \\ 0 & A \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} -A^* & C^*C \\ 0 & A \end{smallmatrix}\right)$. We prove this result next.

Theorem 3.3. *Let A be a closed operator with dense domain $D(A)$, and let $C \in \mathcal{L}(D(A), Y)$. Assume further that the Lyapunov equation*

$$\langle Az_1, Xz_2\rangle + \langle X^*z_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle_Y, \qquad z_1, z_2 \in D(A) \tag{22}$$

has a bounded solution. Define the following operators:

$$A_1 = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \tag{23}$$

with $D(A_1) = D(A) \times D(A^*)$ and

$$A_2 = \begin{pmatrix} -A^*_{\text{ext}} & C^*C \\ 0 & A \end{pmatrix} \tag{24}$$

with $D(A_2) = \{(z_1, z_2) \in Z \times Z \mid z_2 \in D(A) \text{ and } A^*_{\text{ext}} z_1 - C^*Cz_2 \in Z\}$.

If $\rho(A) \cap \rho(-A^*) \neq \emptyset$, then the operators A_1^* and A_2 are are similar. More precisely,

$$D(A_2) = \{(z_1, z_2) \in Z \times Z \mid z_2 \in D(A) \text{ and } z_1 + Xz_2 \in D(A^*)\} \tag{25}$$

and

$$A_2 = \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} -A^* & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}. \tag{26}$$

Proof. The proof is divided into several steps. In the first step we show that A_1 and A_2 satisfy a Lyapunov equation with an invertible solution. In the second step we show that the conditions on the spectrum as formulated in Lemma 2.1 holds. In the last step we conclude (25) and (26) from this lemma.

Step 1: Define on $Z \times Z$ the operator

$$X_2 = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}. \tag{27}$$

It is easy to see that X_2 is bounded and boundedly invertible. Furthermore, we have for $x_1 = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in D(A_1)$ and $x_2 = \left(\begin{smallmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right) \in D(A_2)$

$$\begin{aligned} &\langle A_1x_1, X_2x_2\rangle + \langle X_2^*x_1, A_2x_2\rangle \\ &\quad= \left\langle \begin{pmatrix} Az_1 \\ -A^*z_2 \end{pmatrix}, \begin{pmatrix} \tilde{z}_1 + X\tilde{z}_2 \\ \tilde{z}_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} z_1 \\ X^*z_1 + z_2 \end{pmatrix}, \begin{pmatrix} -A^*_{\text{ext}}\tilde{z}_1 + C^*C\tilde{z}_2 \\ A\tilde{z}_2 \end{pmatrix} \right\rangle \\ &\quad= \langle Az_1, \tilde{z}_1\rangle + \langle Az_1, X\tilde{z}_2\rangle - \langle A^*z_2, \tilde{z}_2\rangle - \langle z_1, A^*_{\text{ext}}\tilde{z}_1 - C^*C\tilde{z}_2\rangle + \langle X^*z_1, A\tilde{z}_2\rangle + \langle z_2, A\tilde{z}_2\rangle \\ &\quad= \langle Az_1, \tilde{z}_1\rangle + \langle Az_1, X\tilde{z}_2\rangle - \langle z_1, A^*_{\text{ext}}\tilde{z}_1 - C^*C\tilde{z}_2\rangle + \langle X^*z_1, A\tilde{z}_2\rangle. \end{aligned}$$

Using (17), we see that

$$\begin{aligned}\langle z_1, A_{\mathrm{ext}}^*\tilde{z}_1 - C^*C\tilde{z}_2\rangle &= \langle z_1, A_{\mathrm{ext}}^*\tilde{z}_1 - C^*C\tilde{z}_2\rangle_{D(A)\times Z_{-1,d}} \\ &= \langle z_1, A_{\mathrm{ext}}^*\tilde{z}_1\rangle_{D(A)\times Z_{-1,d}} - \langle z_1, C^*C\tilde{z}_2\rangle_{D(A)\times Z_{-1,d}} \\ &= \langle Az_1, \tilde{z}_1\rangle - \langle Cz_1, C\tilde{z}_2\rangle_Y,\end{aligned}$$

where we have used (18) and (19). Thus we find that

$$\begin{aligned}&\langle A_1x_1, X_2x_2\rangle + \langle X_2^*x_1, A_2x_2\rangle \\ &\quad = \langle Az_1, \tilde{z}_1\rangle + \langle Az_1, X\tilde{z}_2\rangle - \langle Az_1, \tilde{z}_1\rangle + \langle Cz_1, C\tilde{z}_2\rangle_Y + \langle X^*z_1, A\tilde{z}_2\rangle \\ &\quad = 0, \end{aligned}\tag{28}$$

where we have used the Lyapunov equation.

Step 2: It is easy to see that $\rho(A_1) = \rho(A) \cap \rho(-A^*)$. Let s be an element of this set. we show that $s \notin \sigma_p(-A_2^*)$. If s would be in this point spectrum, then there exists a nonzero $z = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in D(-A_2^*)$ such that $-A_2^*z = sz$. Thus for all $\tilde{z} = \left(\begin{smallmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right) \in D(A_2)$ we have that

$$0 = \langle \tilde{z}, (sI + A_2^*)z\rangle = \langle (\overline{s}I + A_2)\tilde{z}, z\rangle.$$

Using the definition, we see that for all $\tilde{z} = \left(\begin{smallmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{smallmatrix}\right) \in D(A_2)$

$$0 = \langle \overline{s}\tilde{z}_1 - A_{\mathrm{ext}}^*\tilde{z}_1 + C^*C\tilde{z}_2, z_1\rangle + \langle (\overline{s}I + A)\tilde{z}_2, z_2\rangle \tag{29}$$

We see that $D(A^*) \times \{0\} \subset D(A_2)$, and using these element in (29), we find that for all $\tilde{z}_1 \in D(A^*)$

$$0 = \langle (\overline{s}I - A^*)\tilde{z}_1, z_1\rangle$$

This implies that $z_1 = 0$, or $s \in \sigma_p(A)$. By assumption, the latter is not possible. Hence $z_1 = 0$. Applying this result to (29) we find that $\langle (\overline{s}I + A)\tilde{z}_2, z_2\rangle$ for all $\tilde{z}_2 \in D(A)$. Since $s \in \rho(-A^*)$ we conclude that $z_2 = 0$. Concluding we see that $\rho(A_1) \cap \mathbb{C}\backslash\sigma_p(-A_2^*) = \rho(A_1) \neq \emptyset$.

Step 3: From step 1 and 2, we see that A_1 and A_2 satisfies the conditions of Lemma 2.1. Hence we conclude that $D(A_2) = X_2^{-1}D(A_1^*)$ and $A_2 = -X_2^{-1}A_1^*X_2$ on this domain. Using the definition of X_2, we see this is equal (25) and (26), respectively. ■

Remark 3.4. *Using Lemma 2.5, we conclude that the similarity (26) still holds, when the domains of* A_1^* *and* A_2 *are the same. In particular, this holds when* $C \in \mathcal{L}(Z, Y)$*.*

Now we can show the first result concerning the invertible solutions of the Lyapunov equation. In Lemma 3.1 we considered the case when C is bounded, now we shall consider the case when C is unbounded. Using the notation as introduced below Lemma 4.1, we see that C^* maps Y into $Z_{-1,d}$, which is also the range of A_{ext}^*.

Theorem 3.5. *Let* A *with domain* $D(A)$ *be a closed, densely defined operator on the Hilbert space* Z *and let* C *be an element of* $\mathcal{L}(D(A), Y)$*, where* Y *is a second Hilbert space. Assume further that* $\rho(A^*) \cap \rho(-A) \neq \emptyset$ *If* $X \in \mathcal{L}(Z)$ *is an boundedly invertible solution to the Lyapunov equation*

$$\langle Az_1, Xz_2\rangle + \langle X^*z_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle_Y, \qquad z_1, z_2 \in D(A) \tag{30}$$

then the operator A_2 *defined as*

$$A_2 = A^*_{\text{ext}} + C^*CX^{-1} \tag{31}$$

with domain $D(A_2) = \{z \in XD(A) \mid A_2 z \in Z\}$, *is similar to* $-A$. *More, precisely*

$$A_2 = -XAX^{-1} \tag{32}$$

and $D(A_2) = XD(A)$.

The operator X^{-1} *is a solution of the algebraic Riccati equation*

$$\langle A_2z_1X^{-1}z_2\rangle + \langle X^{-*}z_2, A_2z_2\rangle = \langle CX^{-1}z_1, CX^{-1}z_2\rangle_Y \qquad z_1, z_2 \in D(A_2) \tag{33}$$

and of the Lyapunov equation

$$\langle A^*z_1, X^{-1}z_2\rangle + \langle X^{-*}z_1, A_2z_2\rangle = 0, \qquad z_1 \in D(A^*), \quad z_2 \in D(A_2). \tag{34}$$

Furthermore, if A *generates a* C_0*-group, then the same holds for* A_2, *and they satisfy*

$$T(t) = XT_2(-t)X^{-1}, \qquad t \in \mathbb{R}. \tag{35}$$

Proof. Let $s \in \rho(A^*) \cap \rho(-A)$. Since $\rho(A) = \overline{\rho(A^*)}$, we have that $\overline{s} \in \rho(A) \cap \rho(-A^*)$. For this $\overline{s}$ we write (30) as

$$\langle(-\overline{s}I + A)z_1, Xz_2\rangle + \langle X^*z_1, (sI + A)z_2\rangle = -\langle Cz_1, Cz_2\rangle_Y, \qquad z_1, z_2 \in D(A). \tag{36}$$

Since $s \in \rho(-A)$ and $\overline{s} \in \rho(A)$, we see that this equation is equivalent to the following Lyapunov equation

$$X(sI + A)^{-1} + (-sI + A^*)^{-1}X = -\left(C(-\overline{s}I + A)^{-1}\right)^* C(sI + A)^{-1} \quad \text{on } Z. \tag{37}$$

This can be equivalently written as

$$X(sI + A)^{-1} + \left[(-sI + A^*)^{-1} + \left(C(-\overline{s}I + A)^{-1}\right)^* C(sI + A)^{-1}X^{-1}\right] X = 0. \tag{38}$$

Now we define $A_1 := (sI+A)^{-*}$ and $A_3 := (-sI+A^*)^{-1} + \left(C(-\overline{s}I + A)^{-1}\right)^* C(sI+A)^{-1}X^{-1}$, then from (38) we see that A_1 and A_3 are bounded operators and satisfy the Lyapunov equation

$$\langle A_1z_1, X^{-1}z_2\rangle + \langle X^{-*}z_1, A_3z_2\rangle = 0, \qquad z_1, z_2 \in Z.$$

Since A_1 and A_3 are bounded operators, their resolvent sets will intersect. Hence from Lemma 2.1 we find that $\text{ran}(A_3) = X\text{ran}(A_1^*) = \text{ran}((sI + A)^{-1}) = XD(A)$, and

$$A_3 = -XA_1^*X^{-1} = -X(sI + A)^{-1}X^{-1}. \tag{39}$$

It remains to show that A_3 equals the inverse of $-sI+A_2$. Since $\text{ran}(A_3) = XD(A)$, we would have that $\text{ran}(A_3) \subset D(A_2)$ if $A_2\text{ran}(A_3) \subset Z$. This is hard to check directly. Therefor we first extend A_2 to an operator from $XD(A)$ to $Z_{-1,d}$, and show that this operator restricted to the range of A_3 maps into Z. Thus we define

$$\tilde{A}_2 = A^*_{\text{ext}} + C^*CX^{-1} \text{ on } D(\tilde{A}_2) = XD(A).$$

Now we have the following equalities in $Z_{-1,d}$ for $z \in Z$

$$\begin{aligned}(-sI + \tilde{A}_2)A_3 z = {} & (-sI + A^*_{\text{ext}}) \Big((-sI + A^*)^{-1} + \big(C(-\overline{s}I + A)^{-1}\big)^* C(sI + A)^{-1} X^{-1} \Big) z + \\ & C^* C X^{-1} A_3 z \\ = {} & z + (-sI + A^*_{\text{ext}}) \big(C(-\overline{s}I + A)^{-1}\big)^* C(sI + A)^{-1} X^{-1} z - \\ & C^* C (sI + A)^{-1} X^{-1} z,\end{aligned}$$

where we have used (39). Using the definition of A^*_{ext}, we find for all $z_1 \in D(A)$ and $y \in Y$

$$\begin{aligned}\langle z_1, (-sI + A^*_{\text{ext}}) \big(C(-\overline{s}I + A)^{-1}\big)^* y \rangle_{D(A)\times Z_{-1,d}} &= \langle (-\overline{s}I + A) z_1, \big(C(-\overline{s}I + A)^{-1}\big)^* y \rangle \\ &= \langle \big(C(-\overline{s}I + A)^{-1}\big) (-\overline{s}I + A) z_1, y \rangle_Y \\ &= \langle C z_1, y \rangle_Y \\ &= \langle z_1, C^* y \rangle_{D(A)\times Z_{-1,d}}.\end{aligned}$$

Thus we see that

$$(-sI + \tilde{A}_2) A_3 z = z$$

for all z. In particular, we see that $\tilde{A}_2$ maps the range of A_3 into Z. This implies that the range of A_3 is contained in the domain of A_2. Furthermore, we see that

$$(-sI + A_2) A_3 z = z \tag{40}$$

for all z. Using equation (40), it is easy to see that if $z_1 := A_3(-sI + A_2)z$, then

$$(-sI + A_2) z_1 = (-sI + A_2) z.$$

Hence if the kernel of $(-sI + A_2)$ is empty, then $(-sI + A_2)$ equals the inverse of A_3, or equivalently $(-sI + A_2) = A_3^{-1}$. Let $z \in D(A_2)$ lie in the kernel of $(-sI + A_2)$, then

$$\begin{aligned}0 &= \langle z_1, (-sI + A^*_{\text{ext}} + C^* C X^{-1}) z \rangle_{D(A)\times Z_{-1,d}} \\ &= \langle (-\overline{s} + A) z_1, z \rangle + \langle C z_1, C X^{-1} z \rangle_Y \\ &= \langle (-\overline{s}I + A) z_1, z \rangle - \langle A z_1, X X^{-1} z \rangle - \langle X^* z_1, A X^{-1} z \rangle \\ &= -\langle X^* z_1, (sI + A) X^{-1} z \rangle,\end{aligned}$$

where we have used (30). Since this holds for all $z_1 \in D(A)$, we find that $(sI + A)X^{-1}z = 0$. Since $sI + A$ and X are invertible, we conclude that $z = 0$. Hence we have proved the that $D(A_2) = \operatorname{ran}(A_3)$ and $(-sI + A_2) = A_3^{-1}$, or equivalently, $D(A_2) = XD(A)$, and

$$-sI + A_2 = -X(sI + A)X^{-1}.$$

Thus we have shown (32).

Using (32) and the Lyapunov equation (30), the equalities (33) and (34) are easy to prove. The last assertion in the theorem follws also directly from (32). $\blacksquare$

Lemma 3.6. *Let A be the generator of a C_0-semigroup, with $0 \in \rho(A)$. Let C be a bounded operator from $D(A)$ to a second Hilbert space Y. Assume furthermore that X is an invertible, self-adjoint solution to the Lyapunov equation (21). Then the following results hold:*

1. *For all $z \in Z$ and all real $s \in \rho(A)$ we have*
$$\begin{aligned} s^2\|C(sI-A)^{-1}X^{-1}A^{-*}z\|^2 = & -2s\mathrm{Re}\left(\langle (sI-A)^{-1}X^{-1}A^{-*}z, z\rangle\right) + \\ & -2s\|X^{1/2}A(sI-A)^{-1}X^{-1}A^{-*}z\|^2. \end{aligned} \tag{41}$$

2. *There exists an $r > 0$ such that for $s \in [r, \infty)$, we have that the linear operators $sC(sI-A)^{-1}X^{-1}A^{-*}$ and $s^{1/2}X^{1/2}A(sI-A)^{-1}X^{-1}A^{-*}$ are bounded independently of s.*

3. *If $X^{-1}D(A^*) \subset D(C_\Lambda)$, then for all $z_1 \in D(A^*)$ we have*
$$\langle A^*z_1, X^{-1}z_1\rangle + \langle X^{-1}z_1, A^*z_1\rangle \leq -\langle C_\Lambda X^{-1}z_1, C_\Lambda X^{-1}z_1\rangle_Y. \tag{42}$$

4. *If $X^{-1}D(A^*) \cap D(A)$ is dense in Z, then $X^{-1}D(A^*) \subset D(C_\Lambda)$ and*
$$\langle A^*z_1, X^{-1}z_2\rangle + \langle X^{-1}z_1, A^*z_2\rangle = -\langle C_\Lambda X^{-1}z_1, C_\Lambda X^{-1}z_2\rangle_Y \tag{43}$$
for all $z_1, z_2 \in D(A^*)$.

5. *If $D(A) \cap \ker C$ is dense in Z, then $X^{-1}D(A^*) \cap D(A)$ is dense in Z.*

Proof. *Part 1.* Replaying z_1 and z_2 in (21) by $s(sI-A)^{-1}X^{-1}A^{-*}z$, we obtain that
$$\begin{aligned} & -\frac{1}{2}s^2\|C(sI-A)^{-1}X^{-1}A^{-*}z\|^2 \\ & \quad = s^2\mathrm{Re}\left(\langle XA(sI-A)^{-1}X^{-1}A^{-*}z, (sI-A)^{-1}X^{-1}A^{-*}z\rangle\right) \\ & \quad = s\mathrm{Re}\left(\langle XA(sI-A)^{-1}X^{-1}A^{-*}z,\right. \\ & \qquad\qquad \left.(s-A+A)(sI-A)^{-1}X^{-1}A^{-*}z\rangle\right) \\ & \quad = s\mathrm{Re}\left(\langle XA(sI-A)^{-1}X^{-1}A^{-*}z,\right. \\ & \qquad\qquad \left.A(sI-A)^{-1}X^{-1}A^{-*}z\rangle\right) + \\ & \qquad s\mathrm{Re}\left(\langle XA(sI-A)^{-1}X^{-1}A^{-*}z, X^{-1}A^{-*}z\rangle\right) \\ & \quad = s\|X^{1/2}A(sI-A)^{-1}X^{-1}A^{-*}z\|^2 + \\ & \qquad s\mathrm{Re}\left(\langle (sI-A)^{-1}X^{-1}A^{-*}z, z\rangle\right). \end{aligned}$$

So we have proved (41).

Part 2. Since the norm is always positive, we see that for all $z \in Z$
$$\begin{aligned} & s^2\|C(sI-A)^{-1}X^{-1}A^{-*}z\|^2 \\ & \quad \leq -2s\mathrm{Re}\left(\langle (sI-A)^{-1}X^{-1}A^{-*}z, z\rangle\right). \end{aligned} \tag{44}$$

By Hille-Yosida Theorem we have that there exists an $r > 0$ and an $M > 0$ such that $\|s(sI-A)^{-1}\| < M$ for $s \in [r, \infty)$. Since A^{-*} and X^{-1} are bounded, we obtain the result. Combining this with (41) gives the uniform boundedness of the other operator family as well.

Part 3. Since $X^{-1}D(A^*) \subset D(C_\Lambda)$, we have that the limit of the left-hand side of (44) exists. Furthermore, it is known that $s(sI-A)^{-1}z \to z$ for $s \to \infty$. Thus from (44) we obtain that
$$\|C_\Lambda X^{-1}A^{-*}z\|^2 \leq -2\mathrm{Re}\left(\langle X^{-1}A^{-*}z, z\rangle\right).$$

This is the same as (42).

Part 4. Let $\tilde{z} \in X^{-1}D(A^*) \cap D(A)$, then $\tilde{z}$ can be written as $X^{-1}A^{-*}z$, and $AX^{-1}A^{-*}z$ is well-defined. Thus using Hille-Yosida Theorem once more,

$$\|A(sI - A)^{-1}X^{-1}A^{-*}z\|^2 \leq \frac{M^2\|AX^{-1}A^{-*}z\|^2}{(s-r)^2}.$$

So the last term in (41) converges to zero for $s \to \infty$. Since the operators in this last term are uniformly bounded and converges to zero on a dense set, we obtain that they converge to zero for all $z \in Z$. The middle term of (41) converges (see Part 2), and so the first term converges. This implies that $X^{-1}D(A^*) \subset D(C_\Lambda)$. Now using the fact that the last term converges to zero, we find as in Part 3 that the Lyapunov equation (43) holds.

Part 5. Take in (21) z_2 to be any element of $D(A) \cap \ker C$, then we find for all $z_1 \in D(A)$

$$\langle Az_1, Xz_2 \rangle + \langle Xz_1, Az_2 \rangle = 0.$$

This implies that Xz_2 is an element of $D(A^*)$. Thus $z_2 \in X^{-1}D(A^*)$. In other words $D(A) \cap \ker C \subset X^{-1}D(A^*) \cap D(A)$. ■

Remark 3.7. *It may seem that the condition "$D(A) \cap \ker C$ is dense" is very strong. If C is a bounded operator from Z to Y, then this does normally not hold. However, when C is a point evaluation, then it is likely to hold. The domain of A normally consists out of smooth functions satisfying certain boundary conditions. Adding some more conditions at (spatial) points, will not effect the denseness of this set.*

4 INVERTIBLE SOLUTIONS OF THE ARE

Since the algebraic Riccati equation can be transformed into a Lyapunov equation, we have some results for this one as well.

We consider the optimal control problem for an admissible input operator, but with a bounded output operator. Let X be the minimal cost operator. As it is shown in Weiss & Weiss, [10, equation (10.1)], it satisfies the following Lyapunov equation:

$$\langle Az_0, Xx_0 \rangle + \langle z_0, XA^{\mathrm{opt}}x_0 \rangle = -\langle Cz_0, QC_{\Lambda w}x_0 \rangle, ,$$

$z_0 \in D(A), x_0 \in D(A^{\mathrm{opt}})$, where we assume that $N = 0$. Since we assumed that C is bounded and X is self-adjoint, we obtain

$$\langle Az_0, Xx_0 \rangle + \langle Xz_0, A^{\mathrm{opt}}x_0 \rangle = -\langle z_0, C^*QCx_0 \rangle, \tag{45}$$

$z_0 \in D(A), x_0 \in D(A^{\mathrm{opt}})$.

Lemma 4.1. *Let X is a self-adjoint, invertible solution of (45), and assume that $\rho(-A^* - X^{-*}C^*QC) \cap \rho(A^{\mathrm{opt}}) \neq \emptyset$. Then*

1. *$D(A^{\mathrm{opt}}) = X^{-1}D(A^*)$*
2. *A^{opt} and $A + X^{-1}C^*QC$ generate a C_0-group, $T^{\mathrm{opt}}(t)$, and $T_{X^{-1}C^*QC}(t)$, respectively, and $T^{\mathrm{opt}}(t) = X^{-1}T_{X^{-1}C^*QC}(-t)^*X$.*

Proof. The proof is very similar as that of Lemma 3.1. We define $A_1 := A + X^{-1}C^*QC$ and $A_2 = A^{\rm opt}$. Now it is easy to see that (45) is the same as (7). So $D(A^{\rm opt}) = D(A_2) = X^{-1}D(A_1^*) = X^{-1}D(A^*)$.

Now by using Lemma 2.5 we can show (similar as in Lemma 3.1) that A_1 and A_2 both generate a group, and $T_1(-t)^* = XT_2(t)X^{-1}$. Or equivalently, $T^{\rm opt}(t) = X^{-1}T_{X^{-1}C^*QC}(-t)^*X$. ■

The above result shows for a special case that the optimal semigroup can be calculated in different way. In the example of Weiss & Zwart [11], the assumptions of the above lemma are satisfied, and thus it is easier to calculate the domain of $A^{\rm opt}$.

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