Part III: Distributed Parameter Systems

Hans Zwart, University of Twente

Arjan van der Schaft, University of Groningen

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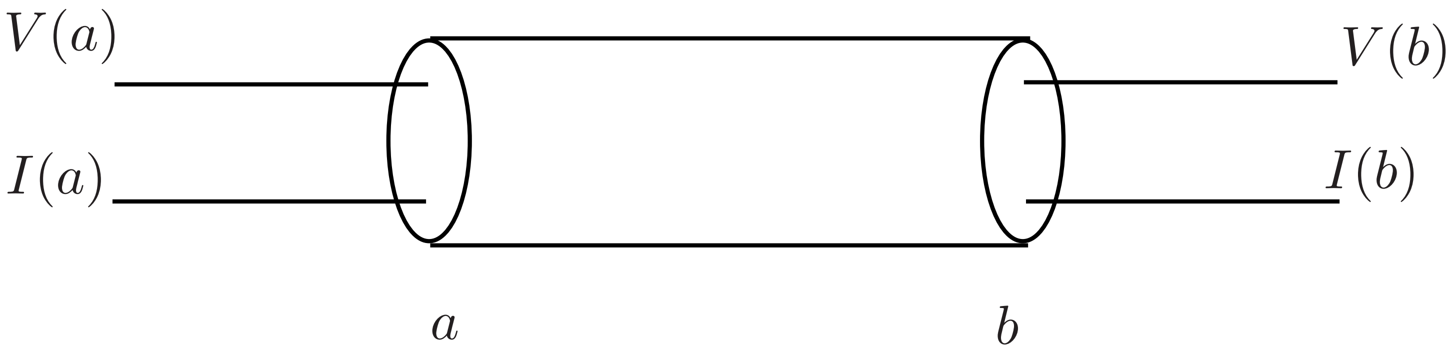
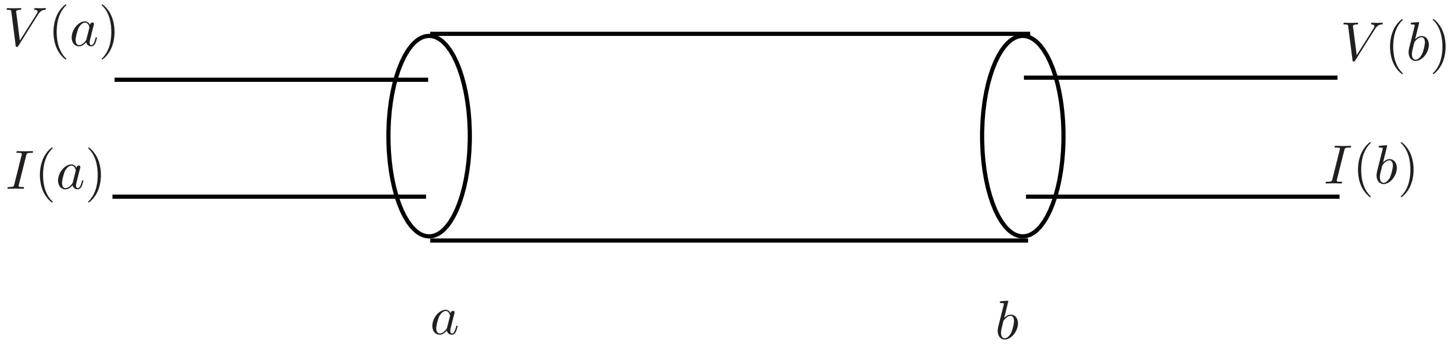
1 A simple example



Consider the transmission line on the spatial interval $[a, b]$

$$\begin{aligned} \frac{\partial Q}{\partial t}(z,t) &= -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \\ \frac{\partial \phi}{\partial t}(z,t) &= -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)}. \end{aligned} \tag{1}$$

1 A simple example



Consider the transmission line on the spatial interval $[a, b]$

$$\frac{\partial Q}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \qquad (1)$$
$$\frac{\partial \phi}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)}.$$

We write $x_1 = Q$ (charge) and $x_2 = \phi$ (flux), and we find that

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(z,t) &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial z}\begin{pmatrix} \frac{1}{C(z)} x_1(z,t) \\ \frac{1}{L(z)} x_2(z,t) \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial z}\left[\begin{pmatrix} \frac{1}{C(z)} & 0 \\ 0 & \frac{1}{L(z)} \end{pmatrix}\begin{pmatrix} x_1(z,t) \\ x_2(z,t) \end{pmatrix}\right] \\
&= P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)(z,t)
\end{aligned}
\qquad (2)
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$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial z}\left[\begin{pmatrix} \frac{1}{C(z)} & 0 \\ 0 & \frac{1}{L(z)} \end{pmatrix}\begin{pmatrix} x_1(z,t) \\ x_2(z,t) \end{pmatrix}\right]$$

$$= P_1\frac{\partial}{\partial z}(\mathcal{L}x)(z,t)$$

So we have the format as seen before, $J = P_1\frac{\partial}{\partial z}$ and $\frac{\partial \mathcal{H}}{\partial x} = \mathcal{L}x$. Thus the Hamiltonian equals $H = \frac{1}{2}\int_a^b x^T\mathcal{L}x dz = \int_a^b \mathcal{H}dz$.

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To recapitulate some work done before, we differentiate the Hamiltonian (energy) along trajectories.

$$
\begin{aligned}
\frac{dH}{dt}(t) &= \frac{1}{2}\int_a^b \frac{\partial x}{\partial t}(z,t)^T \mathcal{L}(z) x(z,t) dz + \frac{1}{2}\int_a^b x(z,t)^T \mathcal{L}(z) \frac{\partial x}{\partial t}(z,t) dz \\
&= \frac{1}{2}\int_a^b \left(P_1 \frac{\partial}{\partial z} \left(\mathcal{L}x\right)(z,t) \right)^T \mathcal{L}(z) x(z,t) dz + \\
&\quad \frac{1}{2}\int_a^b x(z,t)^T \mathcal{L}(z,t) (P_1 \frac{\partial}{\partial z} \left(\mathcal{L}x\right)(z,t) dz \\
&= \frac{1}{2}\int_a^b \frac{\partial}{\partial z} \left[\left(\mathcal{L}x\right)^T (z,t) P_1 \left(\mathcal{L}x\right)(z,t) \right] dz \\
&= \frac{1}{2} \left[\left(\mathcal{L}x\right)^T (z,t) P_1 \left(\mathcal{L}x\right)(z,t) \right]_a^b ,
\end{aligned}
$$

where we have used the symmetry of P_1.

So we have that the time-change of Hamiltonian satisfies

$$\frac{dH}{dt}(t) = \frac{1}{2}\left[\left(\mathcal{L}x\right)^T(z,t)P_1\left(\mathcal{L}x\right)(z,t)\right]_a^b. \qquad (3)$$

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The balance equation (3) also holds for the system

$$\frac{\partial x}{\partial t}(z,t) = P_1\frac{\partial \mathcal{L}x}{\partial z}(z,t) + P_0[\mathcal{L}x](z,t) \tag{4}$$

with P_0 anti-symmetric, i.e., $P_0^T = -P_0$.

Many systems can be written in this format.

2 More examples

Example

Wave equation for the vibrating string:

$$\frac{\partial^2 w}{\partial t^2}(z,t) = \frac{T}{\rho}\frac{\partial^2 w}{\partial z^2}(z,t),$$

where ρ is the mass density, and T is Young's modulus.

This is in our format with

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{pmatrix}$$

The state variable are $x_1 = \rho\frac{\partial w}{\partial t}$ (the momentum) and $x_2 = \frac{\partial w}{\partial z}$ (the strain).

Example

The model of Timoshenko beam is given

$$\begin{aligned} \rho(z)\frac{\partial^2 w}{\partial t^2}(z,t) &= \frac{\partial}{\partial z}\left[K(z)\left[\frac{\partial w}{\partial z}(z,t)-\phi(z,t)\right]\right] \\ I_\rho(z)\frac{\partial^2 \phi}{\partial t^2} &= \frac{\partial}{\partial z}\left[EI(z)\frac{\partial \phi}{\partial z}\right]+K(z)\left[\frac{\partial w}{\partial z}(z,t)-\phi\right], \end{aligned}$$

where $w(z,t)$ is the transverse displacement of the beam and $\phi(z,t)$ is the rotation angle of a filament of the beam. The positive coefficients $\rho(z)$, $I_\rho(z)$, $E(z)$, $I(z)$, and $K(z)$ are the mass per unit length, the rotary moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively.

By introducing the state variables

$$
\begin{aligned}
x_1 &= \frac{\partial w}{\partial z} - \phi : && \text{shear displacement,}\\
x_2 &= \rho \frac{\partial w}{\partial t} : && \text{transverse momentum distribution,}\\
x_3 &= \frac{\partial \phi}{\partial z} : && \text{angular displacement,}\\
x_4 &= I_\rho \frac{\partial \phi}{\partial t} : && \text{angular momentum distribution,}
\end{aligned}
$$

we can write this in our standard format, with

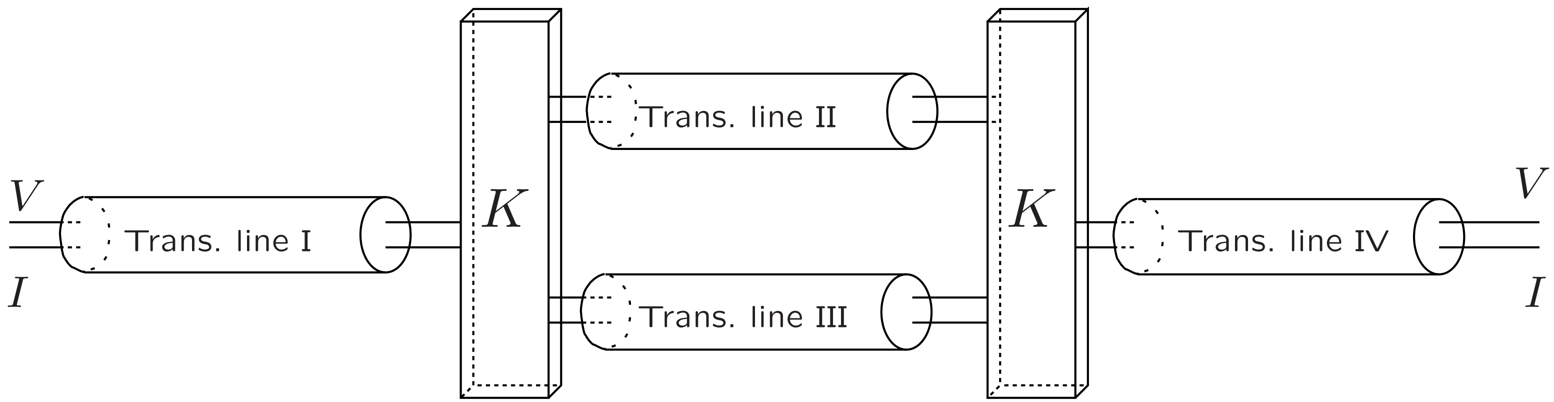
$$
P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

and

$$
\mathcal{L} = \mathrm{diag}\{K, \frac{1}{\rho}, EI, \frac{1}{I_\rho}\}.
$$

Example

Consider transmission lines in a network



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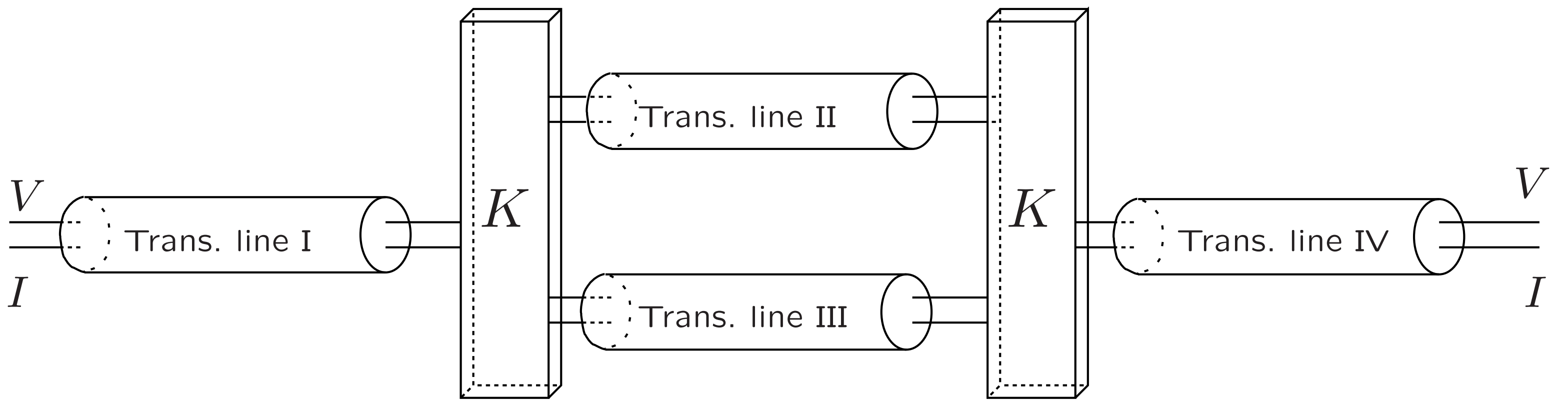

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The P_1 of the big system is the diagonal matrix, build from the uncoupled P_1's (which are all the same). The $\mathcal{L}$ of the coupled system is the diagonal matrix of the uncoupled $\mathcal{L}$.

The coupling is written down as boundary conditions of the p.d.e.

3 Choice of inputs and outputs

For our p.d.e.

$$\frac{\partial x}{\partial t}(z,t) = P_1 \frac{\partial \mathcal{L}x}{\partial z}(z,t) + P_0 \left[\mathcal{L}x\right](z,t)$$

we have

$$\frac{dH}{dt}(t) = \frac{1}{2}\left[\left(\mathcal{L}x\right)^T (z,t) P_1 \left(\mathcal{L}x\right)(z,t)\right]_a^b .$$

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Hence one may influence the Hamiltonian via these boundary variables, and one may observe the system via the boundary. How to choose inputs and outputs?

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Hence one may influence the Hamiltonian via these boundary variables, and one may observe the system via the boundary. How to choose inputs and outputs?

Consider again the transmission line example, and assume that the C and L are constant.

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(z,t) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial z}\left[\begin{pmatrix} \frac{1}{C} & 0 \\ 0 & \frac{1}{L} \end{pmatrix}\begin{pmatrix} x_1(z,t) \\ x_2(z,t) \end{pmatrix}\right].$$

We diagonalize the matrix $P_1\mathcal{L} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{-1}{L} & 0 \end{pmatrix}$.

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Hence

$$\begin{pmatrix} 0 & \frac{-1}{L} \\ \frac{-1}{C} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{L}{C}} & \sqrt{\frac{L}{C}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{LC}} & 0 \\ 0 & -\sqrt{\frac{1}{LC}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{\frac{C}{L}} \\ 1 & \sqrt{\frac{C}{L}} \end{pmatrix} \frac{1}{2}.$$

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Hence

$$\begin{pmatrix} 0 & \frac{-1}{L} \\ \frac{-1}{C} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{L}{C}} & \sqrt{\frac{L}{C}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{LC}} & 0 \\ 0 & -\sqrt{\frac{1}{LC}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{\frac{C}{L}} \\ 1 & \sqrt{\frac{C}{L}} \end{pmatrix} \frac{1}{2}.$$

In the new variables (characteristics)

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} := \begin{pmatrix} 1 & -\sqrt{\frac{C}{L}} \\ 1 & \sqrt{\frac{C}{L}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the transmission line becomes

$$\frac{\partial \xi_1}{\partial t} = \sqrt{\frac{1}{LC}} \frac{\partial \xi_1}{\partial z}, \qquad \frac{\partial \xi_2}{\partial t} = -\sqrt{\frac{1}{LC}} \frac{\partial \xi_2}{\partial z}.$$

Summarizing, we see that if we diagonalize $P_1\mathcal{L}$, then the p.d.e. becomes a set of p.d.e.'s in one variable.

Note that the coupling is now only via the boundary variables.

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Note that the coupling is now only via the boundary variables.

Assume that there is no coupling via the boundary, then we can study the choice of inputs and outputs for a scalar p.d.e.

Consider the shift on the interval $[0,1]$, described by the p.d.e.

$$\frac{\partial w}{\partial t}(z,t) = \frac{\partial w}{\partial z}(z,t), \qquad z \in (0,1). \tag{5}$$

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with boundary condition

$$w(0,t) = 0.$$

The solution would be $w(z,t) = w_0(z+t)$. This is conflicting with the freedom of initial conditions, since
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is good.

A perfect choice for the input/output pair is

$$w(1,t) = u_s(t) \qquad w(0,t) = y_s(t). \tag{6}$$

We summarize the previous.

- The format

$$\frac{\partial x}{\partial t}(z,t) = P_1 \frac{\partial}{\partial z}(\mathcal{L}x)(z,t), \qquad z \in [a,b], \tag{7}$$

  with P_1 symmetric, $\det(P_1) \neq 0$, and $\mathcal{L} > 0$, is a very nice one, and general, see our examples.

  They are port-Hamiltonian systems with a positive (quadratic) Hamiltonian. Note that $J = P_1 \frac{\partial}{\partial z}$, $\frac{\partial \mathcal{H}}{\partial x} = \mathcal{L}x$.

- If we diagonalize $P_1\mathcal{L}$, then we get a set of p.d.e.'s which are only coupled via the boundary. That is we write the system in its characteristics or Riemann coordinates.

- For a scalar p.d.e., we can easily find good inputs and outputs.

- How to find them general?

4 Well-posedness

For $t \geq 0$ and $z \in [a,b]$ we consider the system:

$$\frac{\partial x}{\partial t}(z,t) = P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)(z,t), \qquad x(0,z) = x_0(z) \tag{8}$$

$$0 = M_{11}\left(\mathcal{L}x\right)(b,t) + M_{12}\left(\mathcal{L}x\right)(a,t) \tag{9}$$

$$u(t) = M_{21}\left(\mathcal{L}x\right)(b,t) + M_{22}\left(\mathcal{L}x\right)(a,t) \tag{10}$$

$$y(t) = C_1\left(\mathcal{L}x\right)(b,t) + C_2\left(\mathcal{L}x\right)(a,t). \tag{11}$$

We assume that x takes values in $\mathbb{R}^n$, $P_1^T = P_1$, $\det(P_1) \neq 0$, $\mathcal{L} > 0$, and that $\operatorname{rank}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ C_1 & C_2 \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} C_1 & C_2 \end{bmatrix}$.

Definition

Consider the system (8)–(11). This system is *well-posed* if there exists a $t_f > 0$ and m_f such that the following holds:

1. The homogeneous p.d.e., i.e., $u \equiv 0$ has for any initial condition, $x(0)$ a unique (weak) solution.

2. The following inequality holds for all smooth initial conditions, and all smooth inputs

$$H(t_f) + \int_0^{t_f} \|y(t)\|^2 dt \leq m_f \left[H(0) + \int_0^{t_f} \|u(t)\|^2 dt \right], \tag{12}$$

where H is the Hamiltonian: $\frac{1}{2} \int_a^b x^T \mathcal{L} x dz$.

Remark

- So well-posedness gives you that you have a unique solution for every square integrable input function and every initial condition. Furthermore, the output signal is always square integrable.
- Every well-posed system has a transfer function, bounded in some right-half plane.

Theorem

Under the conditions we have imposed, the following holds:

- If condition 1 holds, then automatically condition 2 holds.

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- The transfer function $G(s)$ can also be obtained, and also $\lim_{s\to\infty} G(s)$.
- There is a matrix condition for checking condition 1.
- Same theorem holds for $\frac{\partial x}{\partial t}(z,t) = P_1 \frac{\partial}{\partial z}\left(\mathcal{L}x\right)(z,t) + P_0\left(\mathcal{L}x\right)(z,t)$.

We want to give an idea of the proof. Therefor we do it for our system with the following choice of boundary input and output

$$u(t) = \begin{pmatrix} V(a,t) \\ V(b,t) \end{pmatrix}, \qquad y(t) = \begin{pmatrix} I(a,t) \\ I(b,t) \end{pmatrix}. \tag{13}$$

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Now we preform the following steps:

1. Since $V = Q/C = x_1/C$ and $I = \phi/L = x_2/L$, we have that

$$[M_{21}, M_{22}] = \left(\begin{array}{cc|cc} 0 & 0 & \frac{1}{C} & 0 \\ \frac{1}{C} & 0 & 0 & 0 \end{array}\right), \quad [C_1, C_2] = \left(\begin{array}{cc|cc} 0 & 0 & 0 & \frac{1}{L} \\ 0 & \frac{1}{L} & 0 & 0 \end{array}\right).$$

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2. Write the system in the "characteristic", $\xi_1 = x_1 - \sqrt{\frac{C}{L}} x_2$, $\xi_2 = x_2 + \sqrt{\frac{C}{L}} x_2$. The p.d.e. becomes

$$\frac{\partial \xi_1}{\partial t} = \sqrt{\frac{1}{LC}} \frac{\partial \xi_1}{\partial z}, \qquad \frac{\partial \xi_2}{\partial t} = -\sqrt{\frac{1}{LC}} \frac{\partial \xi_2}{\partial z}.$$

The "perfect" input and output for this system is

$$u_s(t) = \begin{pmatrix} \xi_1(b,t) \\ \xi_2(a,t) \end{pmatrix}, \qquad y_s(t) = \begin{pmatrix} \xi_1(a,t) \\ \xi_2(b,t) \end{pmatrix}. \tag{14}$$

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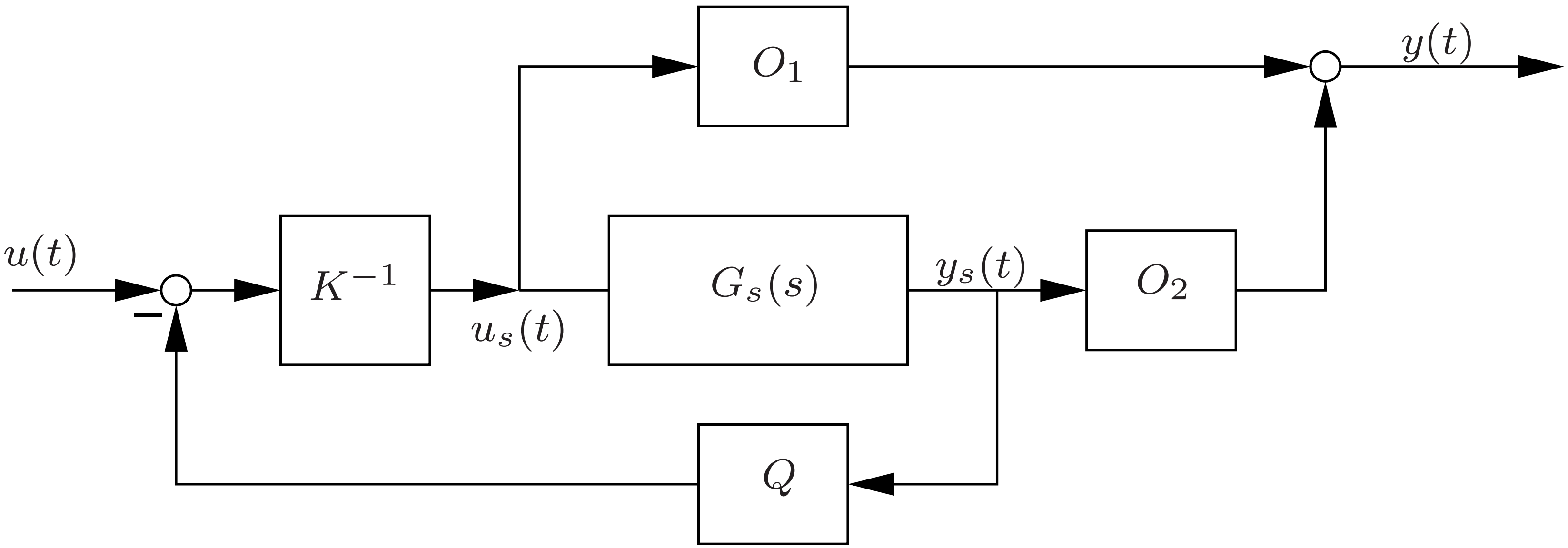
3. Write our input-output pair in this input-output pair.

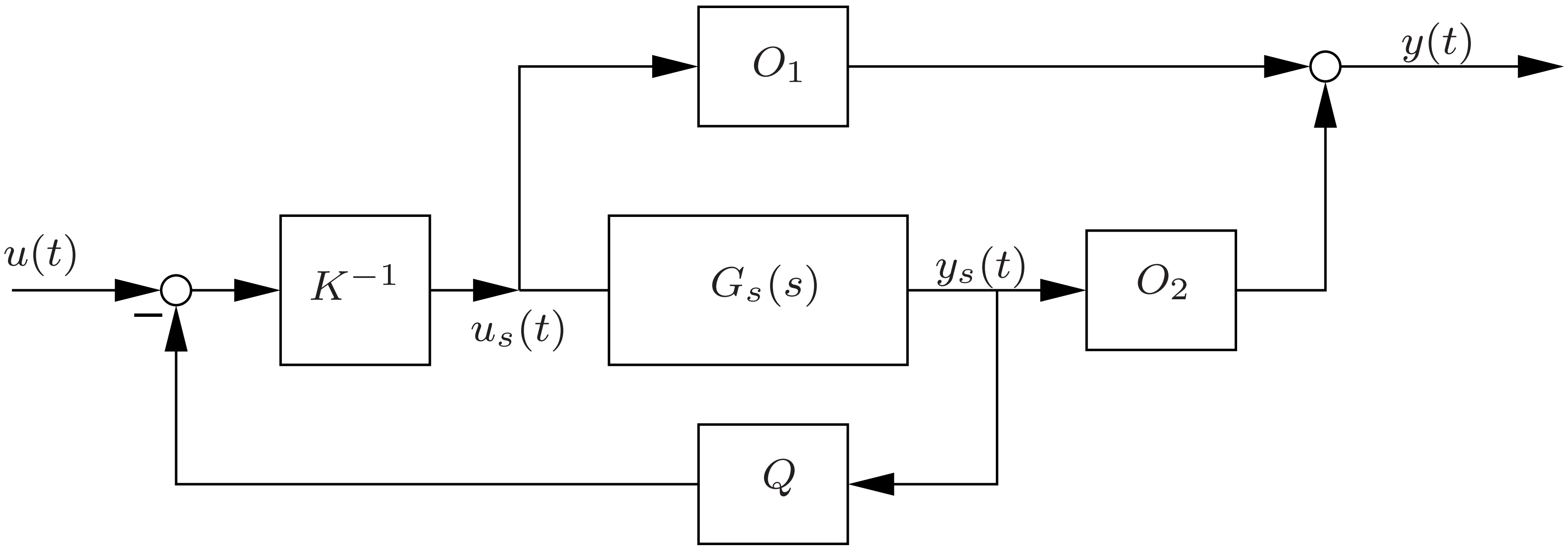
$$\begin{aligned} u(t) &= \begin{pmatrix} 0 & \frac{1}{2C} \\ \frac{1}{2C} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{1}{2C} & 0 \\ 0 & \frac{1}{2C} \end{pmatrix} y_s(t) \\ &= K u_s(t) + Q y_s(t) \end{aligned} \tag{15}$$

$$\begin{aligned} y(t) &= \begin{pmatrix} 0 & \frac{1}{2\sqrt{LC}} \\ \frac{-1}{2\sqrt{LC}} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{-1}{2\sqrt{LC}} & 0 \\ 0 & \frac{1}{2\sqrt{LC}} \end{pmatrix} y_s(t) \\ &= O_1 u_s(t) + O_2 y_s(t). \end{aligned} \tag{16}$$

We regard the system with input/output u, y as a feedback of the system with input/output u_s, y_s, i.e.,



We regard the system with input/output u, y as a feedback of the system with input/output u_s, y_s, i.e.,



Now G_s has feed-through zero, and so any feedback is allowed.

Only condition: K^{-1} exists.

Remark

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- Since

$$G_s(s) = \begin{pmatrix} e^{-s\sqrt{LC}(b-a)} & 0 \\ 0 & e^{-s\sqrt{LC}(b-a)} \end{pmatrix}$$

one can calculate the transfer function from u to y.

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$$G_s(s) = \begin{pmatrix} e^{-s\sqrt{LC}(b-a)} & 0 \\ 0 & e^{-s\sqrt{LC}(b-a)} \end{pmatrix}$$

one can calculate the transfer function from u to y.

- If we choose $u(t) = \begin{pmatrix} V(a,t) \\ I(a,t) \end{pmatrix}$, then not well-posed. The corresponding K is

$$K = \begin{pmatrix} 0 & \frac{1}{2C} \\ 0 & \frac{1}{2\sqrt{LC}} \end{pmatrix}.$$

5 Stability

For our class of system, i.e., $\frac{\partial x}{\partial t} = P_1 \frac{\partial \mathcal{L}x}{\partial z}$, we have

$$\frac{dH}{dt}(t) = \frac{1}{2}\left[(\mathcal{L}x)^T(z,t)P_1(\mathcal{L}x)(z,t)\right]_a^b.$$

Furthermore, we have written our boundary conditions, inputs, and outputs as linear combinations of these boundary variables. One would expect exponential stability of the uncontrolled system when $\frac{dH}{dt} < 0$ for zero inputs. This is the subject of the following theorem.

Theorem

Consider the p.d.e. $\frac{\partial x}{\partial t} = P_1 \frac{\partial \mathcal{L}x}{\partial z}$ with Hamiltonian $H = \frac{1}{2}\int_a^b x^T \mathcal{L}x dz$. If the homogeneous p.d.e., i.e., $u \equiv 0$ satisfies

- $\frac{dH}{dt} \leq -\kappa \|\mathcal{L}x(b)\|^2$, or
- $\frac{dH}{dt} \leq -\kappa \|\mathcal{L}x(a)\|^2$

for some $\kappa > 0$, then the (uncontrolled) system is exponentially stable.

Proof: We have that for τ_f large enough that (long proof)

$$H(\tau_f) \leq c_f \int_0^{\tau_f} \|\mathcal{L}x(b,t)\|^2 dt.$$

Now from our assumption

$$\begin{aligned} H(\tau_f) - H(0) &= \int_0^{\tau_f} \frac{dH}{dt}(t) dt \\ &\leq -\kappa \int_0^{\tau_f} \|\mathcal{L}x(b,t)\|^2 dt \end{aligned}$$

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Thus

$$H(\tau_f) \leq \frac{c_f}{c_f + \kappa} H(0) < H(0).$$

This proves exponential stability. □

We apply this theorem to our transmission line.

$$\frac{\partial Q}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \qquad (17)$$
$$\frac{\partial \phi}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)}.$$

For this system we have

$$P_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{L}(z) = \begin{pmatrix} \frac{1}{C(z)} & 0 \\ 0 & \frac{1}{L(z)} \end{pmatrix}.$$

Furthermore, $V = Q/C$, $I = \phi/L$. We choose the boundary input $V(a,t) = u(t)$, and we place a resistor at the other end, i.e., $V(b,t) = RI(b,t)$, $R > 0$.

Note that this is of the form (8)–(10), see slide 16.

Since $\mathcal{L}x = \left(\begin{smallmatrix} V \\ I \end{smallmatrix}\right)$, we find that

$$\begin{aligned}
\frac{dH}{dt} &= \frac{1}{2}\left[\left(\mathcal{L}x\right)^T(z)P_1\left(\mathcal{L}x\right)(z)\right]_a^b \\
&= \frac{1}{2}\left[\begin{pmatrix} V(z) & I(z) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} V(z) \\ I(z) \end{pmatrix}\right]_a^b \\
&= -V(b)I(b) + V(a)I(a).
\end{aligned}$$

Since $\mathcal{L}x = \left(\begin{smallmatrix} V \\ I \end{smallmatrix}\right)$, we find that

$$\begin{aligned}
\frac{dH}{dt} &= \frac{1}{2}\left[(\mathcal{L}x)^T(z) P_1 (\mathcal{L}x)(z)\right]_a^b \\
&= \frac{1}{2}\left[\begin{pmatrix} V(z) & I(z) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} V(z) \\ I(z) \end{pmatrix}\right]_a^b \\
&= -V(b)I(b) + V(a)I(a).
\end{aligned}$$

Applying our conditions: $V(a) = 0$, and $V(b) = RI(b)$, we have

$$\frac{dH}{dt} = -RI(b)^2 = -\frac{R}{R^2+1}\left[V(b)^2 + I(b)^2\right].$$

Thus exponentially stable.

6 Diffusive systems

We have that the system $\frac{\partial x}{\partial t} = P_1 \frac{\partial \mathcal{L}x}{\partial z}$, satisfies $\frac{dH}{dt} = \frac{1}{2}\left[(\mathcal{L}x)^T(z)P_1(\mathcal{L}x)(z)\right]_a^b$.

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As explained in Part I, there is an underlying structure.

If (f, e) are related like

$$f(z) = P_1 \frac{\partial e}{\partial z}(z) = Je(z), \tag{18}$$

then

$$\int_a^b f(z)^T e(z) dz = \frac{1}{2}\left[e^T(z) P_1 e(z)\right]_a^b. \tag{19}$$

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Choosing now $f = \frac{\partial x}{\partial t}$, and $e = \frac{\partial \mathcal{H}}{\partial x} = \mathcal{L}x$, we recover our system and balance equation.

However, we can make other choices.

Example

Choose the P_1 from the transmission line, i.e.,

$$P_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Now we choose

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial t} \\ f_2 \end{pmatrix}, \qquad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} x \\ Rf_2 \end{pmatrix}, \tag{20}$$

then we find the system ($f = P_1 \frac{\partial e}{\partial z}$)

$$\frac{\partial x}{\partial t} = -\frac{\partial}{\partial z}[Rf_2] = -\frac{\partial}{\partial z}\left[-R\frac{\partial x}{\partial z}\right] = \frac{\partial}{\partial z}\left[R\frac{\partial x}{\partial z}\right]. \tag{21}$$

This is the p.d.e. which describes diffusion.

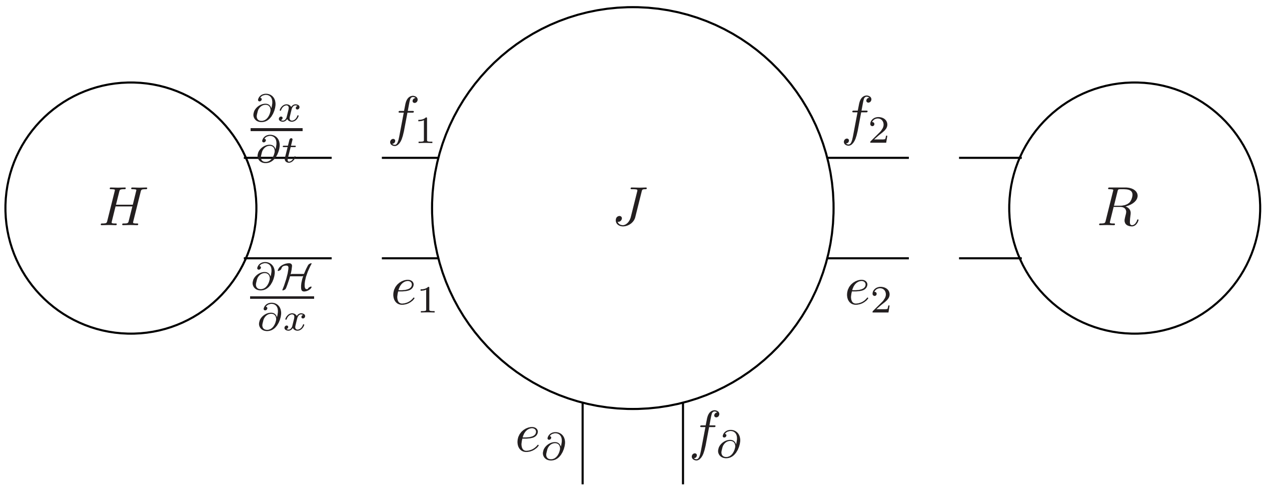

Figure 1: Port-Hamiltonian system with dissipation R

Theorem

Assume that the p.d.e.

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P_1 \frac{\partial}{\partial z}\begin{pmatrix} \mathcal{L}x_1 \\ x_2 \end{pmatrix} \tag{22}$$

with homogeneous boundary conditions has a (mild) solution for every initial condition, and that along these solutions

$$\frac{d}{dt}\left[\int_a^b x_1^T(z,t)\mathcal{L}x_1(z,t) + \|x_2(z,t)\|^2 dz\right] \leq 0. \tag{23}$$

Then the p.d.e.

$$\begin{pmatrix} \frac{\partial x}{\partial t} \\ f_2 \end{pmatrix} = P_1 \frac{\partial}{\partial z}\begin{pmatrix} \mathcal{L}x \\ Rf_2 \end{pmatrix} \tag{24}$$

has a solution, provided $R > \varepsilon I$. Furthermore, $\frac{d}{dt}\int_a^b x^T(z,t)\mathcal{L}x(z,t)dz \leq 0$.

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