Standard diffusive systems as well-posed linear systems

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Abstract

In this paper we show that every diffusive system is a well-posed system in the sense of Salamon and Weiss. Furthermore, we characterize several systems theoretic properties of these systems, such as stability, controllability, and observability. Instead of referring to general results on well-posed linear system, we prove these results directly. Hence we hope that this paper will serve as a tutorial to well-posed linear systems.

Keywords: diffusive representations, well-posed linear systems, fractional calculus, stability, controllability, observability.

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1 Introduction

The class of well-posed linear systems as introduced by Salamon in [17] has become a well-understood class of systems, see e.g. the work of Weiss and the book of Staffans [19]. Many partial partial differential equations with boundary control and point observation can be formulated as a well-posed linear system. Parallel to the development of well-posed linear system the class of diffusive systems has been developed, by [18, §5] and [16] independently. These systems are used to model phenomena for which the impulse response has a "long tail", i.e., decays slowly, or systems with a diffusive nature, like the Lokshin model, in which both viscous and thermal losses at the boundary of a duct are taken into account in the propagation of the acoustic waves, see [8]. Another class of models are the fractional differential equations, i.e., systems which have fractional powers of s in their transfer function, see e.g. [13].

The aim of this paper is to show that standard diffusive systems form a subset of the class of well-posed linear systems. It turns out that the proof of this is not complicated, and hence they form a more or less easy class within the general class of well-posed linear systems. This has three important consequences. Firstly, the theory as developed for well-posed linear systems, like feedback interconnection, etc, is directly applicable to diffusive systems. Secondly, (new) notions and results for the class of well-posed systems can be illustrated by means of diffusive systems, without having to use deep results on functional analysis or partial differential equations. Thirdly, this paper could be used as an introduction into well-posed linear systems.

The paper is organized as follows: in section 2, we recall what a well-posed linear system is, and in section 3 we define diffusive linear systems and give simple examples of them in the form of PDE's. We then prove in section 4 that every standard diffusive linear system is a regular linear system, and we address stability questions and illustrate different cases by means of our examples. Finally, section 5 presents system theoretic properties of these systems, such as controllability and observability. Some results which we present are consequences of earlier published results. This will be indicated, however, we present our own proof if this fits the tutorial nature of this paper better.

2 Abstract linear systems

The theory of well-posed linear systems basically wants to give meaning to the abstract differential equation

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \quad z(0) = z_0 \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

Basically, we want that for every initial condition z_0 and every locally square integrable input u, there exists a (weak) solution of (1) such that the state trajectory z is continuous and the output y is locally square integrable. In order to do so, we first introduce some notation. The Hilbert space Z will be our state space, U is the Hilbert space of input values, and Y is the Hilbert space of output values. Let $D(A)$ denotes the domain of A, and let Z_{-1} denote the dual of the domain of A^*. $\mathcal{L}(Z_1, Z_2)$ is defined as the space of all linear, bounded operators from the Hilbert space Z_1 to the Hilbert space Z_2.

Following [12] we can now define a well-posed linear system.

Definition 2.1. Consider the system (1). We say that the system is *well-posed* if the following four conditions hold:

a. A is the infinitesimal generator of a C_0-semigroup on Z, which we denote by $(T(t))_{t\geq 0}$.

b. $B \in \mathcal{L}(U, Z_{-1})$ and there exists some $t_1 > 0$ such that for all $u \in L^2([0, t_1); U)$ we have that $\int_0^{t_1} T(t_1 - \tau)Bu(\tau)\,\mathrm{d}\tau \in Z$.

c. $C \in \mathcal{L}(D(A), Y)$ and there exists a $t_1 > 0$ such that the map $z_0 \mapsto CT(\cdot)z_0$ can be extended to a bounded map from Z to $L^2([0, t_1); Y)$.

d. There exists a $\rho > 0$ such that any G satisfying

$$\frac{G(s) - G(\lambda)}{\lambda - s} = C(sI - A)^{-1}(\lambda I - A)^{-1}B \qquad s, \lambda \in \mathbb{C}_\rho^+, s \neq \lambda \tag{2}$$

is uniformly bounded on $\mathbb{C}_\rho^+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \rho\}$.

Some remarks are appropriate.

Remark 2.2.
- If the abstract linear system is well-posed, then its solutions have the properties mentioned below equation (1), see [3] where it is proved that this definition of a well-posed system is equivalent to the standard one introduced by Weiss [21], [22].

- If item b. holds for some $t_1 > 0$, then using the semigroup property of $(T(t))_{t\geq 0}$ it is not hard to show that it holds for all $t_1 > 0$. A similar remark holds for item c.
- If $G_0(s)$ is a solution of (2), then all G's satisfying this equation are given by $G(s) = G_0(s) + Q$ with $Q \in \mathcal{L}(U, Y)$. Hence the uniform boundedness condition in item d. is independent of the particular choice of G.

A special function G satisfying (2) is given by the transfer function of the system. This is defined as follows. For $s \in \mathbb{C}$ with real part larger than the growth bound of the semigroup, one takes the input $u(t) = u_0 e^{st}$, for $t \geq 0$ and $u_0 \in U$. For this input one calculates a state and output trajectory of the form $z_0(s)e^{st}$ and $H(s)u_0e^{st}$, respectively. The function $H(s)$ is defined to be the *transfer function* of our system, see [24]. It is not hard to show that this function satisfies (2).

A special class of well-posed linear systems is the class of regular linear systems; a well-posed system is *regular* if $\lim_{s\to\infty} H(s)u_0$ exists for all $u_0 \in U$. If the system is regular, then this limit can be denoted by Du_0 with $D \in \mathcal{L}(U, Y)$.

3 Diffusive linear systems with examples

We begin by defining our class of linear systems.

Definition 3.1. Let μ be a positive measure on $[0, \infty)$, with σ-algebra the Borel sets. Furthermore, assume that

$$c_\mu := \int_0^\infty \frac{1}{1+\xi} \, \mathrm{d}\mu(\xi) < \infty. \tag{3}$$

The *diffusive system* associated with μ is defined as the following parameterized differential equation on $[0, \infty)$:

$$\begin{array}{rcll} \frac{\partial}{\partial t}\phi(\xi, t) &=& -\xi\,\phi(\xi, t) + \mathbb{1}^+(\xi)\,u(t), \quad \xi \geq 0, & \qquad \phi(\cdot, 0) = \phi_0(\cdot), \\ y(t) &=& \int_0^\infty \phi(\xi, t)\,\mathrm{d}\mu(\xi), & \end{array} \tag{4}$$

where u is the input and y is the output. We denote by $\mathbb{1}$ the function equal to 1 everywhere, $\mathbb{1}^+$ the Heaviside step function, and $\mathbb{1}_V$ the indicator function of the set V in the sequel.

In the next section we show that every diffusive system is a (regular) well-posed system. In the rest of the present section we give examples of diffusive systems.

Example 3.2. Consider the following partial differential equation on the unbounded domain $I = \mathbb{R}$:

$$\begin{array}{rcl} \frac{\partial}{\partial t}\theta(x,t) & = & \frac{\partial^2}{\partial x^2}\theta(x,t) + u(t)\,\delta_0(x), \quad x \in \mathbb{R}, \qquad \theta(\cdot,0) = \theta_0(\cdot), \\ y(t) & = & 2\,\theta(0,t), \end{array} \tag{5}$$

with input u and output y. We assume that θ_0 is a real and *even* function.

Taking the Fourier transform with respect to the space variable x gives the following parameterized differential equation

$$\begin{array}{rcl} \frac{\partial}{\partial t}\hat{\theta}(f,t) & = & -4\pi^2 f^2\,\hat{\theta}(f,t) + \mathbb{1}(f)\,u(t), \quad f \in \mathbb{R}, \qquad \hat{\theta}(\cdot,0) = \hat{\theta}_0(\cdot), \\ y(t) & = & 2\int_{-\infty}^{+\infty}\hat{\theta}(f,t)\,\mathrm{d}f, \end{array} \tag{6}$$

Since θ_0 is real and even, we have that $\hat{\theta}_0$ is an *even* function of f. Since $\mathbb{1} : f \mapsto 1$ is also even, it is clear from the first equation in (6) that $\hat{\theta}(f,t)$ is an even function of f. Hence one is allowed to consider system (6) for $f \geq 0$ only and perform the change of variables $\xi = 4\pi^2 f^2$. Thus with $\phi(\xi,t) := \hat{\theta}(\frac{1}{2\pi}\sqrt{\xi},t)$, we find the equivalent system:

$$\begin{array}{rcl} \frac{\partial}{\partial t}\phi(\xi,t) & = & -\xi\,\phi(\xi,t) + \mathbb{1}^+(\xi)\,u(t), \quad \xi \geq 0, \qquad \phi(\cdot,0) = \phi_0(\cdot), \\ y(t) & = & \int_0^\infty \frac{1}{\pi\sqrt{\xi}}\,\phi(\xi,t)\,\mathrm{d}\xi, \end{array} \tag{7}$$

where the following measure μ_1 appears, which is absolutely continuous with respect to the Lebesgue measure:

$$\mathrm{d}\mu_1(\xi) = \frac{1}{\pi\sqrt{\xi}}\,\mathrm{d}\xi.$$

Note that condition (3) is fulfilled for μ_1, since $\int_0^\infty \frac{1}{\sqrt{\xi}\,(1+\xi)}\,\mathrm{d}\xi < +\infty$.

Example 3.3. Consider now the following partial differential equation on the bounded interval $I = [0,1]$:

$$\begin{array}{rcl} \frac{\partial}{\partial t}\theta(x,t) & = & \frac{\partial^2}{\partial x^2}\theta(x,t) \quad 0 \leq x \leq 1, \qquad \theta(\cdot,0) = \theta_0(\cdot) \\ \frac{\partial}{\partial x}\theta(0,t) & = & 0, \quad \text{and} \quad \frac{\partial}{\partial x}\theta(1,t) = u(t), \\ y(t) & = & \theta(1,t), \end{array} \tag{8}$$

with input u and output y, as in [4, Example 4.3.12].

We want to rewrite this partial differential equation as a diffusive system. Therefore we introduce the following set of functions $v_k(x) = (-1)^k\,\cos(k\pi x)$

for $k \geq 0$ and we define $\psi_k(t) = \int_0^1 \theta(x,t) v_k(x)\,\mathrm{d}x, t \geq 0$. These functions satisfy

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\psi_k(t) &= \int_0^1 \frac{\partial}{\partial t}\theta(x,t)v_k(x)\,\mathrm{d}x \\
&= \int_0^1 \frac{\partial^2}{\partial x^2}\theta(x,t)v_k(x)\,\mathrm{d}x \\
&= \left[\frac{\partial}{\partial x}\theta(x,t)v_k(x)\right]_0^1 - \int_0^1 \frac{\partial}{\partial x}\theta(x,t)v_k'(x)\,\mathrm{d}x \\
&= 1\,u(t) - \left[\theta(x,t)v_k'(x)\right]_0^1 + \int_0^1 \theta(x,t)v_k''(x)\,\mathrm{d}x \\
&= -k^2\pi^2\,\psi_k(t) + u(t),
\end{aligned}$$

where we have used that $v_k''(x) = -k^2\pi^2 v_k(x)$. Since the $\{v_k\}_{k\geq 0}$ form a complete orthogonal family of the functions in $L^2(0,1)$, [4, Example A.4.22], we get the following inversion formula:

$$\theta(x,t) = \psi_0(t)v_0(x) + 2\sum_{k=1}^{\infty} \psi_k(t)v_k(x).$$

With this inversion formula we can write the output of our system as $y(t) = \psi_0(t) + 2\sum_{k=1}^{\infty}\psi_k(t)$. Hence, the new system now reads:

$$\begin{aligned}
\tfrac{d}{dt}\psi_k(t) &= -k^2\pi^2\,\psi_k(t) + 1\,u(t), \quad k \geq 0, \qquad \psi_k(0) = \psi_k^0, \\
y(t) &= \psi_0(t) + 2\textstyle\sum_{k=1}^{\infty}\psi_k(t),
\end{aligned} \tag{9}$$

Defining

$$\phi(\xi,t) = \begin{cases} \psi_k(t) & \xi = k^2\pi^2, \quad k \in \{0,1,2\ldots\} \\ 0 & \xi \neq k^2\pi^2 \end{cases}$$

and

$$\phi_0(\xi) = \begin{cases} \psi_k(0) & \xi = k^2\pi^2, \quad k \in \{0,1,2\ldots\} \\ 0 & \xi \neq k^2\pi^2 \end{cases}$$

system (9) is now of the form (4) with the discretely supported measure μ_2:

$$\mu_2(\xi) = \delta_0(\xi) + \sum_{k=1}^{\infty} 2\,\delta_{k^2\pi^2}(\xi),$$

for which condition (3) is fulfilled, since $1 + 2\sum_{k=1}^{\infty}(1+k^2\pi^2)^{-1} < \infty$.

Example 3.4. Finally, consider the same partial differential equation as in Example 3.2, but with different boundary conditions:

$$\begin{aligned} \frac{\partial}{\partial t}\theta(x,t) &= \frac{\partial^2}{\partial x^2}\theta(x,t) \quad 0 \le x \le 1, \qquad \theta(\cdot,0)=0, \\ \theta(0,t) &= 0, \quad \text{and} \quad \frac{\partial}{\partial x}\theta(1,t)=u(t), \\ y(t) &= \theta(1,t), \end{aligned} \tag{10}$$

with input u and output y.

In order to rewrite this partial differential equation as a diffusive system, we introduce the set of functions $w_k(x) = (-1)^k \sin((k+\frac{1}{2})\pi x)$ for $k \ge 0$ and we define $\psi_k(t) = \int_0^1 \theta(x,t) w_k(x)\, \mathrm{d}x$, $t \ge 0$. Similar as in the previous example we can show that these functions satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_k(t) = -(k+\frac{1}{2})^2\pi^2\, \psi_k(t) + u(t),$$

where we have used that $w_k''(x) = -(k+\frac{1}{2})^2\pi^2 w_k(x)$. Since the $\{w_k\}_{k\ge 0}$ are the eigenfunctions of the operator $\frac{\partial^2}{\partial x^2}$ with boundary conditions $\theta(0)=0$ and $\frac{\partial}{\partial x}\theta(1)=0$, and since this operator is self-adjoint and has compact resolvent, we know that they form a complete orthogonal family in $L^2(0,1)$, see e.g. [4, Theorem A.4.20]. Hence we have the following inversion formula:

$$\theta(x,t) = 2\sum_{k=0}^{\infty} \psi_k(t) w_k(x).$$

With this inversion formula we can write the output of our system as $y(t) = 2\sum_{k=0}^{\infty}\psi_k(t)$. Hence the new system now reads:

$$\begin{aligned} \frac{d}{dt}\psi_k(t) &= -(k+\tfrac{1}{2})^2\pi^2\, \psi_k(t) + 1\, u(t), \quad k \ge 0, \qquad \psi_k(0)=\psi_k^0, \\ y(t) &= 2\textstyle\sum_{k=0}^{\infty}\psi_k(t), \end{aligned} \tag{11}$$

Defining

$$\phi(\xi,t) = \begin{cases} \psi_k(t) & \xi = (k+\frac{1}{2})^2\pi^2, \quad k \in \{0,1,2\ldots\} \\ 0 & \xi \ne (k+\frac{1}{2})^2\pi^2 \end{cases}$$

and

$$\phi_0(\xi) = \begin{cases} \psi_k(0) & \xi = (k+\frac{1}{2})^2\pi^2, \quad k \in \{0,1,2\ldots\} \\ 0 & \xi \ne (k+\frac{1}{2})^2\pi^2 \end{cases}$$

system (11) is now of the form (4) with the discretely supported measure μ_3:

$$\mu_3(\xi) = \sum_{k=0}^{\infty} 2\,\delta_{(k+\frac{1}{2})^2\pi^2}(\xi),$$

for which condition (3) is fulfilled, since $2\sum_{k=0}^{\infty}(1+(k+\frac{1}{2})^2\pi^2)^{-1} < \infty$.

We remark that taking the new state function $\phi(\xi,t)$ zero outside the support of the measure, as we did in Examples 3.3 and 3.4, is not essential. In the next section we take as our state space $L^2([0,\infty),\mathrm{d}\mu)$. In this set any two functions which are equal on the support of the measure $\mathrm{d}\mu$ are equal. Hence the values of ϕ outside the support of the measure play no role.

4 Diffusive linear systems are regular linear systems

In order to show that a diffusive system as defined in Definition 3.1 is a regular system, we first have to identify the state space, Z. As state space, Z, we take $L^2([0,\infty),\mathrm{d}\mu)$. On this space we define A to be the multiplication operator by $-\xi$. Thus

$$(A\phi)(\xi) = -\xi\phi(\xi), \qquad \xi \in [0,\infty), \tag{12}$$

with domain $D(A) = \{\phi \in L^2([0,\infty),\mathrm{d}\mu) \mid \xi\phi(\xi) \in L^2([0,\infty),\mathrm{d}\mu)\}$. From Proposition I.4.10 of Engel and Nagel [5] we obtain the following properties of A.

Lemma 4.1. *For the operator A as defined in equation (12) we have*

a. The spectrum of A, $\sigma(A)$, is given by

$$\sigma(A) = \{s \in \mathbb{R} \mid \mu([-s-\varepsilon, -s+\varepsilon]) > 0 \text{ for all } \varepsilon > 0\}.$$

b. The point spectrum of A, $\sigma_p(A)$, is given by

$$\sigma_p(A) = \{s \in \mathbb{R} \mid \mu(\{s\}) > 0\}.$$

c. For s in the resolvent set of A, we have that

$$\left((sI-A)^{-1}\phi\right)(\xi) = \frac{1}{s+\xi}\phi(\xi). \tag{13}$$

Furthermore, A generates a C_0-semigroup, for which the stability properties are easy to characterize.

Lemma 4.2. *The operator A as defined by (12) is the infinitesimal generator of a C_0-semigroup on $Z = L^2([0,\infty), \mathrm{d}\mu)$, which is given by*

$$(T(t)\phi)(\xi) = e^{-\xi t}\phi(\xi), \qquad \xi \in [0,\infty). \tag{14}$$

This semigroup is self-adjoint, analytic and contractive.
Furthermore, the stability of the semigroup can be characterized as follows:

a. It is exponentially stable if and only if $\mu([0,\varepsilon)) = 0$ for some $\varepsilon > 0$.

b. It is strongly stable, i.e., for all $z_0 \in Z$, $T(t)z_0 \to 0$, if and only if $\mu(\{0\}) = 0$.

Proof. From Engel and Nagel [5], Proposition I.4.11 and Theorem II.4.32 we conclude that A generates an analytic C_0-semigroup given by (14). The fact that this semigroup is self-adjoint follows from the realness of ξ and t in the expression (14). Thus it remains to show that the semigroup is a contraction. This is easy since for $t \geq 0$ we have that

$$\int_0^\infty |e^{-\xi t}\phi(\xi)|^2 \mathrm{d}\mu(\xi) \leq \int_0^\infty |\phi(\xi)|^2 \mathrm{d}\mu(\xi). \tag{15}$$

Now for stability, item a. follows from the fact that $\|T(t)\| = \exp(-\varepsilon t)$. Item b. follows from Example V.2.19.iii of Engel and Nagel [5]. ■

Now for our examples, the above lemmas lead to the following results:

- The spectrum of A in Example 3.2 equals $(-\infty, 0]$. Hence the semigroup is not exponentially stable. However, since $\mu_1(\{0\}) = 0$ this semigroup is strongly stable.

- The spectrum of A in Example 3.3 equals $\{-k^2\pi^2, k = 0, 1, 2, \ldots\}$. Since $\mu_2((\{0\}) = 1$, we see that the semigroup is not strongly stable.

- The spectrum of A in Example 3.4 equals $\{-(k+\frac{1}{2})^2\pi^2, k = 0, 1, 2, \ldots\}$. Since $\mu_3([0, \frac{\pi^2}{4})) = 0$, we see that the semigroup is exponentially stable.

Next we prove the main theorem of this section, namely that every standard diffusive system is a regular well-posed linear system. As is clear from Definition 2.1,we have to introduce Z_{-1}, as the dual of $D(A^*)$. Since A is self-adjoint, this domain equals the domain of A. It is easy to see that $D(A)$

equals the Hilbert space $L^2\left([0,\infty),(1+\xi^2)\,\mathrm{d}\mu(\xi)\right)$. Hence the space Z_{-1} is given by

$$Z_{-1} = L^2\left([0,\infty), \frac{1}{1+\xi^2}\,\mathrm{d}\mu(\xi)\right)$$

with duality product

$$\langle f, z\rangle_{Z_{-1},D(A)} = \int_0^\infty f(\xi)\overline{z(\xi)}\,\mathrm{d}\mu(\xi).$$

Looking at equation (1) and (4), it is easy to see that we have to choose

$$Bu = \mathbb{1}^+ \cdot u, \quad \text{and} \quad C\phi_0 = \int_0^\infty \phi_0(\xi)\,\mathrm{d}\mu(\xi). \tag{16}$$

These operators lie in the spaces as given in Definition 2.1 as is shown next.

Lemma 4.3. *The operator B and C as defined in (16) have the following properties:*

a. $B \in \mathcal{L}(\mathbb{C}, Z_{-1})$,

b. $C \in \mathcal{L}(D(A), \mathbb{C})$,

c. $B = C^*$.

Proof. For $u \in \mathbb{C}$ we have that $Bu = \mathbb{1}^+ \cdot u = u$. Furthermore,

$$\|Bu\|^2_{Z_{-1}} = \int_0^\infty |u|^2 \frac{1}{1+\xi^2}\,\mathrm{d}\mu(\xi) \le \int_0^\infty \frac{2}{1+\xi}\,\mathrm{d}\mu(\xi)|u|^2 = 2c_\mu |u|^2.$$

Hence $B \in \mathcal{L}(\mathbb{C}, Z_{-1})$.

For $\phi_0 \in D(A)$, we apply the Cauchy-Schwarz inequality to find that

$$\begin{aligned} |C\phi_0|^2 &\le \left[\int_0^\infty \sqrt{1+\xi^2}|\phi_0(\xi)|\,\frac{\mathrm{d}\mu(\xi)}{\sqrt{1+\xi^2}}\right]^2 \\ &\le \int_0^\infty \frac{\mathrm{d}\mu(\xi)}{1+\xi^2}\int_0^\infty (1+\xi^2)|\phi_0(\xi)|^2\,\mathrm{d}\mu(\xi) \\ &\le 2c_\mu\,\|\phi_0\|^2_{D(A)}. \end{aligned}$$

Thus $C \in \mathcal{L}(D(A), \mathbb{C})$.

Let ϕ_0 be an element of $D(A)$ and $u \in \mathbb{C}$. Then we have that

$$\begin{aligned}\langle Bu, \phi_0 \rangle_{Z_{-1},D(A)} &= \int_0^\infty \mathbb{1}^+(\xi) u \overline{\phi_0(\xi)} \, \mathrm{d}\mu(\xi) \\ &= u \cdot \overline{\int_0^\infty \phi_0(\xi) \, \mathrm{d}\mu(\xi)} \\ &= u \cdot \overline{(C\phi_0)} = \langle u, C\phi_0 \rangle_{\mathbb{C}}.\end{aligned}$$

Thus $B^* = C$. ■

Now that we know what are the B and C operators for our system, we can prove that this system is well-posed.

Theorem 4.4. *Consider the diffusive system (4) with the measure μ satisfying (3). This system is a well-posed linear system on the state space $L^2([0,\infty), \mathrm{d}\mu)$ and input and output space $\mathbb{C}$. Furthermore, its transfer function is given by*

$$H(s) = \int_0^\infty \frac{1}{s+\xi} \, \mathrm{d}\mu(\xi), \qquad s \in \mathbb{C}_0^+ \tag{17}$$

This function has the following properties

a. $H(s)$ is positive real, i.e., $Re(H(s)) > 0$ for $Re(s) > 0$;

b. It satisfies

$$\lim_{\mathrm{Re}(s)\to\infty} H(s) = 0.$$

In particular, this transfer function is regular.

Proof. From Definition 2.1 we see that we have to prove four properties. The first property has already been shown in Lemma 4.2. So it remains to show the other three items.

Step 1. Choose a $t_1 > 0$ and an input $u \in L^2(0, t_1)$, then we have that

$$
\begin{aligned}
&\| \int_0^{t_1} T(t_1 - \tau) B u(\tau) \, \mathrm{d}\tau \|_Z^2 \\
&\quad = \int_0^\infty \left| \int_0^{t_1} e^{-\xi(t_1 - \tau)} \, \mathbb{1}^+(\xi) u(\tau) \, \mathrm{d}\tau \right|^2 \mathrm{d}\mu(\xi) \\
&\quad \leq \int_0^\infty \left[\int_0^{t_1} e^{-2\xi(t_1 - \tau)} \, \mathrm{d}\tau \int_0^{t_1} |u(\tau)|^2 \mathrm{d}\tau \right] \mathrm{d}\mu(\xi) \\
&\quad = \int_0^\infty \frac{1 - e^{-2\xi t_1}}{2\xi} \, \mathrm{d}\mu(\xi) \|u\|_{L^2(0,t_1)}^2 \\
&\quad = \left[\int_0^1 \frac{1 - e^{-2\xi t_1}}{2\xi} \, \mathrm{d}\mu(\xi) + \int_1^\infty \frac{1 - e^{-2\xi t_1}}{2\xi} \, \mathrm{d}\mu(\xi) \right] \|u\|_{L^2(0,t_1)}^2 \\
&\quad \leq \left[\int_0^1 t_1 \, \mathrm{d}\mu(\xi) + \int_1^\infty \frac{1}{1 + \xi} \, \mathrm{d}\mu(\xi) \right] \|u\|_{L^2(0,t_1)}^2 \\
&\quad \leq [2t_1 + 1] \, c_\mu \|u\|_{L^2(0,t_1)}^2,
\end{aligned}
$$

where we have used equation (3). Thus item b. of Definition 2.1 holds.
Step 2. The output operator is given by

$$
C\phi_0 = \int_0^\infty \phi_0(\xi) \, \mathrm{d}\mu(\xi).
$$

Now choose a $t_1 > 0$ and an initial state $\phi_0 \in Z$, then we have that

$$
\begin{aligned}
\int_0^{t_1} |CT(\tau)\phi_0|^2 \, \mathrm{d}\tau &= \int_0^{t_1} \left| \int_0^\infty e^{-\xi\tau} \phi_0(\xi) \, \mathrm{d}\mu(\xi) \right|^2 \mathrm{d}\tau \\
&\leq \int_0^{t_1} \left[\int_0^\infty e^{-2\xi\tau} \, \mathrm{d}\mu(\xi) \int_0^\infty |\phi_0(\xi)|^2 \, \mathrm{d}\mu(\xi) \right] \mathrm{d}\tau \\
&= \int_0^\infty \left[\int_0^{t_1} e^{-2\xi\tau} \, \mathrm{d}\tau \right] \mathrm{d}\mu(\xi) \, \|\phi_0\|_Z^2 \\
&= \int_0^\infty \frac{1 - e^{-2\xi t_1}}{2\xi} \, \mathrm{d}\mu(\xi) \, \|\phi_0\|_Z^2 \\
&\leq [2t_1 + 1] \, c_\mu \, \|\phi_0\|_Z^2.
\end{aligned}
$$

Thus item c. of Definition 2.1 holds.
Step 3. Let us compute the transfer function of our system. Take $u(t) = u_0 e^{st}$ for $t \geq 0$, $u_0 \in \mathbb{C}$, and some $s \in \mathbb{C}_0^+$. Next we want to find a state of the form $\phi(\xi, t) = z_0(\xi, s) \, e^{st}$ which satisfies the differential equation in (4)

for this input. An easy calculation gives $z_0(\xi, s) = \frac{u_0}{s+\xi}$. Finally, replacing the state in the observation equation of (4) by this expression, gives $y(t) = \int_0^\infty \frac{1}{s+\xi}\, \mathrm{d}\mu(\xi)\, u_0\, e^{st}$. Hence we have that the transfer function is given by $H(s) = \int_0^\infty \frac{1}{s+\xi}\, \mathrm{d}\mu(\xi)$.

Using condition (3) once more, we see that H is uniformly bounded in $\mathbb{C}_\rho^+$ for any $\rho > 0$. Note that it is a solution to (2), since by equation (13) we have that

$$C(sI - A)^{-1}(\lambda I - A)^{-1}B = \int_0^\infty \frac{1}{(s+\xi)(\lambda+\xi)}\, \mathrm{d}\mu(\xi) = \frac{H(s) - H(\lambda)}{\lambda - s}.$$

Hence we have shown that the diffusive system is a well-posed linear system. It remains to show that it is regular.

Step 4. Using the expression (17) we see that

$$\mathrm{Re}(H(s)) = \int_0^\infty \mathrm{Re}\left(\frac{1}{s+\xi}\right) \mathrm{d}\mu(\xi) = \int_0^\infty \frac{\mathrm{Re}(s) + \xi}{|s+\xi|^2}\, \mathrm{d}\mu(\xi).$$

From this expression it is clear that $\mathrm{Re}(H(s)) > 0$ if $\mathrm{Re}(s) > 0$.

We have that

$$\left|\frac{1}{s+\xi}\right| \leq \frac{1}{\mathrm{Re}(s) + \xi}$$

Hence $\frac{1}{s+\xi}$ converges pointwise (in ξ) to zero for $\mathrm{Re}(s) \to \infty$. Using condition (3), and applying Lebesgue dominated convergence theorem, we get

$$\lim_{\mathrm{Re}(s)\to\infty} H(s) = \lim_{\mathrm{Re}(s)\to\infty} \int_0^\infty \frac{1}{s+\xi}\, \mathrm{d}\mu(\xi) = 0.$$

Thus the system is regular, with feed-through operator $D = 0$. ■

Remark 4.5. Since $B^* = C$ and since $A = A^*$, we know that the admissibility of B is equivalent to the admissibility of C. However, we have chosen to present them both, since the proofs are very simple.

Thanks to formula (17), we can compute the transfer functions of the three examples, defined in $\mathbb{C}_0^+$ at least:

- the transfer function of the system in example 3.2 is:

$$H_1(s) = \int_0^\infty \frac{1}{\pi\sqrt{\xi}(s+\xi)}\, d\xi = \frac{1}{\sqrt{s}},$$

- the transfer function of the system in example 3.3 is:

$$H_2(s) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{2}{s + n^2\pi^2} = \frac{\cosh(\sqrt{s})}{\sqrt{s}\,\sinh(\sqrt{s})},$$

- the transfer function of the system in example 3.4 is:

$$H_3(s) = \sum_{n=0}^{\infty} \frac{2}{s + (n+\frac{1}{2})^2\pi^2} = \frac{\sinh(\sqrt{s})}{\sqrt{s}\,\cosh(\sqrt{s})}.$$

A proof of the second and third closed-form expressions can be found in [2, chap. 5].

When a system is well-posed, then it is not hard to see that for input zero, $\int_0^t \|y(\tau)\|^2 \mathrm{d}\tau$ converges to zero as $t \downarrow 0$ for every initial condition. Based on the proof of Theorem 4.4, we show that for diffusive systems this convergence is uniform.

Lemma 4.6. *Let $\mathcal{C}^{t_1}$ be defined as*

$$\left(\mathcal{C}^{t_1}\phi_0\right)(t) = \int_0^\infty e^{-\xi t}\phi_0(\xi)\,\mathrm{d}\mu(\xi), \qquad t \in [0, t_1). \tag{18}$$

This is a bounded linear mapping from Z to $L^2(0, t_1)$, and

$$\lim_{t_1 \downarrow 0} \|\mathcal{C}^{t_1}\| = 0.$$

By $\mathcal{B}^{t_1}$ we denote the mapping from $L^2(0, t_1)$ to Z given by

$$\mathcal{B}^{t_1}u = \int_0^{t_1} e^{-\xi(t_1-\tau)}u(\tau)\,\mathrm{d}\tau. \tag{19}$$

This is a bounded linear mapping from $L^2(0, t_1)$ to Z, and

$$\lim_{t_1 \downarrow 0} \|\mathcal{B}^{t_1}\| = 0.$$

Proof. The boundedness and linearity follows from Theorem 4.4. From step 2 in this proof, we have that

$$\|\mathcal{C}^{t_1}\|^2 \le \int_0^\infty \frac{1 - e^{-2\xi t_1}}{2\xi}\,\mathrm{d}\mu(\xi).$$

Since $\lim_{t_1 \downarrow 0} \frac{1-e^{-2\xi t_1}}{2\xi} = 0$ on $[0, \infty)$ and since

$$\frac{1-e^{-2\xi t_1}}{2\xi} \leq \begin{cases} 1 & t_1 \leq 1,\ \xi \in (0,1) \\ \frac{1}{1+\xi} & \xi > 1 \end{cases}$$

we obtain by condition (3) and the Lebesgue dominated convergence theorem that $\lim_{t_1 \downarrow 0} \|\mathcal{C}^{t_1}\|^2 = 0$.

The proof for $\mathcal{B}$ is similar, since B and C are dual operators. ∎

In [19] one may find the concept of a system node. This is a more general concept than well-posedness. Just for completeness, we state the system node for our diffusive system (4).

Lemma 4.7. *The system node for the system (4) is given by*

$$S \begin{pmatrix} \phi_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \xi\phi_0(\xi) + \mathbb{1}^+(\xi)u_0 \\ \int_0^\infty \phi_0(\xi)\,\mathrm{d}\mu(\xi) \end{pmatrix} \tag{20}$$

with domain

$$D(S) = \{(\phi_0, u_0) \in Z \times \mathbb{C} \mid \xi\phi_0(\xi) + \mathbb{1}^+(\xi)u_0 \in Z\}. \tag{21}$$

We end this session by proving an energy balance for the diffusive system (4).

Lemma 4.8. *The system (4) satisfies for all $u \in L^2(0,t)$ and $\phi_0 \in Z$ the following energy balance*

$$\frac{1}{2}\|\phi(t)\|_Z^2 + \int_0^t \int_0^\infty |\phi(\xi,\tau)|^2\, \xi\, \mathrm{d}\mu(\xi)\, \mathrm{d}\tau = \frac{1}{2}\|\phi_0\|_Z^2 + \mathrm{Re}\Big(\int_0^t y(\tau)\overline{u(\tau)}\,\mathrm{d}\tau\Big). \tag{22}$$

In particular, the system is impedance passive.

Proof. For $\phi(t,\xi)$ being a classical solution of (4), we differentiate the left-hand side of (22) with respect to t. We find

$$\frac{1}{2}\left[\int_0^\infty \phi(t,\xi)\overline{\frac{\partial\phi}{\partial t}(t,\xi)}\,\mathrm{d}\mu(\xi) + \int_0^\infty \frac{\partial\phi}{\partial t}(t,\xi)\overline{\phi(t,\xi)}\,\mathrm{d}\mu(\xi)\right] + \int_0^\infty |\phi(\xi,t)|^2 \xi\,\mathrm{d}\mu(\xi).$$

Using the equation (4), we see that this equals

$$\frac{1}{2}\left[\int_0^\infty \phi(t,\xi)\overline{u(t)}\,\mathrm{d}\mu(\xi) + \int_0^\infty u(t)\overline{\phi(t,\xi)}\,\mathrm{d}\mu(\xi)\right].$$

By the definition of the output, this equals

$$\frac{1}{2}\left[y(t)\overline{u(t)} + u(t)\overline{y(t)}\right].$$

So we see that if we differentiate the right-hand side of (22), we obtain the derivative of its left-hand side. Furthermore, (22) trivially holds for $t = 0$. Thus we conclude that the energy balance holds for classical solutions. Now the set of initial conditions and inputs giving a classical solution is dense in the set of all initial conditions, and all L^2-inputs, and so we have that (22) holds for all ϕ_0 and $u \in L^2(0,t)$. ■

This energy balance gives another way of proving the well-posedness. Namely one shows that the diffusive system (4) is a system node, and that its transfer function is bounded in some right-half plane. Theorem 5.1 of [20] now gives that an impedance passive system satisfying these two conditions is well-posed.

5 System theoretic properties of diffusive systems

We give some properties of the diffusive systems. Before we do so, we recapulate some definitions.

Definition 5.1. Consider the system (1), and assume that it is well-posed. Then

a. Assume that the input $u \equiv 0$. It is approximately observable on $[0,t_1]$, if $y = 0$ in $L^2((0,t_1),Y)$ implies that $z_0 = 0$.

b. If is approximately controllable on $[0,t_1]$, if the set of reachable states in time t_1 is dense in Z. A state z_1 is said to be reachable in time t_1 when $z_1 = \int_0^{t_1} T(t_1-\tau)Bu(\tau)\,\mathrm{d}\tau$ for some $u \in L^2((0,t_1),U)$.

c. Assume that the input $u \equiv 0$. It is exactly observable on $[0,t_1]$, if there exists a $c > 0$ such $\int_0^{t_1} \|y(\tau)\|^2 \mathrm{d}\tau \geq c\|z_0\|^2$ for all $z_0 \in Z$.

d. If is exactly controllable on $[0,t_1]$, if the set of reachable states in time t_1 equals Z.

e. The system is output stable, when for $u \equiv 0$ we have that $y \in L^2((0,\infty),Y)$ for all $z_0 \in Z$.

f. The system is input stable, when $\int_0^{\infty} T(\tau)Bu(\tau)\,\mathrm{d}\tau \in Z$ for all $u \in L^2((0,\infty),U)$.

Lemma 5.2. *For the system given by (4), one has the following properties:*

1. *For all* $t_1 > 0$ *we have that the system is approximately controllable on* $[0, t_1]$.

2. *For all* $t_1 > 0$ *we have that the system is approximately observable on* $[0, t_1]$.

Proof. Since B and C are dual operators, and since controllability and observability are dual notions, we only have to show one of the two properties. We choose to prove the second one.

Let $t_1 > 0$. Assume that for $\phi_0 \in Z$, we have that $y = 0$ on $L^2(0, t_1)$. This is equivalent to saying that

$$\int_0^\infty e^{-\xi t} \phi_0(\xi) \, \mathrm{d}\mu(\xi) = 0 \qquad \text{a.e. on } [0, t_1]. \tag{23}$$

In the above expression, we recognize the Laplace transform w.r.t. the measure μ of the function ϕ_0. Since the Laplace transform gives an analytic function, we conclude from (23) that the Laplace transform must be zero for all t. Now by the injectivity of the Laplace transform w.r.t. a measure, see Theorem 6.2.3 in [10], we conclude that $\phi_0 = 0$. ■

So we see that our systems are always approximately observable and controllable. It turns out that they are only exactly observable/controllable in finite time when the measure μ consists out of finitely many point measure, i.e., when Z is finite-dimensional.

Theorem 5.3. *Consider the diffusive system (4) with measure* μ *satisfying (3). The following conditions are equivalent:*

1. *The system is exactly observable on* $[0, t_1]$.

2. *The system is exactly controllable on* $[0, t_1]$.

3. *The dimension of* Z *is finite.*

Proof. The proof consists out of five steps. In the first step we construct a solution to a Lyapunov equation. Using this equation, we show in the second step that when the support of a state ϕ_0 is small, then it cannot lie in the kernel of C. Using this we conclude that the spectrum of A is pure point spectrum without finite accumulation point, step 3, and cannot converge to infinity either, step 4. In the final step we conclude that the dimension of Z must be finite when the system is exactly observable or controllable.

Step 1 By the well-posedness of the system, we have that the operator Q defined as

$$\langle Q\phi_1, \phi_2\rangle = \int_0^\infty \left[\int_0^\infty e^{-(\xi+1)t}\phi_1(\xi)\,\mathrm{d}\mu(\xi) \overline{\int_0^\infty e^{-(\xi+1)t}\phi_2(\xi)\,\mathrm{d}\mu(\xi)} \right] \mathrm{d}t$$

is a bounded and self-adjoint operator from Z to Z. If the system is exactly observable on $[0, t_1]$, then this operator is coercive, i.e., $Q \geq qI$ for some $q > 0$. Furthermore, this operator satisfies the Lyapunov equation, see e.g. [7]

$$\langle (A-I)\phi_1, Q\phi_2\rangle + \langle Q\phi_1, (A-I)\phi_2\rangle = -\langle C\phi_1, C\phi_2\rangle, \quad \phi_1, \phi_2 \in D(A). \tag{24}$$

Step 2 In this part we show the following: If $\theta_2 > \theta_1 \geq 0$ are such that

$$\frac{\theta_2 - \theta_1}{\theta_1 + 1}\|Q\| < q, \tag{25}$$

and if ϕ_0 is such that $C\phi_0 = 0$ and $\int_{\theta_1}^{\theta_2} |\phi_0(\xi)|^2\,\mathrm{d}\mu(\xi) = \|\phi_0\|^2$, then $\phi_0 = 0$. Note that since ϕ_0 has compact support, it lies in the domain of A, and $C\phi_0$ is well-defined.

We take in (24) $\phi_1 = \phi_2 = \phi_0$, and we find that

$$\int_{\theta_1}^{\theta_2} (-\xi - 1)\phi_0(\xi)\overline{Q(\phi_0)(\xi)}\,\mathrm{d}\mu(\xi) + \int_{\theta_1}^{\theta_2} (-\xi - 1)Q(\phi_0)(\xi)\overline{\phi_0(\xi)}\,\mathrm{d}\mu(\xi) = 0. \tag{26}$$

This is equivalent to

$$\begin{aligned} 2(\theta_1 + 1)\int_{\theta_1}^{\theta_2} \phi_0(\xi)\overline{Q(\phi_0)(\xi)}\,\mathrm{d}\mu(\xi) = &\int_{\theta_1}^{\theta_2} (\theta_1 - \xi)\phi_0(\xi)\overline{Q(\phi_0)(\xi)}\,\mathrm{d}\mu(\xi) + \\ &\int_{\theta_1}^{\theta_2} (\theta_1 - \xi)Q(\phi_0)(\xi)\overline{\phi_0(\xi)}\,\mathrm{d}\mu(\xi). \end{aligned} \tag{27}$$

By the Cauchy-Schwarz inequality, the right-hand side can be majorized by $2|\theta_2 - \theta_1|\|Q\|\|\phi_0\|^2$. Thus we see that

$$q\|\phi_0\|^2 \leq \int_{\theta_1}^{\theta_2} \phi_0(\xi)\overline{Q(\phi_0)(\xi)}\mathrm{d}\mu(\xi) \leq \frac{\theta_2 - \theta_1}{\theta_1 + 1}\|Q\|\|\phi_0\|^2. \tag{28}$$

Now (25) implies that $\phi_0 = 0$.

Step 3 From the above property we conclude that the operator A has only point spectrum, and this point spectrum has no (finite) accumulation point. Since if we assume the contrary, then in an interval $[\theta_1, \theta_2]$ with θ_1, θ_2 satisfying (25) it would be possible to construct two disjunct subintervals V_1 and V_2 such that $\mu(V_i) \neq 0$, $i = 1, 2$. The function $\phi_0(\xi) = \mu(V_2)\, \mathbb{1}_{V_1}(\xi) - \mu(V_1)\, \mathbb{1}_{V_2}(\xi)$ has the property that $C\phi_0 = 0$, and $\int_{\theta_1}^{\theta_2} |\phi_0(\xi)|^2 \, \mathrm{d}\mu(\xi) = \|\phi_0\|^2$. Since $\phi_0 \neq 0$, this gives a contradiction.

Step 4 Let λ_n be an eigenvalue of A. It is easy to see that $\phi_n = \mathbb{1}_{\{-\lambda_n\}}$ is the eigenvector corresponding to this eigenvalue. We choose in (24) ϕ_1 and ϕ_2 to be this eigenvector. By doing so, we find

$$\begin{aligned} -\mu(\{-\lambda_n\})^2 &= -\|C\phi_{\lambda_n}\|^2 \\ &= \langle (A-I)\phi_n, Q\phi_n\rangle + \langle Q\phi_n, (A-I)\phi_n\rangle \\ &= (\lambda_n - 1)\langle \phi_n, Q\phi_n\rangle + (\lambda_n - 1)\langle Q\phi_n, \phi_n\rangle. \end{aligned}$$

Since Q is self-adjoint, we have that

$$\begin{aligned} \mu(\{-\lambda_n\})^2 &= 2(1-\lambda_n)\langle \phi_n, Q\phi_n\rangle \\ &\geq 2(1-\lambda_n) q \|\phi_n\|^2 = 2(1-\lambda_n) q \mu(\{-\lambda_n\}). \end{aligned}$$

So we obtain that for all n

$$\mu(\{-\lambda_n\}) \geq 2q(1-\lambda_n). \tag{29}$$

Let $\mathbb{I}$ be the index set of all eigenvalues. By equation (3) and (29), we have that

$$\infty > c_\mu = \int_0^\infty \frac{1}{1+\xi}\, \mathrm{d}\mu(\xi) = \sum_{n\in\mathbb{I}} \frac{1}{1-\lambda_n} \mu(\{-\lambda_n\}) \geq \sum_{n\in\mathbb{I}} 2q.$$

This clearly implies that $\mathbb{I}$ can only be a finite set.

Step 5 From the previous two steps, we conclude that the spectrum of A consists of pure point spectrum without any accumulation point, and it lies in a compact interval. By Lemma 4.1, we conclude that the measure μ is only a finite collection of delta-measures, and so $L^2([0,\infty), \mathrm{d}\mu)$ is similar to $\mathbb{C}^n$ for some n. In other words, if we assume that the system is exactly observable on $[0, t_1]$ for some $t_1 > 0$, then the dimension of the state space is finite. Since in a finite-dimensional state space, approximate and exact observability are equivalent, we see that we have proved the assertion. ■

Sometimes, one sees the concepts of exactly observable in infinite time, that is there exists a $c > 0$ such that $\int_0^\infty \|y(\tau)\|^2 \mathrm{d}\tau > c\|\phi_0\|^2$ for all $\phi_0 \in D(A)$. Note that the integral may be infinite. Concerning this concept, the following remarks can be made.

- Exact observability on $[0, t_1]$ always implies exact observability in infinite time.
- If the system is exponentially stable, then it is exactly observable on $[0, t_1]$ for some (finite) time t_1 if and only if it is exactly observable in infinite time.
- If the system is not exponentially stable, then the two concepts can be different, see also the last item in this list.
- For our class of diffusive systems it is not hard to show that exact observability in infinite-time implies that the measure is pure point spectrum. One simply restricts the state space to the invariant subspace of all states with support on $[\varepsilon, \infty)$, $\varepsilon > 0$. On this subspace the system is exponentially stable. Next one let ε go to zero. Hence similar as in the proof of Theorem 5.3 the measure has support in the compact interval $[0, \xi_0]$ for some $\xi_0 > 0$. The only accumulation point can be zero. Let $\lambda_n < 0$ be a sequence of eigenvalues converging to zero, and let $\phi_n = \mathbb{1}_{\{-\lambda_n\}}$ be the corresponding eigenvector. By the definition of exact observability on infinite-time, we find that $\frac{\mu(\{-\lambda_n\})^2}{-\lambda_n} \geq c\mu(\{-\lambda_n\})$. Thus $\mu(\{-\lambda_n\}) \geq -c\lambda_n$. By condition (3) we find that λ_n must be summable, see also step 4 of the above proof.
- If we additionally assume that the system is output (or input) stable, then we have the following necessary and sufficient condition for the system (4) to be exactly observable in infinite time, see Jacob and Zwart [11]. There exists constants $\gamma_1, \gamma_2 > 0$ such that $\mu(\{-\lambda_n\}) \geq -\gamma_1\lambda_n$ and $\frac{|\lambda_n - \lambda_m|}{-\lambda_n} \geq \gamma_2$ for all n, m with $n \neq m$. For instance, for the measure $\mu(\xi) = \sum_{n=0}^{\infty} 2^{-n}\delta(\xi - 2^{-n})$ the system is exactly observable in infinite time.

In the above remarks, we have used the concept of output stability. The following theorem gives a necessary and sufficient condition.

Theorem 5.4. *The system is output and input stable if and only if there exists a constant $\gamma > 0$ such that for all $h > 0$ we have that*

$$\mu(\{[0, h]\}) \leq \gamma h. \tag{30}$$

Proof. Since input and output stability are dual notions, and since $A^* = A$, $B^* = C$, we see that we only have to show input or output stability. For simplicity, we do input stability. Hence we have to show that $z_1 := \int_0^\infty T(t)Bu(t)\,\mathrm{d}t \in Z$ for all $u \in L^2(0,\infty)$. Using our system, we see that

$$z_1(\xi) = \int_0^\infty e^{-\xi t}\,\mathbb{1}^+(\xi)u(t)\,\mathrm{d}t = \hat{u}(\xi),$$

where $\hat{u}$ denotes the Laplace transform of u. Thus we have to show that there exists a constant, $\gamma_1 > 0$, such that for all $u \in L^2(0,\infty)$

$$\int_0^\infty |\hat{u}(\xi)|^2\,\mathrm{d}\mu(\xi) \le \gamma_1 \int_0^\infty |u(t)|^2\,\mathrm{d}t.$$

Since the L^2-norm of u is equal (up to a constant) to the H^2-norm of $\hat{u}$, we see that proving the above inequality is equivalent to proving

$$\int_0^\infty |\hat{u}(\xi)|^2\,\mathrm{d}\mu(\xi) \le \gamma_2 \int_{-\infty}^\infty |\hat{u}(i\omega)|^2\,\mathrm{d}\omega \tag{31}$$

for all $\hat{u} \in H^2$, where γ_2 may not depend on $\hat{u}$. By the Carleson theorem, [6, p. 31], we see that this holds if and only if $\mu_{\mathbb{C}}$ is a Carleson measure, where $\mu_{\mathbb{C}}$ is the positive measure on the right-half plane defined as

$$\mu_{\mathbb{C}}(V) = \mu\big(V \cap \{s \in \mathbb{C} \mid \mathrm{Im}(s) = 0\}\big), \tag{32}$$

with V a Borel set in the right-half plane. Now $\mu_{\mathbb{C}}$ is a Carleson measure if and only if there exists a constant γ_μ such that

$$\mu_{\mathbb{C}}(R) \le \gamma_\mu h \tag{33}$$

where $R = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \in (0,h),\ \mathrm{Im}(s) \in (\omega_0, \omega_0 + h)\}$. Combining this with (32), we see that $\mu_{\mathbb{C}}$ is a Carleson measure if and only if (30) holds. ∎

Since A is self-adjoint, and the input space is one-dimensional, this result can also be found in [9]. Since, we felt that for our system, a direct proof was more insightful, we decided to include it.

It is easy to see that if the system is exponentially stable, then it is input and output stable. Hence the system of Example 3.4 is input and output stable. Since the system of Example 3.3 has an eigenvalue at zero, we have that $\mu_2(\{0\}) > 0$. Hence we conclude from Theorem 5.4 that this system is not input nor output stable.

Example 5.5. Consider the system of Example 3.2. Using Theorem 5.4 for the measure μ_1 s.t. $\mathrm{d}\mu_1(\xi) = \frac{1}{\pi\sqrt{\xi}}\,\mathrm{d}\xi$, we see that

$$\mu_1([0,h]) = \int_0^h \frac{1}{\pi\sqrt{\xi}}\,\mathrm{d}\xi = \frac{2}{\pi}\sqrt{h}.$$

Since this latter integral cannot be estimated by a constant times h, for small h, we conclude that the system is not input nor output stable.

We conclude this section by showing that static output feedback always (strongly) stabilizes the system.

Lemma 5.6. *For the system (4) the feedback $u(t) = -ky(t)+v(t)$ with $k > 0$ gives a well-posed (closed-loop) system with a strongly stable, contraction semigroup.*

Proof. By Theorem 4.4 we know that the transfer function converges to zero for $\mathrm{Re}(s) \to \infty$. By [23] we have that the feedback $u = -ky + v$ gives a well-posed systems for all $k \in \mathbb{R}^+$. Using the energy balance (22), we see that by closing the loop,

$$\begin{aligned}\frac{1}{2}\|\phi(t)\|_Z^2 + \int_0^t \int_0^\infty |\phi(\xi,\tau)|^2 \xi\,\mathrm{d}\mu(\xi)\,\mathrm{d}\tau = \\ \frac{1}{2}\|\phi_0\|_Z^2 - k\int_0^t |y(\tau)|^2\,\mathrm{d}\tau + \mathrm{Re}\Big(\int_0^t y(\tau)\overline{v(\tau)}\,\mathrm{d}\tau\Big). \qquad (34)\end{aligned}$$

Hence for $v \equiv 0$, we have that $\|\phi(t)\| \leq \|\phi_0\|$, or equivalently, the closed loop semigroup is contractive. Furthermore, we conclude from (34) that the closed-loop system is output stable.

It is not hard to show that the closed-loop generator, A_{cl}, is self-adjoint, and so the intersection of its spectrum with the imaginary axis, consists of at most one point. Since the system is approximately observable, we have that there cannot be eigenvalues on the imaginary axis. Hence we have a generator of a bounded semigroup which has countably many spectral points on the imaginary axis and these points do not belong to the point spectrum. From [1], we conclude that this (closed-loop) semigroup is strongly stable. ■

Remark 5.7. Diffusive systems can be used as infinite-dimensional diagonal realizations of some irrational transfer functions, such as fractional ones. Then, the coupling of a conservative linear system with such a diffusive system can be easily analyzed on an augmented state space. But, since the diffusive part induces a lack of compactness, LaSalle's invariance principle cannot be applied to study the asymptotic stability of the augmented system. Hence resorting to [1, Theorem 1] proves necessary, as shown in [15] for an ODE and in [14] for a PDE.

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