Having a coercive solution to the Lyapunov inequality does not imply stability

Here we given an example of a strongly continuous semigroup which is not strongly stable, but the Lyapunov inequality
\[
\langle Az, Lz \rangle + \langle Lz, Az \rangle < 0
\]
has a coercive solution.

**Example**

Take the Hilbert space
\[
Z = \left\{ f : [0, \infty) \mapsto \mathbb{C} \mid \int_0^{\infty} |f(x)|^2 \left[ e^{-x} + 1 \right] dx < \infty \right\}.
\]
It inner product is given by:
\[
\langle f, g \rangle = \int_0^{\infty} f(x) \overline{g(x)} \left[ e^{-x} + 1 \right] dx.
\]
Note that \( Z \) “is” \( L^2(0, \infty) \).

As semigroup, we choose the shift:
\[
(T(t)f)(x) = \begin{cases} f(x-t) & x > t \\ 0 & x \in [0, t) \end{cases} = f(x-t)1_{[0, \infty]}(x-t).
\]

There holds
\[
\|T(t)f\|^2 = \int_0^{\infty} |f(x-t)1_{(0, \infty)}(x-t)|^2 \left[ e^{-x} + 1 \right] dx
\]
\[
= \int_0^{\infty} |f(\xi)|^2 \left[ e^{-z(\xi+t)} + 1 \right] d\xi
\]
\[
\geq \int_0^{\infty} |f(\xi)|^2 d\xi
\]
\[
\geq \frac{1}{2} \int_0^{\infty} |f(\xi)|^2 \left[ e^{-\xi} + 1 \right] d\xi
\]
\[
= \frac{1}{2} \|f\|^2.
\]

Hence \( (T(t))_{t \geq 0} \) is not strongly stable. The infinitesimal generator is given by
\[
Af = -\frac{df}{dx}
\]
with domain
\[ D(A) = \{ f \in Z \mid f \text{ is absolutely continuous} \}
\]
\[ \frac{df}{dx} \in Z, \text{ and } f(0) = 0 \} . \]

Next we evaluate
\[ \langle Az, z \rangle + \langle z, Az \rangle \]
for \( z \in D(A) \) and \( z \neq 0 \).

For \( A = -\frac{d}{dx} \) with boundary condition \( z(0) = 0 \), we find
\[
\langle Az, z \rangle + \langle z, Az \rangle
= \int_0^\infty (z(x) \frac{dz}{dx}(x) [e^{-x} + 1]) dx +
\int_0^\infty z(x)(-1) \frac{dz}{dx}(x) [e^{-x} + 1] dx
= - \int_0^\infty \frac{d}{dx} ([|z(x)|^2] [e^{-x} + 1]) dx.
\]

Hence
\[
\langle Az, z \rangle + \langle z, Az \rangle
= - [|z(x)|^2 [e^{-x} + 1]]_0^\infty +
\int_0^\infty |z(x)|^2 [-e^{-x}] dx
= 0 - \int_0^\infty |z(x)|^2 e^{-x} dx
< 0.
\]

Concluding, we have constructed an infinitesimal generator \( A \) for which the semigroup is not strongly stable, but the Lyapunov inequality
\[ \langle Az, z \rangle + \langle z, Az \rangle < 0, \quad z \in D(A), z \neq 0 \]
holds. Note that \( L = I \).

**Remark**

If the trajectories, \( t \mapsto T(t)z_0 \), are pre-compact, then this Lyapunov inequality implies strong stability.