Having a coercive solution to the Lyapunov inequality does not imply stability

Here we given an example of a strongly continous semigroup which is not strongly stable, but the Lyapunov inequality

$$\langle Az, Lz\rangle + \langle Lz, Az\rangle < 0$$

has a coercive solution.

Example
Take the Hilbert space

$$Z = \left\{ f : [0,\infty) \mapsto \mathbb{C} \mid \int_0^\infty |f(x)|^2 \left[e^{-x} + 1\right] dx < \infty \right\}.$$

It inner product is given by:

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)} \left[e^{-x} + 1\right] dx.$$

Note that Z "is" $L^2(0,\infty)$.

As semigroup, we choose the shift:

$$\begin{aligned} (T(t)f)(x) &= \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t) \end{cases} \\ &= f(x-t)1_{[0,\infty]}(x-t). \end{aligned}$$

There holds

$$\begin{aligned} \|T(t)f\|^2 &= \int_0^\infty |f(x-t)1_{[0,\infty]}(x-t)|^2 \left[e^{-x} + 1\right] dx \\ &= \int_0^\infty |f(\xi)|^2 \left[e^{-(\xi+t)} + 1\right] d\xi \\ &\geq \int_0^\infty |f(\xi)|^2 d\xi \\ &\geq \frac{1}{2}\int_0^\infty |f(\xi)|^2 \left[e^{-\xi} + 1\right] d\xi \\ &= \frac{1}{2}\|f\|^2. \end{aligned}$$

Hence $(T(t))_{t\geq 0}$ is not strongly stable. The infinitesimal generator is given by

$$Af = -\frac{df}{dx}$$

with domain

$$D(A) = \left\{ f \in Z \mid f \text{ is absolutely continuous } \frac{df}{dx} \in Z, \text{ and } f(0) = 0 \right\}.$$

Next we evaluate

$$\langle Az, z\rangle + \langle z, Az\rangle$$

for $z \in D(A)$ and $z \neq 0$.

For $A = -\frac{d}{dx}$ with boundary condition $z(0) = 0$, we find

$$\begin{aligned}
&\langle Az, z\rangle + \langle z, Az\rangle \\
&\quad = \int_0^\infty (-1)\frac{dz}{dx}(x)\overline{z(x)}\left[e^{-x}+1\right]dx + \\
&\qquad \int_0^\infty z(x)(-1)\overline{\frac{dz}{dx}(x)}\left[e^{-x}+1\right]dx \\
&\quad = -\int_0^\infty \frac{d}{dx}\left(|z(x)|^2\right)\left[e^{-x}+1\right]dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle Az, z\rangle + \langle z, Az\rangle &= -\left[|z(x)|^2\left[e^{-x}+1\right]\right]_0^\infty + \\
&\quad \int_0^\infty |z(x)|^2\left[-e^{-x}\right]dx \\
&= 0 - \int_0^\infty |z(x)|^2 e^{-x} dx \\
&< 0.
\end{aligned}$$

Concluding, we have constructed an infinitesimal generator A for which the semigroup is not strongly stable, but the Lyapunov inequality

$$\langle Az, z\rangle + \langle z, Az\rangle < 0, \qquad z \in D(A), z \neq 0$$

holds. Note that $L = I$.

Remark

If the trajectories, $t \mapsto T(t)z_0$, are pre-compact, then this Lyapunov inequality implies strong stability.