A Space-Time Discontinuous Galerkin Method for the Time-Dependent Oseen Equations

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Abstract

A space-time discontinuous Galerkin finite element method for the Oseen equations on time-dependent flow domains is presented. The algorithm results in a higher order accurate conservative discretization on moving and deforming meshes and is well suited for $hp$-adaptation. A detailed analysis of the stability of the numerical discretization is given which shows that the algorithm is unconditionally stable, also when equal order polynomial basis functions for the pressure and velocity are used. The accuracy of the space-time discretization is investigated using a detailed $hp$-error analysis and computations on a model problem.

Key words: space-time discontinuous Galerkin method, Oseen equations, stability, $hp$-error analysis, ALE methods

1 Introduction

Many problems in fluid dynamics require moving and deforming meshes. Important examples are fluid-structure interaction, water waves and multi-fluid flows with clearly defined interfaces. In all of these problems the computational mesh has to follow the boundary motion and interior points need to move in order to maintain a consistent mesh without grid folding. Also, when the boundary deformation becomes too large it is no longer possible to simply move the interior mesh points and a completely new mesh has to be generated. An important problem one has to face on these dynamic meshes is how to maintain a conservative numerical discretization, which preserves accuracy and efficiency, despite the mesh deformation.

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An excellent technique to achieve this is the space-time discontinuous Galerkin finite element method. In this method basis functions are used which are discontinuous at element faces, both in space and time. Also, by defining the problem in four dimensional space, automatically a conservative finite element discretization is obtained on moving and deforming meshes. Since in a space-time DG method the basis functions in an element are only weakly coupled to the basis functions in neighboring elements, the algorithm is well suited for local mesh refinement (h-adaptation) or adjustment of the polynomial order (p-refinement). In addition, if a new mesh has to be generated one only has to transfer time fluxes at the interface between the old and the new mesh and it is not necessary to interpolate data from the old to the new mesh.

In a series of articles, we have developed a solution adaptive space-time DG method for the compressible Euler and Navier-Stokes equations [16,20,21] and applied this technique to a number of aerodynamical applications. In addition, in [19] we have given a detailed error and stability analysis for a space-time DG method for the advection-diffusion equation. The objective of this article is to extend this work to a space-time DG discretization of the time-dependent Oseen equations, which are the linearized Navier-Stokes equations. The Oseen equations are the main component in an iterative scheme to solve the incompressible Navier-Stokes equations, for instance with a Picard iteration. A thorough understanding of the space-time DG algorithm for the Oseen equations is therefore a prerequisite before applying it to the incompressible Navier-Stokes equations.

The discontinuous Galerkin discretization for the steady Stokes, Oseen and incompressible Navier-Stokes has been considered by Cockburn, Kanschat, and Schötzau in a series of articles [8–10]. Also, the analysis in [17,18] provides important information on the construction of DG algorithms for incompressible flows, but space-time DG algorithms for these equations have not been developed so far. The main focus of this article is therefore the extension of the DG method for incompressible flows to the space-time framework and to analyze the stability and accuracy of the resulting algorithm. An important aspect in space-time DG algorithms is also the efficient solution of the algebraic equations resulting from the space-time discretization. An excellent technique for this is provided by multigrid methods, which have been extensively analyzed for discontinuous Galerkin discretizations by Hemker, Hoffmann and van Raalte in [13–15], but the application of these techniques to the Oseen equations is presently still a topic of ongoing research.

There are several important points that have to be considered in the development of a DG discretization for the Oseen equations, which are also relevant for a space-time DG method.

The first issue is the DG discretization of the viscous terms. The obvious choice
is to discretize the viscous terms in a similar way as done for elliptic equations (for a unified analysis, see [3]), but now extended to vector fields. There are basically two approaches, the techniques proposed by Bassi and Rebay [4] and Brezzi et al. [6,7], and the local discontinuous Galerkin method proposed by Cockburn and Shu in [8–10]. Based on our experience with the advection-diffusion equation [19] and the compressible Navier-Stokes equations [16], we will use the Bassi and Rebay and Brezzi approach for the DG discretization of the viscous contribution since this method results in a very compact stencil. The local discontinuous Galerkin method provides, however, also a very useful discretization technique and many of the results in this article also apply to this method.

The second issue is the pressure stabilization. The analysis presented in [17,18] shows the importance of the pressure stabilization operator for the choice of the polynomial degrees in the approximation of the velocity and pressure in a DG discretization. Without a stabilization term, the DG method can only be proven stable when the polynomial degree used in the approximation of the pressure is one less than the polynomial degree for the approximation of the velocity. By adding a stabilization term similar to the one used for elliptic equations, stability is proven when equal polynomial degrees are used for the velocity and the pressure.

The organization of this article is as follows. First, the Oseen equations are written in the space-time formulation in Section 2. After the introduction of the finite element spaces and trace operators in Section 3, we give a complete derivation of the space-time DG formulation for the Oseen equations on time-dependent domains in Section 4. The analysis of the stability and accuracy of the space-time DG discretization for the Oseen equations is presented in Section 5. Numerical experiments on a model problem are given in Section 6. Finally, concluding remarks are drawn in Section 7.

2 The Oseen equations

In this section we introduce the Oseen equations and set some notations. Let \( \Omega_t \) be an open, bounded, time-dependent domain in \( \mathbb{R}^d \) at time \( t \), where \( d \) is the number of spatial dimensions. The closure of \( \Omega_t \) is \( \overline{\Omega}_t \) and the boundary of \( \Omega_t \) is denoted by \( \partial \Omega_t \). Denoting \( \tilde{x} = (x_1, \ldots, x_d) \) as the spatial variables, we consider in \( \Omega_t \) the time-dependent Oseen equations for the velocity field \( u \in \mathbb{R}^d \) and the kinematic pressure \( p := p/\rho \in \mathbb{R} \):

\[
\frac{\partial u}{\partial t} + \nabla \cdot (u \otimes \tilde{w}) - \nu \nabla \cdot \nabla u + \nabla p = f, \quad \text{in } \Omega_t, \tag{1a}
\]

\[
\nabla \cdot u = 0, \quad \text{in } \Omega_t, \tag{1b}
\]
with $\rho$ the fluid density, $\nu \in \mathbb{R}^+$ the kinematic viscosity, $\bar{w} \in \mathbb{R}^d$ a given convective divergence free velocity field, and $f \in \mathbb{R}^d$ the force vector. We introduce the product between two vectors $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ as $a \otimes b \in \mathbb{R}^{m \times n}$ with elements $(a \otimes b)_{ij} = a_i b_j$. The notation $\nabla$ is used for the spatial gradient operator in $\mathbb{R}^d$, and defined as $\nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right)$. We also define the divergence of a tensor $A \in \mathbb{R}^{m \times n}$ as $\nabla \cdot A = \sum_j \frac{\partial}{\partial x_j} A_{ij}$.

The first step to obtain a space-time discretization is to consider the Oseen equations directly in a domain in $\mathbb{R}^{d+1}$. A point $x \in \mathbb{R}^{d+1}$ has coordinates $(x_0, \bar{x})$, with $x_0 = t$ representing time. We introduce the space-time domain $\mathcal{E} \subset \mathbb{R}^{d+1}$. The boundary of the space-time domain $\partial \mathcal{E}$ consists of the hypersurfaces $\Omega_0 := \{x \in \partial \mathcal{E} \mid x_0 = 0\}$, $\Omega_T := \{x \in \partial \mathcal{E} \mid x_0 = T\}$, and $\mathcal{Q} := \{x \in \partial \mathcal{E} \mid 0 < x_0 < T\}$. Introducing the gradient operator in $\mathbb{R}^{d+1}$ as $\nabla = \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right)$ and the vector $w = (1, \bar{w})$, the Oseen equations (1) can be transformed into a space-time formulation as:

$$\nabla \cdot (u \otimes w) - \nu \nabla \cdot \nabla u + \nabla \cdot (I_d w) = f, \quad \text{in } \mathcal{E},$$

$$\nabla \cdot u = 0, \quad \text{in } \mathcal{E},$$

with $I_d$ the $d \times d$ identity matrix.

Since different boundary conditions are imposed on $\partial \mathcal{E}$, we first discuss the subdivision of $\partial \mathcal{E}$ into different parts. The boundary $\partial \mathcal{E}$ is divided into disjoint subsets $\Gamma_m$ and $\Gamma_p$, with:

$$\Gamma_m := \{x \in \partial \mathcal{E} : w \cdot n < 0\}, \quad \Gamma_p := \{x \in \partial \mathcal{E} : w \cdot n \geq 0\}.$$ 

The subscripts $m$ and $p$ denote the inflow and outflow boundaries, respectively, and $n$ refers to the space-time normal vector at $\partial \mathcal{E}$. We subdivide $\Gamma_m$ further into two sets: $\Gamma_{dm}$ and $\Omega_0$, with $\Gamma_{dm}$ the part of $\Gamma_m$ with a Dirichlet boundary condition and $\Omega_0$ the part of $\Gamma_m$ with the initial condition. The part $\Gamma_p$ is divided into three sets: $\Omega_T$, $\Gamma_{dp}$, and $\Gamma_N$, with $\Omega_T$ the part of $\partial \mathcal{E}$ at the final time $T$, $\Gamma_{dp}$ the part of $\Gamma_p$ with a Dirichlet boundary condition and $\Gamma_N$ the part of $\Gamma_p$ with a Neumann boundary condition. Note that $\Gamma_D = \Gamma_{dm} \cup \Gamma_{dp}$ is the part of the space-time boundary with a Dirichlet boundary condition. The boundary conditions on different parts of $\partial \mathcal{E}$ are written as

$$u = u_0 \quad \text{on } \Omega_0, \quad \text{(3a)}$$

$$u = g_D \quad \text{on } \Gamma_D, \quad \text{(3b)}$$

$$\bar{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N, \quad \text{(3c)}$$

$$p = p_N \quad \text{on } \Gamma_N, \quad \text{(3d)}$$

with $u_0$ a given initial velocity field and $g_D, g_N, p_N$ given data defined on (part of) the boundary. There is no boundary condition imposed on $\Omega_T$. 

4
3 Space-time elements, finite element spaces and trace operators

3.1 Definition of space-time slabs, elements and faces

As already introduced in the previous section, we consider the Oseen equations in a space-time domain $\mathcal{E}$. This requires the definition of space-time slabs. For this purpose the time interval $[0, T]$ is partitioned into several parts. The $n$th time interval is denoted by $I_n$, with its length defined as $\triangle n = t_{n+1} - t_n$. A space-time slab is now defined as the domain $\mathcal{E}^n = \mathcal{E} \cap (I_n \times \mathbb{R}^d)$ with boundaries $\Omega_{t_n}, \Omega_{t_{n+1}}$ and $Q^n = \partial \mathcal{E}^n \setminus (\Omega_{t_n} \cup \Omega_{t_{n+1}})$.

Next, we construct the space-time elements $K$ in the space-time slab $\mathcal{E}^n$. Let the domain $\Omega_{t_n}$ be divided into $N_n$ non-overlapping spatial elements $K^n$. At $t_{n+1}$ the spatial elements $K^{n+1}$ are obtained by mapping the elements $K^n$ to their new position. Each space-time element $K$ is then obtained by connecting elements $K^n$ and $K^{n+1}$ using linear interpolation in time. In case of curved domain boundaries, a higher order accurate interpolation is used for elements connected to the domain boundary. We then denote by $h_K$ the radius of the smallest sphere containing each element $K$. The element boundary $\partial K$ is the union of open faces of $K$, which contains three parts $K^n, K^{n+1}$, and $Q^n_k = \partial K \setminus (K^n \cup K^{n+1})$. We denote by $n_K$ the unit outward space-time normal vector on $\partial K$. The definition of the space-time domain is completed with the tessellation $\mathcal{T}^n_h$ in each space-time slab and $\mathcal{T}_h = \cup_n \mathcal{T}^n_h$ is the tessellation in the space-time domain.

We consider several sets of faces in the tessellation. The set of all faces in $\mathcal{E}$ is denoted with $\mathcal{F}$, the set of all interior faces in $\mathcal{E}$ with $\mathcal{F}_{\text{int}}$, and the set of all boundary faces on $\partial \mathcal{E}$ with $\mathcal{F}_{\text{bnd}}$. In the space-time slab $\mathcal{E}^n$ we denote the set of all faces with $\mathcal{F}^n$ and the set of all interior faces with $\mathcal{S}^n_I$. The faces separating two space-time slabs are denoted as $\mathcal{S}^n_S$. Several sets of boundary faces are defined as follows. The set of faces with a Dirichlet boundary condition is denoted with $\mathcal{S}^n_D$. This set can be divided further into the sets $\mathcal{S}^n_{Dm}$ and $\mathcal{S}^n_{Dp}$, which correspond to the faces with a Dirichlet boundary condition on $\Gamma_m$ and $\Gamma_p$, respectively. The set of faces with a Neumann boundary condition is denoted with $\mathcal{S}^n_N$. Further, the sets $\mathcal{S}^n_I$ and $\mathcal{S}^n_D$ are grouped into $\mathcal{S}^n_{ID}$ and the sets $\mathcal{S}^n_I, \mathcal{S}^n_D$ and $\mathcal{S}^n_N$ are grouped into $\mathcal{S}^n_{IDN}$.

3.2 Finite element spaces and trace operators

The space-time DG discretization requires the use of anisotropic Sobolev spaces on the domain $\mathcal{D} \subset \mathbb{R}^{d+1}$, see for instance [11]. The definition of standard Sobolev spaces follows [5]. Here we restrict the anisotropy to the case...
where the Sobolev index can be different for the temporal and spatial variables. All spatial variables have, however, the same index. Let \((s_t, s_s)\) be a pair of non-negative integers, with \(s_t, s_s\) corresponding to the temporal and spatial Sobolev index, respectively. For \(\gamma_t, \gamma_s \geq 0\), the anisotropic Sobolev space of order \((s_t, s_s)\) on \(D\) is defined by

\[
H^{(s_t, s_s)}(D) := \{ v \in L^2(D) : \partial^{\gamma_t} \partial^{\gamma_s} v \in L^2(D) \text{ for } \gamma_t \leq s_t, |\gamma_s| \leq s_s \},
\]

with associated norm and semi-norm:

\[
\|v\|_{s_t, s_s, D} := \left( \sum_{\gamma_t \leq s_t} \|\partial^{\gamma_t} \partial^{\gamma_s} v\|_{0,D}^2 \right)^{1/2}, \quad |v|_{s_t, s_s, D} := \left( \sum_{|\gamma_t| = s_t} \|\partial^{\gamma_t} \partial^{\gamma_s} v\|_{0,D}^2 \right)^{1/2}.
\]

Since the DG method is a non-conforming method, it is necessary to introduce the concept of a broken anisotropic Sobolev space. To each element \(K\) we assign a pair of nonnegative integers \((s_t, s_s)\) and collect them in the vectors \(s_t = \{s_t, K : K \in \mathcal{T}_h\}\) and \(s_s = \{s_s, K : K \in \mathcal{T}_h\}\). Then we assign to \(\mathcal{T}_h\) the broken Sobolev space \(H^{(s_t, s_s)}(\mathcal{E}, \mathcal{T}_h) := \{ v \in L^2(\mathcal{E}) : v|_K \in H^{(s_t, s_s)}(K) \text{, } \forall K \in \mathcal{T}_h \}\), equipped with the broken Sobolev norm and corresponding semi-norm, respectively,

\[
\|v\|_{s_t, s_s, \mathcal{T}_h} := \left( \sum_{K \in \mathcal{T}_h} \|v\|_{s_t, s_s, K}^2 \right)^{1/2}, \quad |v|_{s_t, s_s, \mathcal{T}_h} := \left( \sum_{K \in \mathcal{T}_h} |v|_{s_t, s_s, K}^2 \right)^{1/2}.
\]

For \(v \in H^{(1,1)}(\mathcal{E}, \mathcal{T}_h)\), we define the broken gradient \(\nabla_h v\) of \(v\) by \((\nabla_h v)|_K := \nabla(v|_K), \forall K \in \mathcal{T}_h\) and similarly for \(v \in H^{(0,1)}(\mathcal{E}, \mathcal{T}_h)\) the broken spatial gradient \(\nabla_h v\) with \((\nabla_h v)|_K := \nabla(v|_K), \forall K \in \mathcal{T}_h\).

We now discuss the finite element spaces associated with the tessellation \(\mathcal{T}_h\) that will be used in this article. First, we introduce mappings of the space-time elements. Following the discussion in [11], we assume that each element \(K \in \mathcal{T}_h\) is an image of a fixed master element \(\hat{K}\), with \(\hat{K}\) an open unit hypercube in \(\mathbb{R}^{d+1}\), constructed via two mappings \(Q_K \circ F_K\), where \(F_K : \hat{K} \to \mathcal{K}\) is an affine
mapping and \( Q_K : \tilde{K} \to K \) is a (regular enough) diffeomorphism (see Fig. 1).

To each element \( K \) we assign a pair of nonnegative integers \((p_t, p_s, K)\) as local polynomial degrees, where the subscripts \( t \) and \( s \) denote time and space, respectively, and collect them into vectors \( p_t = \{ p_{t,K} : K \in T_h \} \) and \( p_s = \{ p_{s,K} : K \in T_h \} \). We define \( Q_{p_t,K,p_s,K}(K) \) as the set of all tensor-product polynomials on \( \tilde{K} \) of degree \( p_{t,K} \) in the time direction and degree \( p_{s,K} \) in each spatial coordinate direction. The finite element spaces of discontinuous polynomial functions are defined as follows

\[
V_h^{(p_t,p_s)} := \{ v \in L^2(\mathcal{E})^d : v|_K \circ Q_K \circ F_K \in \left[ Q_{(p_{t,K},p_{s,K})}(\tilde{K}) \right]^d, \forall K \in T_h \},
\]

\[
Q_h^{(p_t,p_s)} := \{ q \in L^2(\mathcal{E}) : q|_K \circ Q_K \circ F_K \in \left[ Q_{(p_{t,K},p_{s,K})}(\tilde{K}) \right]^d, \forall K \in T_h \}.
\]

In the derivation and analysis of the numerical discretization we also make use of auxiliary spaces \( \Sigma_h^{(p_t,p_s)} \) and \( \Sigma_h^{(p_t,p_s)} \):

\[
\Sigma_h^{(p_t,p_s)} := \{ \tau \in L^2(\mathcal{E})^{d \times (d+1)} : \tau|_K \circ Q_K \circ F_K \in \left[ Q_{(p_{t,K},p_{s,K})}(\tilde{K}) \right]^{d \times (d+1)}, \forall K \in T_h \},
\]

\[
\Sigma_h^{(p_t,p_s)} := \{ \tau \in L^2(\mathcal{E})^{d \times d} : \tau|_K \circ Q_K \circ F_K \in \left[ Q_{(p_{t,K},p_{s,K})}(\tilde{K}) \right]^{d \times d}, \forall K \in T_h \}.
\]

The so-called traces of \( v \in V_h^{(p_t,p_s)} \) on \( \partial K \) are defined as: \( v_K^\pm = \lim_{\varepsilon \downarrow 0} v(x \pm \varepsilon n_K) \). The traces of \( q \in Q_h^{(p_t,p_s)}, \tau \in \Sigma_h^{(p_t,p_s)} \), and \( \tau \in \Sigma_h^{(p_t,p_s)} \) are defined similarly.

Next, we define several trace operators for the sets \( \mathcal{F}_{\text{int}} \) and \( \mathcal{F}_{\text{bnd}} \). Note that functions \( v \in V_h^{(p_t,p_s)}, q \in Q_h^{(p_t,p_s)}, \tau \in \Sigma_h^{(p_t,p_s)} \) and \( \tau \in \Sigma_h^{(p_t,p_s)} \) are in general multivalued on a face \( S \in \mathcal{F}_{\text{int}} \). Introducing the functions \( v_i := v|_K, q_i := q|_K, \tau_i := \tau|_K, \tilde{\tau}_i := \tilde{\tau}|_K \), we define the average operator \( \langle \cdot \rangle \) on \( S \in \mathcal{F}_{\text{int}} \) as:

\[
\langle v \rangle = (v_i + v_j)/2, \quad \langle q \rangle = (q_i + q_j)/2, \quad \langle \tau \rangle = (\tau_i + \tau_j)/2, \quad \langle \tilde{\tau} \rangle = (\tilde{\tau}_i + \tilde{\tau}_j)/2,
\]

while on \( S \in \mathcal{F}_{\text{bnd}} \), we set accordingly

\[
\langle v \rangle = v, \quad \langle q \rangle = q, \quad \langle \tau \rangle = \tau, \quad \langle \tilde{\tau} \rangle = \tilde{\tau}.
\]

We also introduce the jump operators \( [\cdot] \) and \( \langle \cdot \rangle \). For functions \( q \in Q_h^{(p_t,p_s)} \), \( \tau \in \Sigma_h^{(p_t,p_s)} \) and \( \tilde{\tau} \in \Sigma_h^{(p_t,p_s)} \), the jump operators are defined on \( S \in \mathcal{F}_{\text{int}} \) as:

\[
[q] = q_i n_i + q_j n_j, \quad [\tau] = \tau_i \cdot n_i + \tau_j \cdot n_j,
\]

\[
\langle q \rangle = q_i \tilde{n}_i + q_j \tilde{n}_j, \quad \langle \tilde{\tau} \rangle = \tilde{\tau}_i \cdot \tilde{n}_i + \tilde{\tau}_j \cdot \tilde{n}_j,
\]

with \( n_i, \tilde{n}_i \) the outward normal vector on \( \partial K_i \) and its spatial part, respectively.
For functions $v \in V_h^{(p_i,p_j)}$, we define the jump operators on $S \in \mathcal{F}_{\text{int}}$ as follows:

$$[v] = v_i \otimes n_i + v_j \otimes n_j, \quad \langle v \rangle = v_i \otimes \bar{n}_i + v_j \otimes \bar{n}_j, \quad \langle v \rangle = v_i \cdot \bar{n}_i + v_j \cdot \bar{n}_j.$$  

By taking all functions from the neighboring element equal to zero, the definitions of jump operators are also valid on boundary faces $S \in \mathcal{F}_{\text{bnd}}$. Note that $\langle v \rangle$ is scalar, $[q] \in \mathbb{R}^{d+1}$, $\langle q \rangle \in \mathbb{R}^d$ are vectors, and $[v] \in \mathbb{R}^{d \times (d+1)}$, $\langle v \rangle \in \mathbb{R}^{d \times d}$ are matrices. The jumps $[\tau], \langle \tau \rangle \in \mathbb{R}^d$ are vectors.

### 3.3 Lifting operators

The derivation of the space-time DG formulation requires several lifting operators. The main purpose of the lifting operator is to eliminate the auxiliary variables $\sigma$, introduced in Section 4.1, by extending data defined at the element faces into the whole space-time domain.

First, we introduce the local lifting operator $\mathcal{L}_S : (L^2(S))^{d \times (d+1)} \to \Sigma_h^{(p_i,p_j)}$:

$$\int_{\mathcal{E}} \mathcal{L}_S(\vartheta) : \tau \, d\mathcal{E} = \int_S \vartheta : \{\tau\} \, dS, \quad \forall \tau \in \Sigma_h^{(p_i,p_j)}, \forall S \in \cup_n \mathcal{S}_{ID}^n, \quad (4)$$

where the dyadic product between two matrices $A, B \in \mathbb{R}^{m \times n}$ is defined as: $A : B = \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$. Note, the local lifting operator $\mathcal{L}_S$ is only non-zero in the two elements connected to the face $S$. The global lifting operator $\mathcal{L} : (L^2(\cup_n \mathcal{S}_{ID}^n))^{d \times (d+1)} \to \Sigma_h^{(p_i,p_j)}$ is now introduced as

$$\int_{\mathcal{E}} \mathcal{L}(\vartheta) : \tau \, d\mathcal{E} = \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \int_{\mathcal{E}} \mathcal{L}_S(\vartheta) : \tau \, d\mathcal{E}, \quad \forall \tau \in \Sigma_h^{(p_i,p_j)}. \quad (5)$$

We also specify the above lifting operators for the Dirichlet boundary condition. On faces $S \in \cup_n \mathcal{S}_D^n$ we have

$$\int_{\mathcal{E}} \mathcal{L}_S(\mathcal{P}g_D \otimes n) : \tau \, d\mathcal{E} = \int_S g_D \otimes n : \tau \, dS, \quad \forall \tau \in \Sigma_h^{(p_i,p_j)}, \forall S \in \cup_n \mathcal{S}_D^n,$$

with $\mathcal{P}$ the $L^2$ projection on $\Sigma_h^{(p_i,p_j)}$. For the global lifting operators, we proceed as follows. We replace $\vartheta$ by $\mathcal{P}g_D \otimes n$ in (4) and (5) to obtain the following global lifting operator for the Dirichlet boundary:

$$\int_{\mathcal{E}} \mathcal{L}(\mathcal{P}g_D \otimes n) : \tau \, d\mathcal{E} = \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S g_D \otimes n : \tau \, dS, \quad \forall \tau \in \Sigma_h^{(p_i,p_j)}. \quad (6)$$
Using (5) and (6), we then introduce $L_{ID} : (L^2(\cup_n S_{ID}))^{d\times(d+1)} \rightarrow \Sigma_h^{(p_n,p_s)}$ as:

$$L_{ID}(\vartheta) = -L(\vartheta) + L(P g_D \otimes n). \quad (7)$$

Later in this article, we will also use the spatial part of the lifting operators, denoted by $\bar{L}, \bar{L}_S$, which are obtained by eliminating the first component of $L, L_S$, respectively.

4 Space-time DG discretization for the Oseen equations

In this section we give a derivation of the space-time DG formulation for the Oseen equations (2). Since the DG basis functions are discontinuous at the element faces we cannot directly obtain a weak formulation for the Oseen equations because this requires the trace of first order derivatives at the element faces, which are not uniquely defined for discontinuous basis functions. For this purpose we transform the Oseen equations into a first order system by the introduction of an auxiliary variable $\sigma = \nabla u$:

$$\sigma = \nabla u, \quad \text{in } \mathcal{E}, \quad (8a)$$

$$\nabla \cdot (u \otimes w) - \nu \nabla \cdot \sigma + \nabla \cdot (I_0 p) = f, \quad \text{in } \mathcal{E}, \quad (8b)$$

$$\nabla \cdot u = 0, \quad \text{in } \mathcal{E}. \quad (8c)$$

These equations are completed with the boundary conditions (3a)-(3d). In the next three sections we give the derivation of the space-time DG formulation for (8).

In the remainder of this article, we will assume that $w \in H(\text{div } 0, \mathcal{E}) := \{ v \in (H^{(1,1)}(\mathcal{E}))^{d+1} : \nabla \cdot v = 0 \}, \ f \in (L^2(\mathcal{E}))^d, \ g_D \in (L^2(\cup_n S_B))^d, \ g_N, p_N \in L^2(\cup_n S_N), \ u_0 \in (L^2(\Omega_0))^d$.

4.1 Space-time DG formulation for the auxiliary variable

The space-time DG formulation for the auxiliary equation (8a) is obtained by multiplying (8a) with an arbitrary test function $\bar{\tau} \in \bar{\Sigma}_h^{(p_n,p_s)}$; substituting $\sigma, u$ with the approximations $\sigma_h \in \Sigma_h^{(p_n,p_s)}, \ u_h \in V_h^{(p_n,p_s)}$, and integration over a space-time element $\mathcal{K} \in T_h^n$. Next, we perform integration by parts on the right hand side with respect to $x_1, \ldots, x_d$ twice and, after summation over all elements $\mathcal{K} \in T_h^n$, we obtain for all $\bar{\tau} \in \bar{\Sigma}_h^{(p_n,p_s)}$:

$$\int_{\mathcal{E}_h} \sigma_h : \bar{\tau} \, d\mathcal{E} = \int_{\mathcal{E}_h} \nabla_h u_h : \bar{\tau} \, d\mathcal{E} + \sum_{\mathcal{K} \in T_h^n} \int_{Q_{\mathcal{K}}} (\hat{u}_h^\tau - u_h) \otimes \bar{n} : \bar{\tau}^- \, d\partial\mathcal{K}. \quad (9)$$
The variable \( \hat{u}^\sigma_h \) is the \textit{numerical flux} that must be defined to account for the multivalued trace at \( \mathcal{Q}_K \). Note that since we perform the integration by parts on the spatial variables, we only have to consider the weak formulation in the space-time slab \( \mathcal{E}^n \) since there are no fluxes between the different space-time slabs.

We recall the following relation, which is an extension of the identity introduced in [3] to tensors \( \bar{\tau} \) and vectors \( v \), piecewise smooth on \( \mathcal{T}_h \):

\[
\sum_{K \in \mathcal{T}_h} \int_{\mathcal{Q}_K^n} v \otimes \bar{n} : \bar{\tau}^- \, d\partial K = \sum_{S \in \mathcal{S}_{IDN}^n} \int_S \langle \langle v \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS + \sum_{S \in \mathcal{S}_I^n} \int_S \langle \langle v \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS,
\]

and can be proved by a straightforward calculation. When applied to the last term in (9), this results in

\[
\sum_{K \in \mathcal{T}_h} \int_{\mathcal{Q}_K^n} (\hat{u}^\sigma_h - u_h) \otimes \bar{n} : \bar{\tau}^- \, d\partial K = \sum_{S \in \mathcal{S}_{IDN}^n} \int_S \langle \langle \hat{u}^\sigma_h - u_h \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS
\]

\[
+ \sum_{S \in \mathcal{S}_I^n} \int_S \langle \langle \hat{u}^\sigma_h - u_h \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS. \quad (11)
\]

For the numerical flux \( \hat{u}^\sigma_h \), we make a similar choice as in [6,19], now applied to vector functions:

\[
\hat{u}^\sigma_h = \{ u_h \} \text{ on } \mathcal{S}_I^n, \quad \hat{u}^\sigma_h = g_D \text{ on } \mathcal{S}_D^n, \quad \hat{u}^\sigma_h = u_h \text{ on } \mathcal{S}_N^n. \quad (12)
\]

Replacing \( \hat{u}^\sigma_h \) in (11) with the numerical flux (12), the weak formulation (9) is now equal to:

\[
\int_{\mathcal{E}^n} \sigma_h : \bar{\tau} \, d\mathcal{E} = \int_{\mathcal{E}^n} \nabla_h u_h : \bar{\tau} \, d\mathcal{E} - \sum_{S \in \mathcal{S}_{ID}^n} \int_S \langle \langle u_h \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS + \sum_{S \in \mathcal{S}_D^n} \int_S g_D \otimes \bar{n} : \bar{\tau} \, dS
\]

\[
+ \sum_{S \in \mathcal{S}_N^n} \int_S \langle \langle \hat{u}^\sigma_h - u_h \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS. \quad (13)
\]

We sum (13) over all space-time slabs and the next step is to derive an equation for \( \sigma_h \) which can be used in the DG formulation for (8b). The main benefit of the elimination of the auxiliary variables at the discrete level is a significant reduction in the number of equations and variables in the DG discretization. For this purpose, the last two terms on the right hand side of (13) are replaced with the spatial part of the lifting operator \( \mathcal{L}_{ID} \), defined in (7), to obtain:

\[
\int_{\mathcal{E}} \mathcal{L}_{ID}(\langle \langle u_h \rangle \rangle) : \bar{\tau} \, d\mathcal{E} = - \sum_{S \in \mathcal{U}_h \mathcal{S}_{ID}^n} \int_S \langle \langle u_h \rangle \rangle : \langle \langle \bar{\tau} \rangle \rangle \, dS + \sum_{S \in \mathcal{U}_h \mathcal{S}_D^n} \int_S g_D \otimes \bar{n} : \bar{\tau} \, dS.
\]

Introducing (14) into (13) and using the fact that this relation must be valid
for arbitrary test functions \( \tilde{\tau} \), we can express \( \sigma_h \in \Sigma_h^{(p_t,p_s)} \) as

\[
\sigma_h = \nabla_h u_h + \mathcal{L}_{ID}(\mathcal{L}(u_h)), \quad \text{a.e.} \ \forall x \in \mathcal{E}.
\] (15)

### 4.2 Space-time DG formulation for the momentum equations

The space-time DG formulation for the momentum equations in the Oseen equations is obtained by multiplying (8b) with arbitrary test functions \( v \in V_h^{(p_t,p_s)} \) and integrating over an element \( K \in T_h^n \), such that for all \( v \in V_h^{(p_t,p_s)} \) the following relation is satisfied:

\[
\int_K (\nabla \cdot (u \otimes w)) \cdot v \, dK - \int_K (\nu \nabla \cdot \sigma) \cdot v \, dK + \int_K (\nabla \cdot I_d p) \cdot v \, dK = \int_f f \cdot v \, dK. \tag{16}
\]

The functions \( u,\sigma,p \) are then replaced by their approximations \( u_h \in V_h^{(p_t,p_s)} \), \( \sigma_h \in \Sigma_h^{(p_t,p_s)} \), \( p_h \in Q_h^{(p_t,p_s)} \), respectively. Next, we integrate by parts each term on the left hand side of (16). For the first term, integration by parts is with respect to \( x_0, \ldots, x_d \), while for the second and third term the integration by parts is with respect to \( x_1, \ldots, x_d \). After summation over all elements \( K \in T_h \), we have for all \( v \in V_h^{(p_t,p_s)} \):

\[
T_c + T_d + T_p = \int_{\mathcal{E}} f \cdot v \, d\mathcal{E}, \tag{17}
\]

with

\[
T_c := -\int_{\mathcal{E}} u_h \otimes w : \nabla_h v \, d\mathcal{E} + \sum_{K \in T_h} \int_{\partial K} \hat{u}_h^c \otimes w : v \otimes n \, d\partial K, \tag{18a}
\]

\[
T_d := \int_{\mathcal{E}} \nu \sigma_h : \nabla_h v \, d\mathcal{E} - \sum_{K \in T_h} \int_{Q_h^v} \nu \hat{\sigma}_h : v \otimes \bar{n} \, d\partial K, \tag{18b}
\]

\[
T_p := -\int_{\mathcal{E}} I_d p_h : \nabla_h v \, d\mathcal{E} + \sum_{K \in T_h} \int_{Q_h^v} I_d \hat{p}_h : v \otimes \bar{n} \, d\partial K, \tag{18c}
\]

related, respectively, to the convective, diffusive, and pressure terms. Here we replaced \( u_h,\sigma_h,p_h \) at \( \partial K \) with the numerical fluxes \( \hat{u}_h^c, \hat{\sigma}_h, \hat{p}_h \), to account for the multivalued traces at \( \partial K \). The next step is to find appropriate choices for the numerical fluxes and we discuss the derivation for each term separately.

First, we consider the convective term \( T_c \) (18a), which includes the convective flux \( \hat{u}_h^c \). The obvious choice is an upwind flux, as in [9]. However, for simplicity of proving stability of the discretization, the upwind flux is written as the sum of an average plus a jump penalty, as in [7,19]. Thus, we write the numerical flux \( \hat{u}_h^c \otimes w \) as:

\[
\hat{u}_h^c \otimes w = \mathcal{L}(u_h) \otimes w + C_S \| u_h \|.
\] (19)
The parameter $C_S$ is chosen as:

$$C_S = \frac{1}{2}|w \cdot n| \quad \text{on } S \in \mathcal{F}_{\text{int}}. \quad (20)$$

For conciseness of the proof discussed later in Section 5, we extend the definition of $C_S$ to the boundary of the space-time domain as:

$$C_S = \begin{cases} -w \cdot n/2, & \text{on } S \subset \Gamma_m, \\ +w \cdot n/2, & \text{on } S \subset \Gamma_p. \end{cases} \quad (21)$$

We recall identity (10), this time for tensors $\tau$ and vectors $v$, piecewise smooth on $T_h$:

$$\sum_{K \in T_h} \int_{\partial K} v \otimes n : \tau \, d\partial K = \sum_{S \in \mathcal{F}} \int_{S} \{\{ v \} \} : \{\{ \tau \} \} \, dS + \sum_{S \in \mathcal{F}_{\text{int}}} \int_{S} \{\{ v \} \} \cdot \{\{ \tau \} \} \, dS. \quad (22)$$

If we substitute $\tau$ in relation (22) with (19) and use the boundary conditions on $\Gamma_D$ and $\Omega_0$, we obtain the final form of $T_c$:

$$T_c = -\int_{\Omega} u_0 \cdot v \, dS. \quad (23)$$

Next, we consider the diffusive term $T_d$ (18b). We recall identity (10) again. When applied to the second term in (18b), we obtain:

$$\sum_{K \in T_h} \int_{Q_K} v \otimes \hat{\sigma}_h : v \otimes \bar{n} \, d\partial K = \sum_{S \in \mathcal{F}_{\text{int}}} \int_{S} \nu \{\{ \hat{\sigma}_h \} \} : \{\{ v \} \} \, dS + \sum_{S \in \mathcal{F}_{\text{int}}} \int_{S} \nu \langle \langle \hat{\sigma}_h \rangle \rangle \cdot \{\{ v \} \} \, dS. \quad (24)$$

For the numerical flux $\hat{\sigma}_h$, we make the same choice as in [6,19], now applied to tensor functions:

$$\hat{\sigma}_h = \{\{ \sigma_h \} \} \text{ on } S^n_D, \quad \hat{\sigma}_h = \sigma_h \text{ on } S^n_D \cup S^n_N, \quad (25)$$

and substitute (25) into (24). Using the boundary condition (3c) on $S \in \cup_n S^n_N$.
and after replacement of $\sigma_h$ with (15), the term $T_d$ becomes:

$$T_d = \int_E \nu \nabla h u_h : \nabla h v \, d\mathcal{E} + \int_E \nu \bar{L}_{ID}(\langle \langle u_h \rangle \rangle) : \nabla h v \, d\mathcal{E} - \sum_{S \in \cup_n \mathcal{S}_D} \int_S \nu g_N \cdot v \, dS$$

$$- \sum_{S \in \cup_n \mathcal{S}_D} \int_S \nu \langle \langle \nabla h u_h \rangle \rangle : \langle \langle v \rangle \rangle \, dS - \sum_{S \in \cup_n \mathcal{S}_D} \int_S \nu \langle \langle \bar{L}_{ID}(\langle \langle u_h \rangle \rangle) \rangle \rangle : \langle \langle v \rangle \rangle \, dS.$$

We can further evaluate the second and fourth term in (26). Using the lifting operator $\bar{L}_{ID}$ defined in (14) with $\bar{\tau} = \nabla h v$ we have the relation

$$\int_E \nu \bar{L}_{ID}(\langle \langle u_h \rangle \rangle) : \nabla h v \, d\mathcal{E} = - \sum_{S \in \cup_n \mathcal{S}_D} \int_S \nu \langle \langle u_h \rangle \rangle : \langle \langle \nabla h v \rangle \rangle \, dS$$

$$+ \sum_{S \in \cup_n \mathcal{S}_D} \int_S \nu g_D \otimes \bar{n} : \nabla h v \, dS,$$

and by considering only the spatial part of the lifting operators $\mathcal{L}, \mathcal{L}_{ID}$, defined in (5) and (7), we obtain

$$\sum_{S \in \cup_n \mathcal{S}_D} \int_S \nu \langle \langle \bar{L}_{ID}(\langle \langle u_h \rangle \rangle) \rangle \rangle : \langle \langle v \rangle \rangle \, dS = - \int_E \nu \bar{L}(\langle \langle u_h \rangle \rangle) : \bar{L}(\langle \langle v \rangle \rangle) \, d\mathcal{E}$$

$$+ \int_E \nu \bar{L}(\mathcal{P} g_D \otimes \bar{n}) : \bar{L}(\langle \langle v \rangle \rangle) \, d\mathcal{E}.$$ (27)

In order to ensure that only contributions from neighboring elements occur in the discretization, which improves both the computational efficiency and memory use, the contributions from the global lifting operator $\bar{L}$ in (28) are replaced with the local lifting operator $\bar{L}_S$ (defined in (4) and (3.3)), using the following simplifications

$$\int_E \nu \bar{L}(\langle \langle u_h \rangle \rangle) : \bar{L}(\langle \langle v \rangle \rangle) \, d\mathcal{E} \approx \sum_{S \in \cup_n \mathcal{S}_D} \sum_{K \in \mathcal{T}_h} \eta^u_K \int_K \nu \bar{L}_S(\langle \langle u_h \rangle \rangle) : \bar{L}_S(\langle \langle v \rangle \rangle) \, dK,$$ (29a)

$$\int_E \nu \bar{L}(\mathcal{P} g_D \otimes \bar{n}) : \bar{L}(\langle \langle v \rangle \rangle) \, d\mathcal{E} \approx \sum_{S \in \cup_n \mathcal{S}_D} \sum_{K \in \mathcal{T}_h} \eta^n_K \int_K \nu \bar{L}_S(\mathcal{P} g_D \otimes \bar{n}) : \bar{L}_S(\langle \langle v \rangle \rangle) \, dK,$$ (29b)

with the parameter $\eta^u_K$ a positive constant. Later in Section 5 we discuss the minimum value for $\eta^u_K$ in order to have a stable method. Introducing the
relations (27)-(29b) into (26), the term \( T_d \) can be written in its final form

\[
T_d = \int_{\mathcal{E}} \nu \nabla_h u_h : \nabla_h v \, d\mathcal{E} - \sum_{S \in \cup_n \mathcal{S}_N^n} \nu g_N : v \, dS
- \sum_{S \in \cup_n \mathcal{S}_{1D}^n} \int_S \nu \langle \langle u_h \rangle \rangle : \langle \langle \nabla_h v \rangle \rangle \, dS
- \sum_{S \in \cup_n \mathcal{S}_{1D}^n} \int_S \nu \langle \langle \nabla_h u_h \rangle \rangle : \langle \langle v \rangle \rangle \, dS
+ \sum_{S \in \cup_n \mathcal{S}_D^n} \sum_{K \in T_h} \eta^u \int_K \nu \tilde{L}_S(\langle \langle u_h \rangle \rangle) : \tilde{L}_S(\langle \langle v \rangle \rangle) \, dK
+ \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S \nu g_D \otimes \tilde{n} : \nabla_h v \, dS
- \sum_{S \in \cup_n \mathcal{S}_D^n} \sum_{K \in T_h} \eta^v \int_K \nu \tilde{L}_S(\mathcal{P} g_D \otimes \tilde{n}) : \tilde{L}_S(\langle \langle v \rangle \rangle) \, dK.
\]

Finally, we consider the last term \( T_p \) given by (18c). We use the relation (10), this time for vectors \( v \) and scalars \( q \), and replace \( \hat{p}_h \) in (18c) with:

\[
\hat{p}_h = \langle \langle p_h \rangle \rangle \text{ on } \mathcal{S}_I^n, \quad \hat{p}_h = p_h \text{ on } \mathcal{S}_{1D}^n, \quad \hat{p}_h = p_N \text{ on } \mathcal{S}_N^n.
\]

Note that on faces \( S \in \cup_n \mathcal{S}_{1D}^n \), we follow a similar approach as described in [10,17]. The condition on \( S \in \cup_n \mathcal{S}_N^n \) is required to have a well-posed problem for certain flow conditions. We obtain then the following expression for \( T_p \):

\[
T_p = -\int_{\mathcal{E}} p_h \nabla_h \cdot v \, d\mathcal{E} + \sum_{S \in \cup_n \mathcal{S}_{1D}^n} \int_S \langle \langle p_h \rangle \rangle \langle \langle v \rangle \rangle \, dS + \sum_{S \in \cup_n \mathcal{S}_N^n} \int_S p_N v \cdot \tilde{n} \, dS.
\]

Introducing all terms in (23), (30) and (32) into (17), we obtain the space-time DG discretization for the momentum equations (8b) in the Oseen equations:

Find \((u_h, p_h) \in V_h^{(p, p^*)} \times Q_h^{(p, p^*)} / \mathbb{R}\) such that \( \forall v \in V_h^{(p, p^*)} \):

\[
\mathcal{O}_h(u_h, v; w) + \mathcal{A}_h(u_h, v) + \mathcal{B}_h(p_h, v) = N_h(v; w) + F_h(v) + G_h(v).
\]

Here, the forms \( \mathcal{O}_h : V_h^{(p, p^*)} \times V_h^{(p, p^*)} \times H(\text{div} \mathcal{E}) \rightarrow \mathbb{R}, \mathcal{A}_h : V_h^{(p, p^*)} \rightarrow \mathbb{R}, \text{ and } \mathcal{B}_h : Q_h^{(p, p^*)} / \mathbb{R} \times V_h^{(p, p^*)} \rightarrow \mathbb{R} \) are defined as:

\[
\mathcal{O}_h(u_h, v; w) := -\int_{\mathcal{E}} u_h \otimes w : \nabla_h v \, d\mathcal{E} + \sum_{S \in \mathcal{F}_{\text{int}}} \int_S (\langle \langle u_h \rangle \rangle \otimes w + C_S \| u_h \|) : \langle \langle v \rangle \rangle \, dS
+ \sum_{S \in \mathcal{F}_p} \int_S u_h \otimes w : v \otimes n \, dS,
\]

\[
\mathcal{A}_h(u_h, v) := \int_{\mathcal{E}} \nu \nabla_h u_h : \nabla_h v \, d\mathcal{E} - \sum_{S \in \cup_n \mathcal{S}_{1D}^n} \int_S \nu \langle \langle u_h \rangle \rangle : \langle \langle \nabla_h v \rangle \rangle \, dS.
\]
\[ - \sum_{S \in \mathcal{U}_h \mathcal{T}_D} \int_{S} \nu \langle \nabla_h u_h \rangle : \langle v \rangle \ dS \]
\[ + \sum_{S \in \mathcal{U}_h \mathcal{T}_D} \sum_{K \in \mathcal{T}_h} \eta^K \int_{K} \nu \tilde{\mathcal{L}}_S(\langle u_h \rangle) : \tilde{\mathcal{L}}_S(\langle v \rangle) \ dK, \]

\[ \mathcal{B}_h(p_h, v) := -\int_{\mathcal{E}} p_h \nabla_h \cdot v \ d\mathcal{E} + \sum_{S \in \mathcal{U}_h \mathcal{T}_D} \int_{S} \langle p_h \rangle \langle v \rangle \ dS, \]  

and the linear forms \( N_h : V_h^{(p_1, p_3)} \times H(\text{div} \ \mathcal{E}) \rightarrow \mathbb{R} \), \( F_h : V_h^{(p_1, p_3)} \rightarrow \mathbb{R} \), \( G_h : V_h^{(p_1, p_3)} \rightarrow \mathbb{R} \) as:

\[ N_h(v; w) := -\sum_{S \in \mathcal{U}_h S_{D}} \int_{S} g_D \otimes w : v \otimes n \ dS + \int_{\partial \Omega_0} u_0 \cdot v \ dS, \]

\[ F_h(v) := -\sum_{S \in \mathcal{U}_h S_{D}} \int_{S} \nu g_D \otimes \bar{n} : \nabla_h v \ dS + \sum_{S \in \mathcal{U}_h S_N} \int_{S} \nu g_N \cdot v \ dS \]
\[ + \sum_{S \in \mathcal{U}_h S_N} \sum_{K \in \mathcal{T}_h} \eta^K \int_{K} \nu \tilde{\mathcal{L}}_S(\mathcal{P} \mathcal{g}_D \otimes \bar{n}) : \tilde{\mathcal{L}}_S(\langle v \rangle) \ dK, \]

\[ G_h(v) := \int_{\mathcal{E}} f \cdot v \ d\mathcal{E} - \sum_{S \in \mathcal{U}_h S_N} \int_{S} p_N v \cdot \bar{n} \ dS. \]

### 4.3 Space-time DG formulation for the continuity equation

The space-time DG for the continuity equation is obtained by multiplying (8c) with arbitrary test functions \( q \in \mathcal{Q}_h^{(p_1, p_3)} \), replacing \( u \) with \( u_h \in V_h^{(p_1, p_3)} \) and integrating over the element \( K \in \mathcal{T}_h \). After integration by parts twice with respect to \( x_1, \ldots, x_d \) and summation over all elements \( K \in \mathcal{T}_h \) we obtain:

\[ \int_{\mathcal{E}} (\nabla_h \cdot u_h) q \ d\mathcal{E} + \sum_{K \in \mathcal{T}_h} \int_{\mathcal{Q}_h^K} (\tilde{u}_h^P - u_h) \cdot \bar{n} q \ d\partial K = 0, \quad \forall q \in \mathcal{Q}_h^{(p_1, p_3)}, \]  

with \( \tilde{u}_h^P \) the numerical flux that has to be introduced to account for the multivalued traces on \( \mathcal{Q}_h^{n} \). Using (10), we can write (40) as

\[ \int_{\mathcal{E}} (\nabla_h \cdot u_h) q \ d\mathcal{E} + \sum_{S \in \mathcal{U}_h \mathcal{T}_{IDN}} \int_{S} \langle \tilde{u}_h^P - u_h \rangle : \langle q \rangle \ dS \]
\[ + \sum_{S \in \mathcal{U}_h \mathcal{T}_{I}} \int_{S} \langle \tilde{u}_h^P - u_h \rangle \langle q \rangle \ dS = 0. \]  

The next step is to find an appropriate numerical flux \( \hat{u}_h^P \). The following approach is considered for the numerical flux \( \hat{u}_h^P \) on \( S \in \mathcal{U}_{I} \mathcal{T}_{I} \):

\[ \hat{u}_h^P = \langle u_h \rangle + \gamma \langle p_h \rangle, \]  

15
with $\gamma > 0$. This approach is first introduced for a DG discretization in [9]. The term containing the pressure is called the pressure stabilization. On boundary faces we choose:

$$\hat{u}_h^p = g_D \text{ on } S_n^D, \quad \hat{u}_h^p = u_h \text{ on } S_n^N.$$  \hspace{1cm} (43)

Introducing the numerical fluxes (42)-(43) into (41), we obtain the final form of the weak formulation for the continuity equation (8c):

Find $(u_h, p_h) \in V_h^{(p_t,p_s)} \times Q_h^{(p_t,p_s)} / \mathbb{R}$ such that $\forall q \in Q_h^{(p_t,p_s)} / \mathbb{R}$:

$$-B_h(q, u_h) + C_h(p_h, q) = H_h(q),$$  \hspace{1cm} (44)

with $B_h$ defined in (36), while $C_h : Q_h^{(p_t,p_s)} / \mathbb{R} \times Q_h^{(p_t,p_s)} / \mathbb{R} \to \mathbb{R}$ and $H_h : Q_h^{(p_t,p_s)} \to \mathbb{R}$ are defined as:

$$C_h(p_h, q) := \sum_{S \in \cup_{n} S_n^D} \int_S \gamma \langle \langle p_h \rangle \rangle \cdot \langle \langle q \rangle \rangle \, dS,$$  \hspace{1cm} (45)

$$H_h(q) / \mathbb{R} := - \sum_{S \in \cup_{n} S_n^D} \int_S g_D \cdot \bar{n} q \, dS.$$  \hspace{1cm} (46)

The space-time DG formulation for the Oseen equations (8) is now stated as:

Find $(u_h, p_h) \in V_h^{(p_t,p_s)} \times Q_h^{(p_t,p_s)} / \mathbb{R}$ such that $\forall (v, q) \in V_h^{(p_t,p_s)} \times Q_h^{(p_t,p_s)} / \mathbb{R}$:

$$O_h(u_h, v; w) + A_h(u_h, v) + B_h(p_h, v) = N_h(v; w) + F_h(v) + G_h(v),$$  \hspace{1cm} (47)

$$-B_h(q, u_h) + C_h(p_h, q) = H_h(q).$$  \hspace{1cm} (48)

5 Stability and error analysis

In this section we discuss the stability and error analysis for the space-time DG formulation for the Oseen equations given by (47)-(48).

The first main result in this section is Theorem 5.9, stating that the space-time DG discretization for the Oseen equations has a unique solution and is unconditionally stable, also when equal order polynomials are used for the velocity and pressure. A central point in the stability analysis will be the derivation of an inf-sup condition and an upper bound for the solution of the Oseen equations in terms of initial and boundary data and the source term.

The second main result is an a priori $hp$-error estimate, given in Theorem 5.13 and Corollary 5.14, which shows the dependence of the error on the mesh size
and polynomial order $p$.

The analysis of the space-time DG formulation (47)-(48) is considerably simplified by the introduction of the following DG norm and norm for vector functions.

**Definition 5.1** The DG norm $\| \cdot \|_{DG}$, related to the bilinear form (35), is defined on $H^{(0,1)}(\mathcal{E}) + V_h^{(p_t,p_s)}$ as:

$$
\| v \|_{DG}^2 = \sum_{K \in T_h} \| v \|^2_{0,0,K} + \sum_{K \in T_h} \| \nabla_h v \|^2_{0,0,K} + \sum_{S \in \bigcup_{n} \mathcal{S}'} \sum_{K \in T_h} \| \hat{L}_S(\langle \langle \cdot \rangle \rangle) \|^2_{0,0,K},
$$

with $H^{(0,1)}(\mathcal{E})$ the anisotropic Sobolev space defined in Section (3.2). The seminorm $| \cdot |_S$, with $S$ is a set of faces, is defined on $Q_h^{(p_t,p_s)}$ as:

$$
| q |^2_S = \sum_{S \in \mathcal{S}} \int_S | \langle q \rangle |^2 dS. \tag{49}
$$

5.1 *Continuity, coercivity, and inf-sup condition*

The first result in this section establishes the continuity property of the bilinear form $\mathcal{A}_h$.

**Lemma 5.2** Let $\nu_m = \max_{x \in \mathcal{E}} \nu(x)$, $\eta_m = \max_{K \in T_h} \eta_K^s$, and $N_f$ be the number of faces of each element $K \in T_h$. Then there exists a constant $\alpha_A = \eta_m + 2\sqrt{N_f} + 1 > 0$, independent of the mesh size $h = \max_{K \in T_h} h_K$, such that

$$
| \mathcal{A}_h(u_h, v) | \leq \nu_m \alpha_A \| u_h \|_{DG} \| v \|_{DG}, \quad \forall u_h, v \in V_h^{(p_t,p_s)}.
$$

**Proof.** We consider the bilinear form $\mathcal{A}_h$ in the form

$$
\mathcal{A}_h(u_h, v) = \sum_{K \in T_h} \int_K \nu \nabla_h u_h : \nabla_h v \ dK - \sum_{K \in T_h} \int_K \nu \nabla_h u_h : \hat{L}(\langle \langle \cdot \rangle \rangle) \ dK
$$

$$
- \sum_{K \in T_h} \int_K \nu \hat{L}(\langle \langle u_h \rangle \rangle) : \nabla_h v \ dK
$$

$$
+ \sum_{S \in \bigcup_{n} \mathcal{S}'} \sum_{K \in T_h} \eta_K^s \int_K \nu \hat{L}_S(\langle \langle u_h \rangle \rangle) : \hat{L}_S(\langle \langle v \rangle \rangle) \ dK. \tag{50}
$$

As a consequence of (5), we have

$$
\| \hat{L}(\langle \langle v \rangle \rangle) \|^2_{0,0,K} \leq N_f \sum_{S \in \bigcup_{n} \mathcal{S}'} \| \hat{L}_S(\langle \langle v \rangle \rangle) \|^2_{0,0,K}. \tag{51}
$$

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with \( N_f \) the number of faces of each element \( K \in T_h \). Application of Schwarz’ inequality and the use of inequality (51) yields:

\[
|A_h(u_h, v)| \leq \nu_m \alpha_A \|u_h\|_{\text{DG}} \|v\|_{\text{DG}},
\]

with \( \nu_m = \max_{x \in E} \nu(x) \), \( \eta_m^u = \max_{K \in T_h} \eta_K \), and \( \alpha_A = \eta_m^u + 2\sqrt{N_f} + 1 \). □

Next, we establish the coercivity of \( A_h \).

\[ \text{Lemma 5.3} \]

Let \( \nu_0 = \min_{x \in E} \nu(x) \), \( \eta_0^u = \min_{K \in T_h} \eta_K \), \( N_f \) be the number of faces of each element \( K \in T_h \). Then there exists a constant \( \bar{\beta}_A > 0 \), independent of the mesh size \( h = \max_{K \in T_h} h_K \), such that

\[ A_h(v, v) \geq \nu_0 \bar{\beta}_A \|v\|_{\text{DG}}^2, \quad \forall v \in V_h^{(m,p_m)}, \]

with \( \bar{\beta}_A = \frac{\beta_A}{2} \min(1, 1/C_p^2) \), where \( \beta_A = \min(1 - \epsilon, \eta_0^u - \frac{N_f}{\epsilon}) \) for \( \epsilon \in (\frac{N_f}{\eta_0^u}, 1) \), and \( C_p^2 \) the coefficient in the discrete Poincaré inequality ([2], Lemma 2.1).

\[ \text{Proof.} \]

We start with replacing \( u_h \) in (50) with \( v \):

\[
A_h(v, v) = \sum_{K \in T_h} \int_K \nu \nabla_h v : \nabla_h v \, dK - 2 \sum_{K \in T_h} \int_K \nu \nabla_h v : \tilde{L}(\langle\langle v\rangle\rangle) \, dK + \sum_{S \in \bigcup_n S_{ID}} \sum_{K \in T_h} \eta_K^u \int_K \nu \tilde{L}_S(\langle\langle v\rangle\rangle) : \tilde{L}_S(\langle\langle v\rangle\rangle) \, dK. \tag{53}
\]

Using the Schwarz and arithmetic-geometric mean inequalities we have the inequality:

\[
2 \int_K \nabla_h v : \tilde{L}(\langle\langle v\rangle\rangle) \, dK \leq \epsilon \|\nabla_h v\|_{0,0,K}^2 + \frac{1}{\epsilon} \|\tilde{L}(\langle\langle v\rangle\rangle)\|_{0,0,K}^2, \tag{54}
\]

with \( \epsilon > 0 \). Introducing inequalities (51) and (54) into (53), we deduce

\[ A_h(v, v) \geq \nu_0 (1 - \epsilon) \sum_{K \in T_h} \|\nabla_h v\|_{0,0,K}^2 + \nu_0 \left( \eta_0 - \frac{N_f}{\epsilon} \right) \sum_{S \in \bigcup_n S_{ID}} \sum_{K \in T_h} \|\tilde{L}_S(\langle\langle v\rangle\rangle)\|_{0,0,K}^2. \tag{55} \]

If we take the parameters \( \eta_0^u > N_f \) and \( \epsilon \in (\frac{N_f}{\eta_0^u}, 1) \), then for \( 0 < \beta_A = \min(1 - \epsilon, \eta_0^u - \frac{N_f}{\epsilon}) \), we obtain

\[ A_h(v, v) \geq \nu_0 \beta_A \sum_{K \in T_h} \|\nabla_h v\|_{0,0,K}^2 + \nu_0 \beta_A \sum_{S \in \bigcup_n S_{ID}} \sum_{K \in T_h} \|\tilde{L}_S(\langle\langle v\rangle\rangle)\|_{0,0,K}^2. \tag{56} \]
Next, we recall the discrete inequality which is first discussed in [3], p. 1763, but now applied to vector functions in the space-time discretization:

$$\|v\|_{0,0,E} \leq C_p \left( \sum_{K \in T_h} \|\nabla h v\|_{0,0,K}^2 + \sum_{S \in \bigcup_{T_D} K \in T_h} \sum_{S \in T_D} \|\mathcal{L}_S(\langle v \rangle)\|_{0,0,K}^2 \right)^{1/2}. \quad (57)$$

Using (57) in (56), we then obtain:

$$A_h(v,v) \geq \nu_0 \left( \frac{\beta_A}{2 C_p} \sum_{K \in T_h} \|v\|_{0,0,K}^2 + \frac{\beta_A}{2} \sum_{K \in T_h} \|\nabla h v\|_{0,0,K}^2 \right. \left. + \frac{\beta_A}{2} \sum_{S \in \bigcup_{T_D} K \in T_h} \sum_{S \in T_D} \|\mathcal{L}_S(\langle v \rangle)\|_{0,0,K}^2 \right). \quad (58)$$

Choosing $\beta_A = \frac{\beta_A}{2} \min(1, 1/C_p^2)$ completes the proof. □

For the analysis of the DG algorithm we also need that the bilinear form $B_h$ (36) is continuous, which is stated in the next lemma.

**Lemma 5.4** Let $N_f$ be the number of faces of each element $K \in T_h$. Then there exists a constant $\alpha_B = \sqrt{N_f + 1}$, independent of the mesh size $h = \max_{K \in T_h} h_K$, such that

$$|B_h(q,v)| \leq \alpha_B \|q\|_{0,0,E} \|v\|_{DG}, \quad \forall (q,v) \in Q_h^{(p_t,p_s)}/\mathbb{R} \times V_h^{(p_t,p_s)}. \quad (61)$$

**Proof.** First, we consider the bilinear form $B_h(q,v)$ in the form

$$B_h(q,v) = - \sum_{K \in T_h} \int_K I_d q : \nabla h v \, dK + \sum_{K \in T_h} \int_K I_d q : \mathcal{L}(\langle v \rangle) \, dK. \quad (59)$$

Application of Schwarz’ inequality and inequality (51) yields:

$$|B_h(q,v)| \leq \|q\|_{0,0,E} \|v\|_{DG} + \sqrt{N_f} \|q\|_{0,0,E} \|v\|_{DG}. \quad (60)$$

Choosing $\alpha_B = \sqrt{N_f + 1}$ completes the proof. □

We also need to show that the bilinear form $B_h$ satisfies the inf-sup condition. First, we introduce the inf-sup condition for the Stokes equations in the domain $\Omega_t$ as follows:

$$\inf_{0 \neq q \in L^2(\Omega_t)/\mathbb{R}} \sup_{0 \neq v \in (H^1_0(\Omega_t))^d} \frac{- \int_{\Omega_t} q \nabla h \cdot v \, d\Omega}{\|v\|_{1,\Omega_t} \|q\|_{0,\Omega_t}} \geq C_{\Omega_t} > 0, \quad (61)$$

with the constant $C_{\Omega_t}$ depending only on $\Omega_t$. For a proof see [12]. If we fix now $q \in Q_h^{(p_t,p_s)}/\mathbb{R}$, then the inf-sup condition (61) guarantees that there exists a
\[ z(t) \in (H^1_0(\Omega_t))^d, \text{ with } t \in [0,T], \text{ such that:} \]

\[ -\int_{\Omega_t} q \nabla h \cdot z(t) \, d\Omega = \|q\|_{L^2(\Omega_t)}^2, \quad \text{with} \quad \|z(t)\|_{H^1(\Omega_t)} \leq C^{-1}_{\Omega_t} \|q\|_{L^2(\Omega_t)}, \quad (62) \]

where we use the Poincaré inequality to change \( |z(t)|_{H^1(\Omega_t)} \) into \( \|z(t)\|_{H^1(\Omega_t)} \). Integrating in time from \( t = 0 \) to \( t = T \), then results in the relation:

\[ -\int_{\mathcal{E}} q \nabla h \cdot z \, d\mathcal{E} = \|q\|_{L^2(\mathcal{E})}^2, \quad \text{with} \quad \|z\|_{H^1(\mathcal{E})} \leq C^{-1}_{\mathcal{E}} \|q\|_{L^2(\mathcal{E})}. \quad (63) \]

In the next lemma, we establish an \textit{inf-sup} condition for the bilinear form \( B_h(\cdot, \cdot) \), defined in (36).

**Lemma 5.5** The following \textit{inf-sup} condition holds for \((q,v) \in Q_{h}^{(p_t,p_s)}/\mathbb{R} \times V_{h}^{(p_t,p_s)}\):

\[ B_h(q,v) \geq C_{\inf} \|q\|_{L^2(\mathcal{E})}^2 \left( 1 - \frac{|q|_{L^2(\mathcal{E})}}{\|q\|_{L^2(\mathcal{E})}} \right), \quad \forall q \in Q_{h}^{(p_t,p_s)}/\mathbb{R}, \quad (64) \]

with \( C_{\inf} > 0 \) solely depending on \( C_{\mathcal{E}}^{-1} \) and the interpolation bounds.

**Proof.** To prove the \textit{inf-sup} condition, we follow similar steps as in [18]. First, we fix \( q \in Q_{h}^{(p_t,p_s)}/\mathbb{R} \) and define the \( L^2 \)-projection \( P_{K(t)} : (L^2(K(t)))^d \to V_{h}^{(0,p_s)}|_{K(t)}, \) with \( K(t) = K \cap \{t\} \) the space-time element at time \( t \), as:

\[ \int_{K(t)} (P_{K(t)}v) \cdot \phi \, dK = \int_{K(t)} v \cdot \phi \, dK, \quad \forall \phi \in V_{h}^{(0,p_s)}|_{K(t)}. \quad (65) \]

Next, for \( q \in Q_{h}^{(p_t,p_s)}/\mathbb{R} \), we consider the bilinear form \( B_h(q,z) \), for \( z \in L^2([0,T] , H^1_0(\Omega_t))^d \):

\[ B_h(q,z) = -\int_{\mathcal{E}} q \nabla h \cdot z \, d\mathcal{E} + \sum_{S \in \cup_{n} S_{ID}^n} \int_S \langle q \rangle \langle z \rangle \, dS. \]

Since \( \langle z \rangle = 0 \) on \( S \in \cup_{n} S_{ID}^n \), we can use (63) to obtain

\[ B_h(q,z) = \|q\|_{L^2(\mathcal{E})}^2. \quad (66) \]

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We consider now the bilinear form $B_h(q, v)$, with $v = P_{K(t)} z$:

\[
B_h(q, v) = B_h(q, z) + B_h(q, P_{K(t)} z - z),
\]

\[
= \|q\|_{0,0,E}^2 - \sum_{S \in \cup_n S_I^T} \int_S \|q\| \|P_{K(t)} z - z\| \, dS + \sum_{S \in \cup_n S_T^D} \int_S \|q\| \|P_{K(t)} z - z\| \, dS,
\]

\[
= \|q\|_{0,0,E}^2 + \sum_{S \in \cup_n S_T^D} \int_S \|P_{K(t)} z - z\| \, dS - \sum_{\kappa \in T_h} \int_{Q_{\kappa}^n} q\bar{n} \cdot (P_{K(t)} z - z) \, d\partial\kappa
\]

\[
+ \sum_{S \in \cup_n S_T^D} \int_S \|q\| \|P_{K(t)} z - z\| \, dS. \tag{67}
\]

The last equation is obtained using integration by parts with respect to $x_1, \ldots, x_d$, and the fact that $\bar{n} = 0$ at $\Omega_0$ and $\Omega_T$. Applying identity (10) (for vectors) into (67), using the orthogonality property of the $L^2$-projection $P_{K(t)}$ and the fact that $z = 0$ at $\partial\Omega_t$, we then obtain:

\[
B_h(q, v) = \|q\|_{0,0,E}^2 - \sum_{S \in \cup_n S_I^T} \int_S \|q\| \|P_{K(t)} z - z\| \, dS, \quad \forall q \in Q_{h}^{(p, p_s)} / \mathbb{R}. \tag{68}
\]

We estimate the second term on the right hand side in (68) as follows:

\[
| \sum_{S \in \cup_n S_T^D} \int_S \|q\| \|P_{K(t)} z - z\| \, dS | \leq \left( \sum_{S \in \cup_n S_T^D} \int_S \|q\|^2 \, dS \right)^{1/2} \times \left( \sum_{S \in \cup_n S_T^D} \int_S \|P_{K(t)} z - z\|^2 \, dS \right)^{1/2},
\]

\[
\leq C \|q\|_{\cup_n S_I^T} \left( \sum_{\kappa \in T_h} \|\nabla_h z\|_{0,0,K}^2 \right)^{1/2}, \tag{69}
\]

using the seminorm defined in (49) for $\gamma \approx h$ and a standard interpolation estimate. Using (69), we obtain the following inequality for $B_h(q, v)$:

\[
B_h(q, v) \geq \|q\|_{0,0,E}^2 - C \|q\|_{\cup_n S_I^T} \left( \sum_{\kappa \in T_h} \|\nabla_h z\|_{0,0,K}^2 \right)^{1/2} = C \|q\|_{0,0,E}^2 \left( 1 - \frac{\|q\|_{\cup_n S_I^T} \|z\|_{0,1,E}}{\|q\|_{0,0,E}^2} \right). \tag{70}
\]

Together with the fact that $\|z\|_{0,1,E} \leq C_{E}^{-1} \|q\|_{0,0,E}$ this completes the proof. \qed

We also show that for a divergence free convective velocity field $w$, the trilinear form $O_h$ (34) satisfies a stability relation.

**Lemma 5.6** For $w \in H(\text{div} \, 0, \mathcal{E})$, the trilinear form $O_h$ satisfies the following stability relation

\[
O_h(v; v; w) = \sum_{S \in \mathcal{F}} \|C_S^{1/2} \|v\|\|w\|_{0,0,S}, \quad \forall v \in V_h^{(p, p_s)},
\]

\[
\forall v \in V_h^{(p, p_s)}. \tag{21}
\]


Proof. First we replace $u_h$ in (34) with $v$:

$$
\mathcal{O}_h(v, v; w) = - \int_E v \otimes w : \nabla_h v \, dE + \sum_{S \in \mathcal{F}_{int}} \int_S \left( \{ v \} \otimes w + C_S \| v \| \right) : \{ v \} \, dS
$$

$$
+ \sum_{S \in \Gamma_p} \int_S v \otimes w : v \otimes n \, dS.
$$

(71)

Using the following relation: $v \otimes w : \nabla_h v = \frac{1}{2} (\nabla_h \cdot w)(v \cdot v) - \frac{1}{2} \nabla_h \cdot (v \otimes w) \cdot v$ and applying the Gauss’ theorem, we can write (71) as

$$
\mathcal{O}_h(v, v; w) = \frac{1}{2} \int_E (\nabla_h \cdot w)(v \cdot v) \, dE - \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} v \otimes w : v \otimes n \, d\partial K
$$

$$
+ \sum_{S \in \mathcal{F}_{int}} \int_S \{ v \} \otimes w : \{ v \} \, dS + \sum_{S \in \mathcal{F}_{int}} \int_S C_S \| v \| : \{ v \} \, dS
$$

$$
+ \sum_{S \in \Gamma_p} \int_S v \otimes w : v \otimes n \, dS.
$$

(72)

Using (22) and the fact that $w$ is a continuous function, we then obtain:

$$
\mathcal{O}_h(v, v; w) = \frac{1}{2} \int_E (\nabla_h \cdot w)(v \cdot v) \, dE - \frac{1}{2} \sum_{S \in \mathcal{F}_{int}} \int_S w \cdot [v \cdot v] \, dS
$$

$$
+ \sum_{S \in \mathcal{F}_{int}} \int_S \{ v \} \otimes w : \{ v \} \, dS + \sum_{S \in \mathcal{F}_{int}} \int_S C_S \| v \| : \| v \| \, dS
$$

$$
- \frac{1}{2} \sum_{S \in \Gamma_m} \int_S v \otimes w : v \otimes n \, dS + \frac{1}{2} \sum_{S \in \Gamma_p} \int_S v \otimes w : v \otimes n \, dS.
$$

(73)

Due to the continuity of $w$, on faces $S \in \mathcal{F}_{int}$, we have:

$$
\int_S \{ v \} \otimes w : \{ v \} \, dS = \frac{1}{2} \int_S w \cdot [v \cdot v] \, dS.
$$

(74)

Finally, as a consequence of (74), the fact that $\nabla_h \cdot w = 0$, and using the definition of $C_S$ in (20)-(21), we obtain that the trilinear form $\mathcal{O}_h$ satisfies the stability relation in Lemma 5.6. \(\square\)

5.2 Global stability

In this section we discuss a global stability result. First, we define the product space $\mathcal{H}_{h}^{(p_t, p_s)} = V_{h}^{(p_t, p_s)} \times Q_{h}^{(p_t, p_s)} / \mathbb{R}$, endowed with the norm:

$$
\|(v, q)\|_{DG}^2 = \|v\|_{DG}^2 + \|q\|_{\mathcal{F}_{int}}^2 + \sum_{S \in \mathcal{F}} \|C_{S}^{1/2}\| \|v\|_{0,0,S}^2.
$$

(75)

If we define the forms $\tilde{A} : \mathcal{H}_{h}^{(p_t, p_s)} \times \mathcal{H}_{h}^{(p_t, p_s)} \to \mathbb{R}$ and $\tilde{L} : \mathcal{H}_{h}^{(p_t, p_s)} \to \mathbb{R}$ as:
\[ \tilde{A}(u_h, p_h; v, q) := A_h(u_h, v) + B_h(p_h, v) - B_h(q, u_h) + C_h(p_h, q) + O_h(u_h, v; w), \quad (76) \]
\[ \tilde{L}(v, q) := N_h(v; w) + F_h(v) + G_h(v) + H_h(q), \quad (77) \]
then (47)-(48) are equivalent with:

Find \((u_h, p_h) \in Z_h^{(p_n, p_s)}\), such that:

\[ \tilde{A}(u_h, p_h; v, q) = \tilde{L}(v, q), \quad \forall (v, q) \in Z_h^{(p_n, p_s)}. \quad (78) \]

First, we discuss the consistency of the space-time DG method (47)-(48). The formulation is consistent when (47)-(48) is also satisfied by \((u, p) \in H^{1,2}(\mathcal{E})^d \times H^{0,1}(\mathcal{E})/\mathbb{R}\), the solution of (2)-(3):

\[ \tilde{A}(u, p; v, q) = \tilde{L}(v, q), \quad \forall (v, q) \in H^{1,1}(\mathcal{E})^d \times L^2(\mathcal{E})/\mathbb{R}. \quad (79) \]

The proof for consistency is straightforward. We replace \((u_h, p_h)\) in (76) by \((u, p)\). Since \((u, p)\) solves (2)-(3), we have \(\|u\| = 0, \langle u \rangle = 0, \langle p \rangle = 0, \{u\} = u, \{p\} = p\) on \(\mathcal{F}_{\text{int}}\). If we introduce these relations into (76), perform integration by parts, and use the boundary conditions (3), we obtain \(\tilde{L}(v, q)\). Subtracting (79) from (78) yields the Galerkin orthogonality property:

\[ \tilde{A}(u - u_h, p - p_h; v, q) = 0, \quad \forall (v, q) \in Z_h^{(p_n, p_s)}. \quad (80) \]

Next, we show the coercivity of \(\tilde{A}\).

**Lemma 5.7** Let \(\nu_0 = \min_{x \in \mathcal{E}} \nu(x)\) and assume that the constant \(\tilde{\beta}_A\) satisfies the conditions given in Lemma 5.3. Then there exists a constant \(C_{\tilde{A}} > 0\) such that

\[ \tilde{A}(v, q; v, q) \geq C_{\tilde{A}} \|v\|_D^2, \quad \forall (v, q) \in Z_h^{(p_n, p_s)}, \quad (81) \]

with \(C_{\tilde{A}} = \min(\nu_0 \tilde{\beta}_A, \gamma, 1)\).

**Proof.** First we replace \((u_h, p_h)\) in (76) with \((v, q)\) to obtain:

\[ \tilde{A}(v, q; v, q) = A_h(v, v) + C_h(q, q) + O_h(v, v; w). \]

Using Lemmas 5.3 and 5.6 and the norm \(\|\cdot\|_{\cup_s \mathcal{S}_I}\), we obtain:

\[ \tilde{A}(v, q; v, q) \geq \nu_0 \tilde{\beta}_A \|v\|_D^2 + \gamma |q|_{\cap_s \mathcal{S}_I}^2 + \sum_{S \in \mathcal{F}} \|C_S^{1/2} \|v\|_{\mathcal{I}_{0,S}}^2. \]
Taking $C_A = \min(v_0, \beta_A, \gamma, 1)$ completes the proof. □

In the next lemma we show that the solution to (47)-(48) is bounded by known data. For this purpose we define the $L^2$-projection $\mathcal{P}_S : (L^2(S))^d \to \gamma_S(V_h^{(m,p^\nu)})$, with $\gamma_S$ the trace at $S \in \cup_n S^n_D$, as:

$$\int_S \mathcal{P}_S v \cdot \psi \, dS = \int_S v \cdot \psi \, dS, \quad \forall \psi \in \gamma_S(V_h^{(m,p^\nu)}),$$

and we introduce the lifting operator $\hat{\mathcal{L}}_S(\gamma_S(V_h^{(m,p^\nu)})) \to V_h^{(m,p^\nu)}$, which satisfies:

$$\hat{\mathcal{L}}_S(\mathcal{P}_S g_D) = \mathcal{P}_S g_D \quad \text{at } S \in \cup_n S^n_D.$$

In addition, we will need the face bubble functions $\Psi_S$, with $S \subset \partial K$, which are defined as $\Psi_S = \tilde{\Psi}_i \circ F_k^{-1} \circ Q_k^{-1}$, with $i$ the index of the face $\tilde{S}_i \subset \partial K$ which is connected to the face $S \subset \partial K$ through the mapping $Q_k \circ F_k$. The face bubble functions $\tilde{\Psi}_i$ satisfy the conditions

$$\tilde{\Psi}_i(\hat{x}) > 0, \quad \forall \hat{x} \in \hat{K},$$

$$\tilde{\Psi}_i = 0, \quad \text{at } S_j, \quad \forall j \in \{1, \ldots, N_f\} \text{ and } j \neq i,$$

hence $\tilde{\Psi}_i = 0$ at $\partial \tilde{S}_i$. For any $q_h \in \mathcal{Q}_h^{(m,p^\nu)}$, we then have the inequality

$$|q_h|_S^2 \leq C_i |\Psi_S q_h|_S^2, \quad \forall S \in \mathcal{F},$$

with $C_i > 0$ a constant independent of the mesh size. The proof of this inequality follows directly from the fact that all norms are equivalent in a finite dimensional space and condition (82). See for instance [1].

**Lemma 5.8** Let $C_A$ satisfy the conditions given in Lemma 5.7, $\hat{C}_A = \frac{C_i}{\max(1, C_i)}$, $\nu_m = \max_{x \in \Omega} \nu(x)$, and $\eta_m = \max_{K \in T_h} \eta_k$. Then the solution to (47)-(48) satisfies the following upper bound:

\[
\begin{align*}
\hat{C}_A^2 \| (u_h, p_h) \|_{DG}^2 &\leq 4 \sum_{K \in T_h} \| f \|_{0,0,K}^2 + 2\nu_m^2 \left( N_f + (\eta_m^2)^2 \right) \sum_{S \in \cup_n S^n_D} \sum_{K \in T_h} \| \hat{\mathcal{L}}_S(\mathcal{P} g_D) \|_{0,0,S}^2 \\
&+ 24 \sum_{S \in \cup_n S^n_D} \| C_S^{1/2} g_D \|_{0,S}^2 + 12 \| C_S^{-1/2} u_0 \|_{0,0,\Omega}^2 + 4\nu_m^2 \sum_{S \in \cup_n S^n_D} \| C_S^{-1/2} g_N \|_{0,0,S}^2 \\
&+ 4 \sum_{S \in \cup_n S^n_D} \| C_S^{-1/2} p_N \|_{0,0,S}^2 + \sum_{S \in \cup_n S^n_D} \| \mathcal{P}_S \hat{\mathcal{L}}_S(\mathcal{P} g_D) \|_{0,1,E}^2 \\
&+ \frac{2\nu_m \hat{C}_A}{C_{inf}} \sum_{S \in \cup_n S^n_D} \| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P} g_D) \|_{DG} + \sum_{S' \in \cup_n S^n_D} \| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P} g_D) \|_{0,1,E} \\
&+ 4 \sum_{S \in \cup_n S^n_D} \| \nabla_h (\Psi_S \hat{\mathcal{L}}_S(\mathcal{P} g_D)) \cdot w \|_{0,0,K}^2
\end{align*}
\]
Lemma 5.7, with \((v, q)\).

We start with (83) with

Proof

inequality

Next, we find an upper bound for

defined in Section 3.3 and inequality (51), we can write

\[ F_h(\Psi S\hat{\mathcal{L}}(P_{SgD})) : \nabla_h u_h \, d\mathcal{E} \]

\[ + \sum_{K \in T_h} \int_K \nu \mathcal{L}(\langle \mathcal{P}_{gD} \rangle) : \nabla_h u_h \, d\mathcal{E} \]

\[ + \nu_m \sum_{K \in T_h} \int_K \nabla_h u_h \, d\mathcal{E} \]

\[ \leq \nu_m \sum_{K \in T_h} \| \nabla_h u_h \|_{0,0,K}^2 + \nu_m \sum_{K \in T_h} \| \mathcal{L}(\langle \mathcal{P}_{gD} \rangle) \|_{0,0,K}^2 \]

Proof. We start with (83) with \(q_h = p_h\), then the DG-norm (75) satisfies the inequality

\[ \|(u_h, p_h)\|_{DG}^2 \leq \max(1, C_i)\|(u_h, \Psi p_h)\|_{DG}^2, \]

with \(\Psi p_h = \sum_{S \in F} \Psi_{Sp_h}\). Next, using (84), the coercivity estimate given by Lemma 5.7, with \((v, q) = (u_h, \Psi p_h)\), and (78) we obtain

\[ C\mathcal{A}\|(u_h, p_h)\|_{DG}^2 \leq \max(1, C_i)\mathcal{A}(u_h, \Psi p_h; u_h, \Psi p_h) = \max(1, C_i)\hat{\mathcal{L}}(u_h, \Psi p_h), \]

where we used the fact that \(\hat{\mathcal{A}}(u_h, \Psi p_h; u_h, \Psi p_h)\) does not depend on \(\Psi p_h\). The final estimate is now obtained by determining an upper bound for each term in \(\hat{\mathcal{L}}\) separately.

First, we consider \(N_h\) given by (37):

\[ N_h(u_h; w) = - \sum_{S \in \cup_n S_{S_m}} \int_S g_D \otimes w : u_h \otimes n \, dS + \int_{\Omega_0} u_0 \cdot u_h \, dS, \]

\[ = 2 \sum_{S \in \cup_n S_{S_m}} \int_S C_S \|g_D\| : \|u_h\| \, dS + \int_{\Omega_0} \|u_h\| \cdot \|u_h\| \, dS, \]

\[ \leq \frac{3}{2} \epsilon_1 \sum_{S \in F} \|C_S^{1/2}\|_{0,0,S}^2 \|u_h\|_{0,0,S} + \frac{1}{\epsilon_1} \sum_{S \in \cup_n S_{S_m}} \|C_S^{1/2} g_D\|_{0,0,S}^2 \]

\[ + \frac{1}{2\epsilon_1} \|C_S^{-1/2} u_0\|_{0,0,\Omega_0}^2. \] (85)

Next, we find an upper bound for \(F_h\) given by (38). Using the lifting operators defined in Section 3.3 and inequality (51), we can write \(F_h\) as:

\[ F_h(u_h) = - \sum_{K \in T_h} \int_K \nu \mathcal{L}(\langle \mathcal{P}_{gD} \rangle) : \nabla_h u_h \, d\mathcal{E} \]

\[ + \sum_{S \in \cup_n S_N} \sum_{K \in T_h} \eta^\nu_{K} \int_K \nu \mathcal{L}(\langle \mathcal{P}_{gD} \rangle) : \mathcal{L}(\langle \mathcal{P}_{gD} \rangle) \, dK \]

\[ + \sum_{S \in \cup_n S_N} \int_S \nu g_N \cdot u_h \, dS, \]

\[ \leq \nu_m \frac{\epsilon_2}{2} \sum_{K \in T_h} \| \nabla_h u_h \|_{0,0,K}^2 + \nu_m \frac{\epsilon_3}{2\epsilon_2} \sum_{S \in \cup_n S_N} \sum_{K \in T_h} \| \mathcal{L}(\langle \mathcal{P}_{gD} \rangle) \|_{0,0,K}^2 \]
An upper bound for $G_h$, given by (39), is obtained as follows:

$$G_h(u_h) = \int_{\mathcal{E}} f \cdot u_h \, d\mathcal{E} - \sum_{S \in \cup_n S_n^D} \int_S p_N u_h \cdot \bar{n} \, dS,$$

$$\leq \frac{\epsilon_5}{2} \sum_{K \in T_h} \|u_h\|_{0,0,K}^2 + \frac{1}{2\epsilon_5} \sum_{K \in T_h} \|f\|_{0,0,K}^2$$

$$+ \frac{\epsilon_6}{2} \sum_{S \in \cup_n S_n^D} \|C_s^{1/2}\| \|u_h\|_{0,0,S}^2 + \frac{1}{2\epsilon_6} \sum_{S \in \cup_n S_n^D} \|C_s^{1/2}p_N\|_{0,0,S}^2. \quad (87)$$

Finally, for the estimation of $H_h$, given by (46), we first use the relation:

$$H_h(\Psi_{p_h}) = - \sum_{S \in \cup_n S_n^D} \int_S \Psi_S \mathcal{P}_{SGD} \cdot \bar{n} p_h \, dS. \quad (88)$$

An estimate for $H_h(\Psi_{p_h})$ can now be obtained by choosing the test function $\psi$ in the weak formulation (33) as $\psi = \Psi_S \hat{L}_S(\mathcal{P}_{SGD})$ in the element $K$ connected to the boundary face $S \in \cup_n S_n^D$ and zero elsewhere:

$$\sum_{S \in \cup_n S_n^D} \int_S \Psi_S \mathcal{P}_{SGD} \cdot \bar{n} p_h \, dS$$

$$= \sum_{S \in \cup_n S_n^D} \left( \int_{\mathcal{E}} p_h \nabla_h \cdot (\Psi_S \hat{L}_S(\mathcal{P}_{SGD})) \, d\mathcal{E} + \int_{\mathcal{E}} u_h \otimes w \cdot \nabla_h (\Psi_S \hat{L}_S(\mathcal{P}_{SGD})) \, d\mathcal{E} \right.$$

$$- \sum_{S' \subset \Gamma_p} \int_{S'} u_h \otimes w : \Psi_S \hat{L}_S(\mathcal{P}_{SGD}) \otimes n \, dS - A_h(u_h, \Psi_S \hat{L}_S(\mathcal{P}_{SGD}))$$

$$+ N_h(\Psi_S \hat{L}_S(\mathcal{P}_{SGD}); w) + F_h(\Psi_S \hat{L}_S(\mathcal{P}_{SGD})) + G_h(\Psi_S \hat{L}_S(\mathcal{P}_{SGD})),$$

where we used the fact that $\Psi_S \hat{L}_S(\mathcal{P}_{SGD}) = 0$ at $\partial \mathcal{K}/S$. We estimate now the individual terms on the right hand side. The first contribution has an upper bound:

$$\sum_{S \in \cup_n S_n^D} \int_{\mathcal{E}} p_h \nabla_h \cdot (\Psi_S \hat{L}_S(\mathcal{P}_{SGD})) \, d\mathcal{E}$$

$$\leq \|p_h\|_{0,0,\mathcal{E}} \sum_{S \in \cup_n S_n^D} \|\Psi_S \hat{L}_S(\mathcal{P}_{SGD})\|_{0,1,\mathcal{E}}$$

$$\leq (p_h_{\cup_n S_n^D} + \alpha \inf_{S' \subset \cup_n S_n^D} \sum_{S' \subset \cup_n S_n^D} \|\Psi_{S'} \hat{L}_{S'}(\mathcal{P}_{S'D})\|_{DG})$$

$$\sum_{S \in \cup_n S_n^D} \|\Psi_S \hat{L}_S(\mathcal{P}_{SGD})\|_{0,1,\mathcal{E}}$$
Collecting all terms, introducing the following coefficients:

\[
\text{solution of (47)-(48) in } \mathbb{Z}
\]

The result of Lemmas 5.7 and 5.8 immediately imply the existence of a unique \( B \) and the boundedness of \( B^5.5 \), and the boundedness of \( \text{discretization given by conditions given in Lemma 5.7} \). Then the space-time discontinuous Galerkin has as upper bound:

\[
\text{Theorem 5.9}
\]

\[\begin{align*}
\text{Let } K \in T, \text{ number of faces of each element } &u \text{ (85)-(87) with } \\
\text{Corollary 5.10} \quad \text{The space-time DG discretization of the Oseen equations, } &\text{given by (47)-(48), is unconditionally stable.}
\end{align*}\]
This result is a direct consequence of Lemma 5.8, which is valid for any value of the spatial mesh size and time step, and Theorem 5.9. □

5.3 $hp$-Error analysis

The results obtained in Sections 5.1 and 5.2 now make it possible to analyze the $hp$-error of the space-time discretization. Since the spatial mesh size $h$ and time step $\Delta t$ are in general quite different we make the dependence of the error on these parameters explicit. For the error analysis we define the projection $P_u : L^2(\mathcal{E}) \to V_h^{(p_t, p_s)}$ as:

$$\sum_{K \in T_h} (P_u z, v)_K = \sum_{K \in T_h} (z, v)_K, \quad \forall v \in V_h^{(p_t, p_s)},$$ (89)

and the projection $P_p : L^2(\mathcal{E}) \to Q_h^{(p_t, p_s)}$ as:

$$\sum_{K \in T_h} (P_p q, q)_K = \sum_{K \in T_h} (p, q)_K, \quad \forall q \in Q_h^{(p_t, p_s)},$$ (90)

which can be used to decompose the global error $(u - u_h, p - p_h)$ as:

$$(u - u_h, p - p_h) = (u - P_u u, p - P_p p) + (P_u u - u_h, P_p p - p_h)$$

$$\equiv (\rho_u, \rho_p) + (\theta_u, \theta_p),$$ (91)

with $(\rho_u, \rho_p)$ the interpolation error and $(\theta_u, \theta_p)$ the discretization error.

In the next lemma we give an estimate for $(\theta_u, \theta_p)$ in terms of $(\rho_u, \rho_p)$.

**Lemma 5.11** Let $C_A$ satisfy the conditions given in Lemma 5.7, $\alpha_A, \alpha_B$ be defined in Lemmas 5.2 and 5.4, respectively, $\bar{w}_m = \max_{K \in T_h} \| \bar{w} \|_{L^\infty(K)}, \nu_m = \max_{x \in \mathcal{E}} \nu(x), C_S$ given by (20) and the stabilization coefficient $\gamma > 0$. Then the functions $(\theta_u, \theta_p)$ satisfy the inequality:

$$\frac{1}{4} C_A^2 \| (\theta_u, \theta_p) \|_{DG}^2 \leq \left( \nu_m^2 \alpha_A + \bar{w}_m^2 + (1 + C_A/C_{inf}) \alpha_B^2 \right) \| \rho_u \|_{DG}^2$$

$$\quad + \alpha_B^2 \| \rho_p \|_{0,0,\mathcal{E}}^2 + \gamma^2 \| \rho_p \|_{0,S_i}^2$$

$$\quad + \sum_{S \in \mathcal{F}_{int}} \| C_{S_i}^{1/2} \| \| \rho_u \|_{0,0,S}^2 + \sum_{S \in \mathcal{F}} \| C_{S_i}^{1/2} \| \| \rho_u \|_{0,0,S}^2.$$

**Proof.** To prove this lemma, we use the orthogonality relation (80) to obtain:

$$\tilde{A}(\rho_u + \theta_u, \rho_p + \theta_p; v, q) = 0, \quad \forall (v, q) \in Z_h^{(p_t, p_s)}.$$ (92)

Taking $(v, q) = (\theta_u, \theta_p)$, we then obtain: $\tilde{A}(\theta_u, \theta_p; \theta_u, \theta_p) = -\tilde{A}(\rho_u, \rho_p; \theta_u, \theta_p)$.
We continue with an estimate for $|\tilde{A}(\rho_u, \rho_p; \theta_u, \theta_p)|$:

$$|\tilde{A}(\rho_u, \rho_p; \theta_u, \theta_p)| \leq |A_h(\rho_u, \theta_u)| + |B_h(\rho_p, \theta_u)| + |C_h(\rho_p, \theta_p)| + |O_h(\rho_u, \theta_u; w)|. \quad (93)$$

The first term in (93) can be estimated using Lemma 5.2, while the second and the third term can be estimated using Lemma 5.4. The fourth term can be estimated easily as follows:

$$|C_h(\rho_p, \theta_p)| \leq \gamma |\rho_p|_{\bigcup_n S^n_1} |\theta_p|_{\bigcup_n S^n_1}. \quad (94)$$

Next, we derive an estimate for the fifth term in (93):

$$|O_h(\rho_u, \theta_u; w)| \leq \int_E \rho_u \cdot \frac{\partial \theta_u}{\partial t} \, d\mathcal{E} + \int_E \rho_u \otimes \tilde{w} : \nabla_h \theta_u \, d\mathcal{E} + \sum_{S \in F_{\text{int}}} \int_S \|\rho_u\| \otimes \theta_u \, dS + \sum_{S \in F_{\text{int}}} \int_S C_S \|\rho_u\| \otimes \|\theta_u\| \, dS + \sum_{S \subseteq F_p} \int_S C_S \rho_u \otimes \theta_u \otimes n \, dS. \quad (95)$$

Since $\theta_u \in V_h^{(\rho_u, \rho_u)}$, which is polynomial, hence $\frac{\partial \theta_u}{\partial t} \in V_h^{(\rho_u, \rho_u)}$, we can use the orthogonality relation (89) to eliminate the first term in (95). The remaining terms in (95) are estimated as follows:

$$|O_h(\rho_u, \theta_u; w)| \leq \sum_{K \in T_h} \|\tilde{w}\|_{L^\infty(K)} \|\rho_u\|_{0,0,K} \|\nabla_h \theta_u\|_{0,0,K} + \sum_{S \in F_{\text{int}}} \|C_S^{1/2} \|\rho_u\|_{0,0,S} \|C_S^{1/2} \|\theta_u\|_{0,0,S} + \sum_{S \in F} \|C_S^{1/2} \|\rho_u\|_{0,0,S} \|C_S^{1/2} \|\theta_u\|_{0,0,S} \leq \bar{w}_m \|\rho_u\|_{DG} \|\theta_u\|_{DG} + \left( \sum_{S \in F_{\text{int}}} \|C_S^{1/2} \|\rho_u\|_{0,0,S} \right)^{1/2} \left( \sum_{S \in F} \|C_S^{1/2} \|\theta_u\|_{0,0,S} \right)^{1/2} + \left( \sum_{S \in F} \|C_S^{1/2} \|\rho_u\|_{0,0,S} \right)^{1/2} \left( \sum_{S \in F} \|C_S^{1/2} \|\theta_u\|_{0,0,S} \right)^{1/2}, \quad (96)$$

with $\bar{w}_m = \max_{K \in T_h} \|\tilde{w}\|_{L^\infty(K)}$. We also need an estimate for $\|\theta_p\|_{0,0,\varepsilon}$, which can be obtained directly using Lemmas 5.4 and 5.5:

$$\|\theta_p\|_{0,0,\varepsilon} \leq \|\theta_p|_{\bigcup_n S^n_1} + (\alpha_B/C_{inj}) \|\rho_u\|_{DG}. \quad (97)$$

Collecting all terms we obtain the estimate:

$$|\tilde{A}(\rho_u, \rho_p; \theta_u, \theta_p)| \leq \nu_m \alpha_A \|\rho_u\|_{DG} \|\theta_u\|_{DG} + \alpha_B \|\rho_p\|_{0,0,\varepsilon} \|\theta_u\|_{DG}$$
Applying the Schwarz’ inequality and arithmetic-geometric mean inequality finally yields:

\[
|\tilde{A}(\rho_u, \rho_p; \theta_u, \theta_p)| \leq \frac{3}{4} \beta \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2 + \frac{1}{\beta} \left( \nu_m^2 \alpha_A^2 + \bar{w}_m^2 \right) \|\rho_u\|_{DG}^2 + \frac{1}{\beta} \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2 + \frac{1}{\beta} \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2 \\
+ \frac{2}{\beta} \left( \nu_m^2 \alpha_A^2 + \bar{w}_m^2 \right) \|\rho_p\|_{0,0,\mathcal{E}}^2 + \frac{1}{\beta} \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2 + \frac{1}{\beta} \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2 \\
+ \frac{2}{\beta} \|\rho_p\|_{0,0,\mathcal{E}}^2 + \frac{1}{\beta} \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2 + \frac{1}{\beta} \frac{\alpha_B^2}{C_{inf}} \|\rho_u\|_{DG}^2. 
\]

(98)

Combining (99) with Lemma 5.7 for \((v,q) = (\theta_u, \theta_p)\), taking \(\beta = C_A\), and multiplying the result with \(C_A\) completes the proof. □

We now present upper bounds for the interpolation error \((\rho_u, \rho_p)\), which are taken from [19].

**Lemma 5.12** Assume that \(\mathcal{K}\) is a space-time element in \(\mathbb{R}^{d+1}\) constructed via two mappings \(Q_{\mathcal{K}}, F_{\mathcal{K}}\), with \(F_{\mathcal{K}} : \hat{\mathcal{K}} \to \mathcal{K}\) and \(Q_{\mathcal{K}} : \hat{\mathcal{K}} \to \mathcal{K}\). Assume also that \(h_{i,\mathcal{K}}, i = 1, \ldots, d\) is the edge length of \(\hat{\mathcal{K}}\) in the \(x_i\) direction, and \(\Delta_{x, t}\) the edge length in the \(x_0\) direction (see illustration in Fig. 1 for \(d = 2\)). Let \(u|_{\mathcal{K}} \in H^{(k_{u, \mathcal{K}}, k_{v, \mathcal{K}} + 1)}(\mathcal{K})\), with \(k_{u, \mathcal{K}}, k_{v, \mathcal{K}} \geq 0\), and \(p|_{\mathcal{K}} \in H^{(k_{p, \mathcal{K}}, k_{v, \mathcal{K}} + 1)}(\mathcal{K})\), with \(k_{p, \mathcal{K}}, k_{v, \mathcal{K}} \geq 0\). Let \(P_u\) denote the \(L^2\) projection of \(u\) onto the finite element space \(V_h^{(p_u,p_v)}\), and \(P_p\) the \(L^2\) projection of \(p\) onto the finite element space \(Q_h^{(p_u,p_v)}\). Then the projection error \(\rho_u\) obeys the error bounds:

\[
\|\rho_u\|_{0,0,\mathcal{K}}^2 \leq C Z_{\mathcal{K}}^u, \quad \|\nabla_h \rho_u\|_{0,0,\mathcal{K}}^2 \leq C N_{\mathcal{K}}^u, \quad \|\rho_u\|_{0,0,\mathcal{K}}^2 \leq C (A_{\mathcal{K}}^u + B_{\mathcal{K}}^u),
\]

where

\[
Z_{\mathcal{K}}^u = \sum_{i=1}^{d} \left( \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u} \right)^{2s_{u,\mathcal{K}}} \|\nabla^2 \theta_{0,0,\mathcal{K}}^u\|_{0,0,\mathcal{K}}^2 + \left( \frac{\Delta_{x, t}^u}{p_{t,\mathcal{K}}^u} \right)^{2s_{u,\mathcal{K}}} \|\nabla^2 \theta_{0,0,\mathcal{K}}^u\|_{0,0,\mathcal{K}}^2.
\]
\[ N^u_K = \sum_{i=1}^{d} \frac{h_{i,K}^{2u}}{(p_{s,K})^{2u}} \| \partial_i^{2u} + 1 u \|_{0,0,K}^2 + \sum_{i=1}^{d} \sum_{j \neq i} \frac{h_{j,K}^{2u+2}}{(p_{s,K})^{2u+2}} \| \partial_j^{2u+1} \partial_i u \|_{0,0,K}^2 \]
\[ + \sum_{i=1}^{d} \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u+2} \| \partial_0^{2u} + 1 \partial_i u \|_{0,0,K}^2, \]
\[ A^u_K = \sum_{i=1}^{d} \left( \frac{h_{i,K}}{p_{s,K}^u} \right)^{2u+1} \| \partial_i^{2u+1} u \|_{0,0,K}^2 + \sum_{i=1}^{d} \sum_{j \neq i} \frac{1}{h_{i,K}} \left( \frac{h_{j,K}}{p_{s,K}^u} \right)^{2u} \| \partial_j^{2u} \partial_i u \|_{0,0,K}^2 \]
\[ + \sum_{i=1}^{d} \sum_{j \neq i} \frac{h_{i,K}}{p_{s,K}^u} \left( \frac{h_{j,K}}{p_{s,K}^u} \right)^{2u} \| \partial_j^{2u} \partial_i u \|_{0,0,K}^2, \]
\[ B^u_K = \sum_{i=1}^{d} \frac{1}{h_{i,K}} \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u} \| \partial_0^{2u} u \|_{0,0,K}^2 + \sum_{i=1}^{d} \frac{h_{i,K}}{p_{s,K}^u} \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u} \| \partial_0^{2u} \partial_i u \|_{0,0,K}^2 \]
\[ + \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u+1} \| \partial_0^{2u} + 1 \partial_i u \|_{0,0,K}^2, \]
\[ + \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u} \| \partial_0^{2u} \partial_i u \|_{0,0,K}^2, \]
\[ with \ p_{s,K}^u \ and \ p_{s,K}^u \ the \ local \ polynomial \ degree \ in \ time \ and \ space \ for \ u, \ respectively, \ on \ element \ K, \ 0 < s_0^u \leq \min(p_{s,K}^u+1, k_{t,K}^u+1), \ 0 < s_0^u \leq \min(p_{s,K}^u+1, k_{t,K}^u+1), \ 0 < s_0^u \leq \min(p_{s,K}^u+1, k_{t,K}^u+1), \ 0 < s_0^u \leq \min(p_{s,K}^u+1, k_{t,K}^u+1), \ 0 < t_0^u \leq \min(p_{s,K}^u, k_{t,K}^u), \ and \ 0 < t_0^u \leq \min(p_{s,K}^u, k_{t,K}^u). \ Further \ we \ have: \]
\[ \sum_{S \in \cup_{K} S_{TD}} \| \tilde{C}_S(\| \rho_u \|) \|_{0,0,E}^2 \leq C \sum_{K \in T_h} (R^u_K + T^u_K), \]

where:
\[ R^u_K = \sum_{i=1}^{d} \frac{(p_{s,K}^u)^2}{h_{i,K}} \left( \frac{h_{i,K}}{p_{s,K}^u} \right)^{2u+1} \| \partial_i^{2u+1} u \|_{0,0,K}^2 + \sum_{i=1}^{d} \sum_{j \neq i} \left( \frac{p_{s,K}^u}{h_{i,K}} \right) \left( \frac{h_{j,K}}{p_{s,K}^u} \right)^{2u+1} \| \partial_j^{2u} \partial_i u \|_{0,0,K}^2 \]
\[ + \sum_{i=1}^{d} \sum_{j \neq i} \frac{h_{i,K}}{p_{s,K}^u} \left( \frac{h_{j,K}}{p_{s,K}^u} \right)^{2u+1} \| \partial_j^{2u} \partial_i u \|_{0,0,K}^2, \]
\[ T^u_K = \sum_{i=1}^{d} \frac{(p_{s,K}^u)^2}{h_{i,K}} \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u} \| \partial_0^{2u} u \|_{0,0,K}^2 + \sum_{i=1}^{d} \frac{p_{s,K}^u}{h_{i,K}} \left( \frac{\Delta_n t}{p_{s,K}^u} \right)^{2u} \| \partial_0^{2u} \partial_i u \|_{0,0,K}^2, \]

The projection error \( \rho_p \) obeys the error bounds:
\[ \| \rho_p \|_{0,0,K}^2 \leq C Z^p_K, \quad \| \rho_p \|_{0,0,0,K}^2 \leq C (A^p_K + B^p_K), \]

where
\[ Z^p_K = \sum_{i=1}^{d} \left( \frac{h_{i,K}}{p_{s,K}^p} \right)^{2p} \| \partial_i^{2p} p \|_{0,0,K}^2 + \left( \frac{\Delta_n t}{p_{s,K}^p} \right)^{2p} \| \partial_0^{2p} p \|_{0,0,K}^2, \]
Using Lemmas 5.11 and 5.12, we immediately obtain the following error estimates:

**Theorem 5.13** Suppose $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_\mathcal{K}, F_\mathcal{K}$, with $F_\mathcal{K} : \mathcal{K} \rightarrow \mathcal{K}$ and $Q_\mathcal{K} : \mathcal{K} \rightarrow \mathcal{K}$. Suppose also that $h_{i,\mathcal{K}}, i = 1, \ldots, d$ is the edge length of $\mathcal{K}$ in the $x_i$ direction, and $\Delta t$ the edge length in the $x_0$ direction. Let $u|_\mathcal{K} \in H^{(k^u_{i,\mathcal{K}}+1,k^u_{s,\mathcal{K}}+1)}(\mathcal{K})$, with $k^u_{i,\mathcal{K}}, k^u_{s,\mathcal{K}} \geq 0$, and $p|_\mathcal{K} \in H^{(k^p_{i,\mathcal{K}}+1,k^p_{s,\mathcal{K}}+1)}(\mathcal{K})$, with $k^p_{i,\mathcal{K}}, k^p_{s,\mathcal{K}} \geq 0$. Let $(u_h, p_h) \in V_h^{(p_{i,\mathcal{K}})} \times Q_h^{(p_{s,\mathcal{K}})} / \mathbb{R}$ be the discontinuous Galerkin approximation to $(u, p)$ in (47)-(48). Then the following error bound holds:

$$\| (u - u_h, p - p_h) \|_{DG}^2 \leq 2C(1 + a_1) \sum_{K \in T_h} (Z^K + N^K + R^K + T^K) + 2C \bar{C}_S (1 + a_4) \sum_{K \in T_h} (A^K + B^K) + 2C a_2 \sum_{K \in T_h} Z^K + 2C (1 + a_3) \sum_{K \in T_h} (A^K + B^K),$$

with $\bar{C}_S = \max_{K \in T_h} \| C_S \|_{L^\infty(\mathcal{K})}$ and

$$a_1 = \frac{4}{C_A^2} \left( \nu^2 \alpha_A^2 + \omega_m^2 + (1 + C_A/C_{inf}) \alpha_B^2 \right), \quad a_2 = \frac{4 \alpha_B^2}{C_A^2}, \quad a_3 = \frac{4 \gamma^2}{C_A^2}, \quad a_4 = \frac{8}{C_A^2}.$$

**Proof.** The estimate follows directly from taking the DG norm of both sides.
Table 1

$L^2$-norm of the error for $u$ and $p$ for different values of stabilization parameter $\gamma$.

<table>
<thead>
<tr>
<th>$(p_t, p_s)$</th>
<th>$\gamma = 10$</th>
<th>$\gamma = 100$</th>
<th>$\gamma = 1000$</th>
<th>$\gamma = 10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2)$</td>
<td>$u$</td>
<td>8.7806E - 05</td>
<td>8.7833E - 05</td>
<td>8.7836E - 05</td>
</tr>
<tr>
<td></td>
<td>$p$</td>
<td>5.2186E - 03</td>
<td>5.2162E - 03</td>
<td>5.2159E - 03</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$u$</td>
<td>8.8268E - 05</td>
<td>8.8297E - 05</td>
<td>8.8300E - 05</td>
</tr>
<tr>
<td></td>
<td>$p$</td>
<td>5.5157E - 03</td>
<td>5.5136E - 03</td>
<td>5.5133E - 03</td>
</tr>
<tr>
<td>$(1, 3)$</td>
<td>$u$</td>
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<td>1.6869E - 06</td>
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<tr>
<td>$(3, 3)$</td>
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<td>1.1899E - 06</td>
<td>1.1899E - 06</td>
</tr>
<tr>
<td></td>
<td>$p$</td>
<td>5.4106E - 05</td>
<td>5.4096E - 05</td>
<td>5.4095E - 05</td>
</tr>
</tbody>
</table>

of (91) and using the triangle inequality to express the error in terms of the interpolation error $(\rho_u, \rho_p)$ and the discretization error $(\theta_u, \theta_p)$. The final error estimate is obtained using Lemma 5.11 and the interpolation estimates given by Lemma 5.12. □

**Corollary 5.14** When $u$ and $p$ are sufficiently smooth, the spatial shapes of all elements $K \in T_h$ are regular: $h = \max_K h_K, \forall K \in T_h$, and uniform polynomial degrees $(p_t^u, p_s^u)$ and $(p_t^p, p_s^p)$ are used for all elements $K \in T_h$, then we obtain the error bound:

$$\| (u - u_h, p - p_h) \|^2_{DG} \leq 2C \left( (1 + a_1) \frac{h^{2p_s^u}}{(p_s^u)^{2p_t^u + 1}} + (p_s^p)^2 \frac{\Delta_t h^{2p_t^u + 1}}{h} + C_S (1 + a_4) \frac{h^{2p_t^p + 1}}{(p_t^u)^{2p_t^u + 1}} \right) |u|^2_{p_t^u + 1, p_s^u + 1, \varepsilon} + 2C \left( a_2 \frac{h^{2p_t^p + 2}}{(p_t^p)^{2p_t^p + 2}} + (1 + a_3) \frac{h^{2p_t^p + 1}}{(p_t^p)^{2p_t^p + 1}} \right) |p|^2_{p_t^p + 1, p_s^p + 1, \varepsilon}. $$

6 Numerical results

In this section we provide several numerical experiments in two spatial dimensions to investigate the order of accuracy of the space-time DG discretization given by (47)-(48). As exact solution in the tests, we choose the following
functions:

\[ u_1(t, x_1, x_2) = -\exp(x_1)(x_2 \cos(x_2) + \sin(x_2)) \exp(-t), \]
\[ u_2(t, x_1, x_2) = \exp(x_1)x_2 \sin(x_2) \exp(-t), \]
\[ p(t, x_1, x_2) = 2 \exp(x_1) \sin(x_2) \exp(-t), \]

which solve the Oseen equations (1) with \( \bar{w} = u \), a suitably chosen source vector \( f \) and initial condition \( u_0 \). The computational domain is taken to be \((-1, 1)^2\) and Dirichlet boundary conditions are imposed on the boundary.

We first study the influence of the choice of the stabilization parameter \( \gamma \) in the bilinear form \( \mathcal{C}_h \), defined in (45), on the accuracy of the DG solution. We conduct therefore simulations for different values of \( \gamma \) on a mesh with \( 8 \times 8 \) elements and different polynomial degrees, both in space and time. The results are shown in Table 1. For each simulation, the polynomial degrees for \( p \) are taken the same as for \( u \). The results show that the choice of the stabilization parameter \( \gamma \) does not influence the accuracy of \( u \) and \( p \). The parameter \( \gamma \) does have, however, a significant influence on the conditioning of the discretization matrix. Larger values of \( \gamma \) gives a better conditioning of the matrix.

Next, we study the order of accuracy of the velocity field \( u \) and the pressure \( p \) on meshes with different mesh sizes and increasing polynomial degrees. Here we use the stabilization term \( \mathcal{C}_h \) with \( \gamma = 10000 \). We first study the error in the \( L^2 \)-norm in the whole space-time domain \( \mathcal{E} \) for the velocity field \( u \). The results are shown in Fig. 2. The plots show that the rate of convergence of the space-time DG method for the velocity field is optimal in the \( L^2 \)-norm. Using linear polynomials in time and higher polynomial degrees in space we observe that, as the mesh becomes finer, the error is dominated by the error in time, but this only happens, however, when the spatial error is already very small.
Fig. 3. $L^2(\Omega_T)$ error for the pressure $p$ in the space-time DG discretization of the Oseen equations under $h$-refinement and for different polynomial degrees.

We also consider the $L^2$-norm of error for the pressure $p$ in $\Omega_T$, the domain at the final time $T$ of the simulation, both when equal polynomial degrees for $u$ and $p$ are used and also for different polynomial degrees. The results are shown in Fig. 3. We observe that when equal polynomial degrees are used for $u$ and $p$ then the $L^2$-norm of the error in the pressure converges at the rate $h^{p_s+0.5}$, with $p_s$ the polynomial degree of the pressure, while when the polynomial degrees for $p$ are one less than the polynomial degrees for $u$, then the pressure converges at the rate $h^{p_s+1}$.

7 Concluding remarks

In this article we present a space-time DG discretization for the Oseen equations in time-dependent flow domains. We prove the continuity, coercivity and stability of the method, which is shown to be unconditionally stable, and investigate the effect of the pressure stabilization operator on the stability. In addition, a full $hp$-error analysis is presented where the dependence of the error on mesh size, time step and polynomial order is made explicit.

The numerical experiments show that the convergence rate of the space-time DG solution for the velocity field is optimal in the $L^2$-norm, while the pressure converges at the rate $h^{p_s+0.5}$, with $p_s$ the polynomial degree of the pressure, when equal polynomial degrees of the velocity and pressure are used and $h^{p_s+1}$ when the polynomial degree of the velocity is larger than for the pressure. The simulations show that the algorithm also performs well for higher polynomial degrees in time.

The extension of the space-time DG discretization discussed in this article
to the incompressible Navier-Stokes equations can be accomplished using a Picard iteration, in which the prescribed velocity field \( w \) in (1a) is replaced by the computed velocity field \( u \) from the previous iteration. The only main addition to algorithm then is the use of a projection operator which ensures that the discrete velocity field is divergence free. Also, the application of the presented algorithm to free boundary problems is under investigation.

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References


