Space-Time Discontinuous Galerkin Method for Large Amplitude Nonlinear Water Waves

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Summary.
A space-time discontinuous Galerkin (DG) finite element method for nonlinear water waves in an inviscid and incompressible fluid is presented. The space-time DG method results in a conservative numerical discretization on time dependent deforming meshes which follow the free surface evolution. The dispersion and dissipation errors of the scheme are investigated and the algorithm is demonstrated with the simulation of waves generated by a wave maker.
Key words: space-time discontinuous Galerkin method, nonlinear water waves

1 Introduction

Large amplitude nonlinear water waves are frequently modeled by considering the fluid as inviscid, incompressible and irrotational. This makes it possible to describe the flow field with a potential function ϕ, which satisfies the Laplace equation. In addition, the potential function must satisfy a kinematic and dynamic boundary condition at the free surface:

$$\frac{\partial \zeta}{\partial t} + \overline{\nabla}_s \phi \cdot \overline{\nabla}_s \zeta - \frac{\partial \phi}{\partial z} = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \overline{\nabla} \phi \cdot \overline{\nabla} \phi + z = 0, \tag{2}$$

with ζ the wave height, t time, and $\overline{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$, $\overline{\nabla}_s = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^T$ defined with respect to a Cartesian coordinate system (x, y, z), where z is the coordinate direction pointing upward from the flat water surface. At a wave maker a prescribed normal velocity is imposed and at other solid surfaces a zero normal velocity. The equations are made dimensionless with an average water depth and the gravitational constant.

The numerical solution of large amplitude nonlinear water waves is non-trivial since the free surface boundary conditions must be imposed at the actual free surface, which must be determined as part of the solution process.

The free surface experiences large deformations during the wave evolution, in particular due to wave interactions. An additional complication is that the mathematical model has no inherent dissipation, which makes it difficult to ensure sufficient stability and robustness of the numerical scheme.

In this paper we will present a new space-time discontinuous Galerkin finite element algorithm to compute large amplitude nonlinear waves. The algorithm uses basis functions which are discontinuous both in space and time, and discretizes the equations directly in four dimensional space, with time as the fourth dimension. This approach combines the benefits of discontinuous Galerkin methods, for instance their suitability for *hp*-mesh adaptation and parallel computing, with an accurate representation of the time-dependent boundary using deforming elements. The space-time DG algorithm is closely related to the so called arbitrary Lagrangian Eulerian technique, which provides more flexibility in the mesh deformation algorithm, see e.g. [VV2002], but the space-time DG method ensures that the discretization remains conservative on deforming meshes. The space-time discontinuous Galerkin method discussed in this paper is an extension to nonlinear waves of the spatial discretization discussed in [VT2005] for linear waves using the space-time techniques presented in [VV2002].

2 Space-time discontinuous Galerkin formulation

In this section we summarize the space-time discontinuous Galerkin finite element discretization.

We consider the space-time domain $\mathcal{E}_h \subset \mathbb{R}^4$ split into space-time slabs $\mathcal{E}_h^n$ with the free-surface boundary $\Gamma_{\mathcal{S}}^n$ for the time intervals (t_n, t_{n+1}) with $n = 0, 1, \cdots$. We introduce a finite element tessellation $\mathcal{T}_h^n$ with space-time elements $\mathcal{K}$ on $\mathcal{E}_h^n$ and define the finite element spaces V_p^n and Σ_p^n associated with the tessellation $\mathcal{T}_h^n$ as:

$$
\begin{aligned}
V_p^n :=&\{v \in L^2(\mathcal{E}_h^n) \,|\, v|_{\mathcal{K}} \in \mathcal{P}_p(\mathcal{K}),\ \forall \mathcal{K} \in \mathcal{T}_h^n\},\\
\Sigma_p^n :=&\{\sigma \in [L^2(\mathcal{E}_h^n)]^3 \,\Big|\, \sigma|_{\mathcal{K}} \in [\mathcal{P}_p(\mathcal{K})]^3,\ \forall \mathcal{K} \in \mathcal{T}_h^n\},
\end{aligned}
$$

with $L^2(\mathcal{E}_h^n)$ the space of Lebesgue square integrable functions on $\mathcal{E}_h^n$ and $\mathcal{P}_p(\mathcal{K})$ the space of polynomials of degree p on $\mathcal{K}$. The finite element space W_p^n associated with the free surface is defined as

$$
W_p^n :=\{v \in L^2(\Gamma_{\mathcal{S}}^n) \,|\, v|_{\mathcal{F}_{\mathcal{S}}^n} \in \mathcal{P}_p(\mathcal{F}_{\mathcal{S}}^n),\ \forall \mathcal{F}_{\mathcal{S}}^n \subset \Gamma_{\mathcal{S}}^n\},
$$

where $L^2(\Gamma_{\mathcal{S}}^n)$ is the space of Lebesgue square integrable functions on $\Gamma_{\mathcal{S}}^n$.

Next, we define some trace operators to manipulate the numerical fluxes in the discontinuous Galerkin formulation. For $v \in V_p^n$ we define the average $\langle v \rangle$ and jump $[\![v]\!]$ operators of v at an internal face $\mathcal{F} \in \mathcal{F}_I^n$ as follows:

$$\langle v\rangle := \frac{1}{2}\left(v_L + v_R\right), \qquad [\![v]\!] := v_L\bar{n}_L + v_R\bar{n}_R, \tag{3}$$

with $v_L := v|_{\partial\mathcal{K}_L}$ and $v_R := v|_{\partial\mathcal{K}_R}$, and $\mathcal{K}_L$, $\mathcal{K}_R$ the elements connected to the face $\mathcal{F} \in \mathcal{F}_I^n$ with outward space normal vectors $\bar{n}_L$ and $\bar{n}_R$, respectively. For $q \in \Sigma_p^n$ we similarly define q_L, q_R, $\langle q\rangle$ and $[\![q]\!]$.

The novel ingredient in the finite element formulation discussed in this paper is the incorporation of the kinematic condition at the free surface (1) as a natural boundary condition in the finite element formulation. The space-time finite element discretization then will automatically account for the mesh movement necessary to follow the free surface waves. In order to accomplish this we need to establish a relation between the function $f = \zeta - z$, describing the free surface, the wave height ζ and the space-time normal vector n. A straightforward calculation using (1) shows that

$$n = (n_t, \bar{n})^T = \frac{\nabla f}{|\nabla f|} = \frac{(\frac{\partial f}{\partial t}, \overline{\nabla} f)^T}{|\nabla f|} = \frac{1}{|\nabla(\zeta - z)|}(\frac{\partial \zeta}{\partial t}, \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, -1)^T, \tag{4}$$

with ∇ the space-time nabla operator defined as $\nabla = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$. We can rewrite the kinematic condition (1) as:

$$\frac{1}{|\nabla f|}\frac{\partial f}{\partial t} + \overline{\nabla}\phi \cdot \frac{\overline{\nabla} f}{|\nabla f|} = 0, \tag{5}$$

which implies using (4) that the space component of the normal velocity at the free surface $\Gamma_{\mathcal{S}}$ is equal to

$$\bar{n} \cdot \overline{\nabla}\phi = \frac{-1}{|\nabla(\zeta - z)|}\frac{\partial \zeta}{\partial t}. \tag{6}$$

This relation can be used directly in the DG formulation of the Laplace equation, which is defined as:

Find a $(\phi_h, \zeta_h) \in V_p^n \times W_p^n$, such that for all $(v_1, v_2) \in V_p^n \times W_p^n$,

$$B_h(\phi_h, v_1) + (\frac{\frac{\partial \zeta_h}{\partial t}}{|\nabla_h(\zeta_h - z)|}, v_1)_{\Gamma_{\mathcal{S}}^n} + (\frac{\zeta_h^+ - \zeta_h^-}{|\nabla_h(\zeta_h^+ - z)|}, v_1^+)_{\Gamma_{\mathcal{S}}(t_n^+)} = L_h(v_1), \tag{7}$$

$$\begin{aligned}(\frac{\partial \phi_h}{\partial t}, v_2)_{\Gamma_{\mathcal{S}}^n} + (\zeta_h, v_2)_{\Gamma_{\mathcal{S}}^n} + (\phi_h^+ - \phi_h^-, v_2^+)_{\Gamma_{\mathcal{S}}(t_n^+)} \\ + (\frac{1}{2}(\overline{\nabla}_h\phi_h - \sum_{\mathcal{F}\in\mathcal{F}_I^n}\mathcal{R}_{\mathcal{F}}([\![\phi_h]\!])) \cdot (\overline{\nabla}_h\phi_h - \sum_{\mathcal{F}\in\mathcal{F}_I^n}\mathcal{R}_{\mathcal{F}}([\![\phi_h]\!])), v_2)_{\Gamma_{\mathcal{S}}^n} = 0,\end{aligned} \tag{8}$$

where $\Gamma_{\mathcal{S}}(t_n^+) = \lim_{\epsilon\downarrow 0}\Gamma_{\mathcal{S}}(t_n + \epsilon)$, $\phi_h^\pm = \lim_{\epsilon\downarrow 0}\phi_h(t_n \pm \epsilon)$, $\zeta_h^\pm = \lim_{\epsilon\downarrow 0}\zeta_h(t_n \pm \epsilon)$, with ϕ_h^- and ζ_h^- the known potential and wave height at $t = t_n$. The operators $B_h : V_p^n \times V_p^n \to \mathbb{R}$ and $L_h : V_p^n \to \mathbb{R}$ are defined as:

$$B_h(u, v) = \int_{\mathcal{E}_h^n}\overline{\nabla}_h u \cdot \overline{\nabla}_h v dx - \int_{\Gamma_0^n}\big([\![u]\!] \cdot \langle\overline{\nabla}_h v\rangle + [\![v]\!] \cdot \langle\overline{\nabla}_h u\rangle\big)ds$$

$$+ \sum_{\mathcal{F} \in \mathcal{F}_I^n} (\eta_{\mathcal{F}} + n_f) \int_{\mathcal{E}_h^n} \mathcal{R}_{\mathcal{F}}([\![u]\!]) \cdot \mathcal{R}_{\mathcal{F}}([\![v]\!]) dx, \tag{9}$$

$$L_h(v) = \int_{\Gamma_{\mathcal{N}}^n} v g_N ds,$$

with the local lifting operator $\mathcal{R}_{\mathcal{F}} : \left[L^2(\mathcal{F})\right]^3 \to \Sigma_p^n$:

$$\int_{\mathcal{E}_h^n} \mathcal{R}_{\mathcal{F}}(q) \cdot \sigma dx = \int_{\mathcal{F}} q \cdot \langle \sigma \rangle \, ds, \qquad \forall \sigma \in \Sigma_p^n. \tag{10}$$

The resulting nonlinear equations are solved with a Newton method with special attention given to the discretization of the free surface terms. During each time step and Newton iteration the mesh is adjusted to accommodate for the free surface motion. For more details about the scheme and the numerical implementation, we refer the reader to [VX2006].

3 Fourier analysis of discrete scheme

In this section, we conduct a Fourier analysis of the two-dimensional space-time discretization for the linear water wave equations as in [VT2005]. This will provide us with information on the dissipation and dispersion error in the wave motion due to the numerical discretization.

We assume a two-dimensional domain with a uniform mesh with $N_x \times N_z$ coordinates $x_j = j\Delta x$, $z_m = m\Delta z$ and periodic boundary conditions in the x-direction. For the linear water wave equations, the space-time DG scheme (7)–(8) reduces to the following form

$$B_h(\phi_h, v_1) + (\frac{\partial \zeta_h}{\partial t}, v_1)_{\Gamma_{\mathcal{S}}^n} + (\zeta_h^+ - \zeta_h^-, v_1^+)_{\Gamma_{\mathcal{S}}(t_n^+)} = L_h(v_1), \tag{11}$$

$$(\frac{\partial \phi_h}{\partial t}, v_2)_{\Gamma_{\mathcal{S}}^n} + (\phi_h^+ - \phi_h^-, v_2^+)_{\Gamma_{\mathcal{S}}(t_n^+)} + (\zeta_h, v_2 t)_{\Gamma_{\mathcal{S}}^n} = 0. \tag{12}$$

We use the Fourier ansatz for the coefficients $\widehat{\phi}_j$ and $\widehat{\zeta}_j$ in the space-time DG discretization at time level $t^n = n\Delta t$:

$$\widehat{\phi}_j(z, t_n) = \lambda^n \exp(ikj\Delta x)\widehat{\phi}^F(z), \quad \widehat{\zeta}_j(t_n) = \lambda^n \exp(ikj\Delta x)\widehat{\zeta}^F \tag{13}$$

with $\lambda^n = \exp(-i\omega n\Delta t)$ the amplification factor, k the wavenumber and $i = \sqrt{-1}$.

If we introduce (13) into the space-time discretization for the linear case (11)–(12), then we obtain the following generalized eigenvalue problem:

$$\lambda \left(\begin{array}{c|c} \mathcal{M}^F & \begin{matrix} 0 \\ \mathcal{P} \end{matrix} \\ \hline 0 \; \mathcal{H} & \mathcal{S} \end{array} \right) \begin{pmatrix} \widehat{\phi}_D^F \\ \widehat{\phi}_S^F \\ \hline \widehat{\zeta}^F \end{pmatrix} = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 \\ \mathcal{Y} \end{matrix} \\ \hline 0 \; \mathcal{V} & 0 \end{array} \right) \begin{pmatrix} \widehat{\phi}_D^F \\ \widehat{\phi}_S^F \\ \hline \widehat{\zeta}^F \end{pmatrix} \tag{14}$$

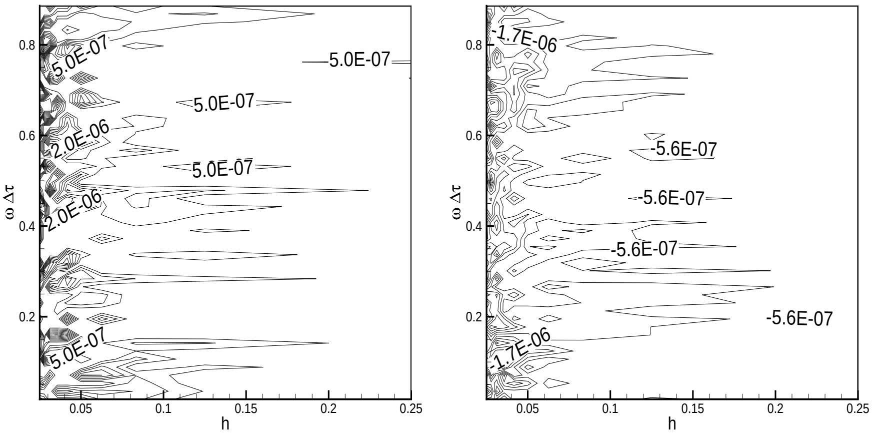

Fig. 1. Dispersion (left) and dissipation (right) for the space-time DG method with quadratic polynomial basis functions.

with $\widehat{\phi}_D^F \in \mathbb{R}^{N_p(N_z-1)}$, $\widehat{\phi}_S^F \in \mathbb{R}^{N_p}$, $\widehat{\zeta}^F \in \mathbb{R}^{N_s}$, where $N_p = (p+1)^3$, $N_s = (p+1)^2$ and p the order of the polynomials. The suffix S refers to the coefficients in elements connected to the free surface and D to elements not connected to the free surface. The matrices $\mathcal{P}$, $\mathcal{Y} \in \mathbb{R}^{N_p \times N_s}$, $\mathcal{H}$, $\mathcal{V} \in \mathbb{R}^{N_s \times N_p}$ and $\mathcal{S} \in \mathbb{R}^{N_s \times N_s}$ are related to the free surface. The matrix $\mathcal{M}^F \in \mathbb{R}^{N_p N_z \times N_p N_z}$ is a Hermitian positive definite block-tridiagonal matrix.

The eigenvalue λ is computed with MATLAB for a wide range of Δt values and $k \in (0, 2\pi]$. For all cases, the modulus of λ is always less than or equal to one, hence the numerical discretization is unconditionally stable.

We also use Fourier analysis to compute the dispersion and dissipation error of the numerical scheme by comparing the frequency and dissipation of the discrete modes with the exact harmonic wave solution. The results for quadratic polynomial basis functions are shown in Fig. 1.

4 Numerical example

As an illustration of the numerical scheme, we consider nonlinear free-surface waves generated by a wave maker at $x = 0$ with time-harmonic frequency $\omega_w = 2$ in a domain $\Omega = [0, 4] \times [-1, 0]$. Homogeneous Neumann boundary conditions are assumed at the bottom $z = -1$ and at the end of the domain at $x = 4$. The initial free-surface height and velocity potential are zero and the computational mesh is constantly updated to follow the free surface motion. The wave profiles in the domain at $T = 7.5$ and $T = 20$ are presented in Figs. 2. The wave profiles compare well for different meshes and different polynomial order indicating that the wave motion is computed accurately.

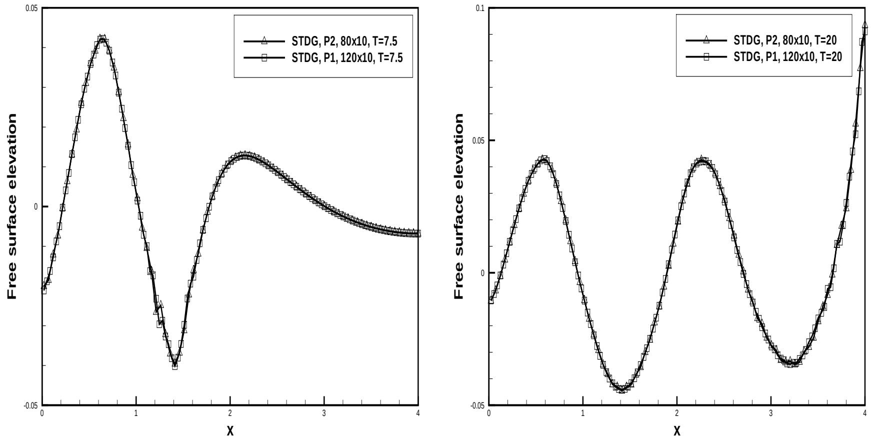

Fig. 2. Wave profile at $T = 7.5$ (left) and $T = 20$ (right) generated by a wave maker at $x = 0$ for polynomial basis functions of degree $p = 1$ and 2 using the space-time DG method.

5 Concluding remarks

In this paper we have presented a space-time discontinuous Galerkin method for nonlinear water waves. This technique results in a higher order accurate conservative numerical scheme on time dependent deforming meshes which are necessary to follow the free surface evolution. A Fourier analysis is given for the linear water wave equations indicating that the discretization is unconditionally stable and shows that the dispersion and dissipation errors of the scheme are minimal. Numerical examples for nonlinear free-surface waves show that the space-time DG method can accurately compute the nonlinear waves generated by a wave maker.

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