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A SPACE-TIME DISCONTINUOUS GALERKIN FINITE-ELEMENT DISCRETIZATION OF THE EULER EQUATIONS USING ENTROPY VARIABLES

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Abstract. *A method to numerically solve the Euler equations for fluids with general equations of state is presented. It is based on a formulation solving the conservation equations for either pressure primitive variables or entropy variables, instead of the commonly used conservation variables. We use a space-time discontinuous Galerkin finite-element discretization, which yields a highly local, potentially higher-order scheme. The algorithm is applied to test cases for compressible fluids to demonstrate its capabilities and the performance of the different variable sets.*

1 INTRODUCTION

Many numerical methods for fluid dynamics are suitable only for idealized types of fluids, like those for compressible flow being tailored to ideal gasses, and on the other hand methods that are based on formulations for (nearly) incompressible media. The inclusion of complicated equations of state, necessary to describe the physics of complex fluids, raises difficulties, as does the computation on domains where compressible and incompressible flow conditions coexist.

One way to derive numerical methods that are suitable both for compressible and incompressible flows is to solve for other variables than the set of conserved variables, for which the equations are originally derived. Two sets of variables that yield well-posed formulations in the incompressible limit are pressure primitive variables and entropy variables[1]. The entropy variables have the additional advantage that they symmetrize the equations and link to nonlinear stability[2]. Another key feature of the entropy variable formulation is its close relation to the underlying thermodynamics, which can be incorporated by expressing the equations of state in terms of two material coefficients, the volume expansivity and isothermal compressibility, which are available in analytical or tabulated form for relevant substances.

The Euler equations are discretized with respect to the abovementioned sets of variables (entropy and pressure primitive) using a discontinuous Galerkin (DG) finite element method. It extends the space-time DG discretization discussed by van der Vegt and van der Ven[3] to general equations of state and a formulation suitable for both compressible and incompressible flows. The algorithm results in a highly local and potentially higher-order discretization. Since space-time finite-element basis functions are used, the method is well-suited for problems with moving and deforming boundaries, and local *hp*-adaptation can be accommodated naturally.

In this article, the algorithm will be demonstrated with test cases for media with ideal gas equations of state.

2 GOVERNING EQUATIONS

2.1 Euler equations

We consider a fluid in the two-dimensional domain Ω during the time interval $\mathtt{T} = [t_s, t_e]$, described by the Euler equations of fluid dynamics written in conservation form, i.e.

$$\frac{\partial U_i}{\partial t} + \frac{\partial F_{ir}}{\partial x_r} = 0\,, \qquad i \in 1,\ldots,4\,, \qquad \forall\,(\mathbf{x},t) \in \Omega \times \mathtt{T}\,, \tag{1}$$

combined with suitable boundary and initial conditions of the form

$$\begin{aligned} U(\mathbf{x},t) &= B(U,U_w)\,, & \forall\,(\mathbf{x},t) &\in \partial\Omega \times \mathtt{T}\,,\\ U(\mathbf{x},t_s) &= U_0(\mathbf{x})\,, & \forall\,\mathbf{x} &\in \Omega\,, \end{aligned}$$

where the summation convention applies to the repeated index r for the space dimensions, $r = 1, 2$. The vector of *conservation variables* U and the flux matrix F are given by

$$U = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho e^{tot} \end{pmatrix}, \quad F = \begin{pmatrix} \rho v_1 & \rho v_2 \\ \rho v_1^2 + p & \rho v_1 v_2 \\ \rho v_2 v_1 & \rho v_2^2 + p \\ v_1(\rho e^{tot} + p) & v_2(\rho e^{tot} + p) \end{pmatrix}. \tag{2}$$

Here ρ denotes density, v_1, v_2 are the velocity components with respect to a Cartesian coordinate system, and p is pressure. The total energy e^{tot} in this case is the sum of internal and kinetic energy,

$$e^{tot} = e + k = e + \tfrac{1}{2} v_r^2\,. \tag{3}$$

2.2 Equations of state

The system (1) for the variables and flux (2) contains two more unknowns than equations, hence it is not closed. One additional relationship has been given by Equation (3), but to complete the description, we have to specify how the pressure p depends on the other variables.

Remark: In the more general context of the Navier-Stokes equations, an additional quantity, the temperature T, is needed to describe the diffusive effect of internal energy: heat conduction.

To remain within the framework of thermodynamics, which relates state variables of divariant fluids (i.e. those whose state is determined by two independent variables, e.g. p and T), we include temperature in our formulation. □

Equations that diagnostically relate state variables of a fluid, for instance ρ, p, e, T, are called *equations of state*. The system (1,2) can be closed by specifying, for example, the functions $e = e(p, T)$ and $p = p(\rho, T)$.

For an ideal gas with constant specific heat (at constant volume), c_v, and the gas constant R, these relations take the form

$$e = e(T) = c_v T\,, \tag{4}$$

$$p = \rho R T\,. \tag{5}$$

A different idealization, the incompressible fluid, represents the limit in which there is only one independent thermodynamic variable as the density ρ is constant.

To characterize the compressibility of a material we use two coefficients, the volume expansivity α_p and the isothermal compressibility β_T, which are defined as the relative changes of the specific volume $v = \frac{1}{\rho}$ as

$$\alpha_p \coloneqq \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p\,, \quad \beta_T \coloneqq -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T\,. \tag{6}$$

General equations of state can be formulated based on these two quantities as functions of the thermodynamic state[4]. For an ideal gas described by Equation (5) the coefficients evaluate to $\alpha_p = 1/T$ and $\beta_T = 1/p$; for incompressible fluids both coefficients α_p and β_T are zero.

2.3 Entropy variable formulation

We now interpret the conservation variables U as dependent on some other set of variables. In particular we are interested in the *primitive variables including pressure* and *entropy variables*,

$$Y = \begin{pmatrix} p \\ v_1 \\ v_2 \\ T \end{pmatrix}, \qquad V = \frac{1}{T}\begin{pmatrix} \mu - \frac{1}{2}v_m^2 \\ v_1 \\ v_2 \\ -1 \end{pmatrix}, \tag{7}$$

respectively, with the chemical potential μ.

By rewriting (1) as quasi-linear system with respect to any suitable set of variables, W, we obtain

$$A_0^W \frac{\partial W}{\partial t} + A_r^W \frac{\partial W}{\partial x_r} = 0\,, \tag{8}$$

with $A_0^W = \frac{\partial U}{\partial W}$, $A_j^W = \frac{\partial F_j}{\partial W}$. Analysis shows that[1]:

- For conserved variables U, the flux Jacobians A_j^U, $(j = 1, 2)$, are not well defined in the incompressible limit $\alpha_p, \beta_T \to 0$ (i.e. they contain entries that diverge or take the form $\frac{0}{0}$). This is connected to the fact that the equation for density in (1) loses its prognostic character for $\rho = const$, and causes the breakdown of many numerical methods using conservation variables when simulating (almost) incompressible fluids.

- The Jacobians A_j^Y, A_j^V, $(j = 0, 1, 2)$, are well defined in the incompressible limit. This corresponds to the suitability of numerical methods based on these sets of variables to compute both compressible *and* incompressible flows (in particular many methods for incompressible flow are based on the pressure variables).

- The matrix A_0^V (for entropy variables) is symmetric and positive definite, in the incompressible case this weakens to semi-definiteness. The matrices A_j^V, $(j = 1, 2)$, are symmetric. Discrete methods based on the entropy variables satisfy the entropy production inequality without (additional) dissipative mechanisms.

These properties led us to adapt a discontinuous Galerkin discretization[3] to use the pressure primitive and in particular the entropy variables, in place of conservation variables. Note that this method is not built on the quasi-linearized form (8), but discretizes the original conservation equations (1). Hence the method is conservative, even in the incompressible limit.

3 DISCONTINUOUS GALERKIN DISCRETIZATION

To apply a finite element discretization, the domain $\Omega \subset \mathbb{R}^2$ is approximated by a suitable mesh $\mathcal{T}^h$ with N_E elements whose minimal element in-circle diameter is h. We extend this space tessellation into the time dimension and consider the problem immediately in $\mathcal{E} \subset \mathbb{R}^3$. The space-time elements K_e, $e = 1, \ldots, N_E$, are related to the reference element $\hat{K} = [-1; 1]^3$ by mappings $G_e : \hat{K} \to K_e$.

The basis functions are defined on the reference element $\hat{K}$ as tensor products of the one-dimensional monomial basis $P^n = \{p : [-1; 1] \to \mathbb{R}, p_k(\xi) = \xi^k, k = 0, \ldots, n\}$, and we allow different maximum orders n_t and n_s for the time and space dimensions. Hence the basis on the reference element is $Q^{n_t,n_s}(\hat{K}) = \{\varphi : \hat{K} \to \mathbb{R} \mid \varphi \in P^{n_t} \times (P^{n_s})^2\}$. Using these basis functions on the reference element, the space of the test and trial functions for the finite element formulation is defined as

$$\mathcal{V}_{\mathcal{T}^h}^{n_t,n_s} = \left\{ W : \mathcal{T}^h \times \mathrm{T} \to \mathbb{R}^4 \mid W|_{K_e} \circ G_e \in \mathrm{span}\{(Q^{n_t,n_s}(\hat{K}))^4\} \quad \forall e = 1, \ldots, N_E \right\}. \tag{9}$$

Functions in this space are represented *per element* as

$$W|_{K_e}(\mathbf{x}, t) = \sum_{i=1}^{(n_t+1)(n_s+1)^2} W_i \, \varphi_i(G_e^{-1}(\mathbf{x}, t)), \tag{10}$$

with $W_i \in \mathbb{R}^4$ and $\varphi_i \in Q^{n_t,n_s}(\hat{K})$. Note that these functions may be discontinuous at the element boundaries.

As we will shortly integrate Equation (1) over the space-time domain $\mathcal{E}$, we rewrite it in space-time divergence form as

$$\sum_{r=0}^{2} \frac{\partial F_{ir}}{\partial x_r} = 0\,, \qquad i \in 1, \ldots, 4\,, \tag{11}$$

with $x_0 = t$ and the flux in the time dimension $F_{\cdot 0}(U) = U$.

The DG discretization is obtained by the following steps: We multiply Equation (11) with an arbitrary test function $\phi \in \mathcal{V}_{\mathcal{T}^h}^{n_t,n_s}$ and chose the expansion of the variables, W_h, from the same space, too. Integrating over each space-time element K_e and summing over all elements we receive the weak formulation:

Find a $W_h \in \mathcal{V}_{\mathcal{T}^h}^{n_t,n_s}$, such that for all $\phi \in \mathcal{V}_{\mathcal{T}^h}^{n_t,n_s}$ the following relation is satisfied:

$$\sum_{e=1}^{N_E} \left(- \int_{K_e} \frac{\partial \phi_p}{\partial x_r} \mathcal{F}_{pr}(U(W_h)) d\tau + \int_{\partial K_e} \phi_p \hat{\mathcal{F}}_{pr}(U(W_h^-), U(W_h^+))\, n_r ds \right) = 0\,. \tag{12}$$

The boundary of each element K_e is denoted as ∂K_e with n_i the components of the exterior pointing normal vector at this surface. Since the discrete function W_h is not necessarily continuous at the element faces we replace the $\mathcal{F}(U)$ in the integral over the element boundaries by a consistent, symmetric numerical flux $\hat{\mathcal{F}}(U_h^-, U_h^+)$. Herein, the limit of the discrete function W_h from the interior of K_e is called W_h^-, whereas the exterior limit is denoted W_h^+; the latter is either based on the u_h in a neighboring element or on the boundary data.

The numerical flux $\hat{\mathcal{F}}$ ensures that the discretization is conservative. We can expose the conservativity of the discretization and indicate the computational structure of the DG finite element method by introducing the faces F_i, $i \in \{1, \ldots, N_F\}$, which are the objects of codimension one that bound the elements. In Equation 12, the integrals over the boundaries of elements are evaluated twice for each internal face F_i. The two integrals differ only in the sign of the normal vector $\mathbf{n}$, so that we can combine the integrands as $\phi_p^L n_r \hat{\mathcal{F}}_{pr} + \phi_p^R (-n_r) \cdot \hat{\mathcal{F}}_{pr} = (\phi_p^L - \phi_p^R) n_r \hat{\mathcal{F}}_{pr}$, where superscripts $(\cdot)^L$ and $(\cdot)^R$ denote values taken from the elements on the (arbitrarily defined) left and right side of the face, respectively. (Because the support of each basis function is limited to one element only one of the contributions ϕ^L, ϕ^R will be nonzero.) Then we have

$$- \sum_{e=1}^{N_E} \int_{K_e} \frac{\partial \phi_p}{\partial x_r} \mathcal{F}_{pr}(U(W_h)) d\tau + \sum_{i=1}^{N_F} \int_{F_i} (\phi_p^L - \phi_p^R)\, n_r \hat{\mathcal{F}}_{pr}(U(W_h^L), U(W_h^R)) ds = 0\,. \tag{13}$$

This constitutes a nonlinear system of equations for the expansion coefficients of W_h in each element, which is solved using a pseudo-time method[5].

As the spatial numerical flux, we use the HLLC approximate Riemann solver for the ideal gas test cases in Section 4; for other cases this may be replaced by the analytical flux of the mean state on the two sides of the face (with an added stabilization term). On the faces which separate the time slabs, an upwind flux is used conforming to causality.

The treatment of boundary conditions through external states in the Riemann solver, which is a bye-product of the DG discretization, considerably simplifies the specification of various conditions, like sub- and supersonic in-/outflow or slip flow. In the spatially continuous discretization of Hauke and Hughes[1], the discrete equation system is solved by a Newton method, which requires not only a costly global system assembly but also the linearization of the boundary conditions.

4 RESULTS

To demonstrate the possibilities arising from the formulation presented above, we give some computational results.

4.1 Flow around a circular cylinder

We consider the subsonic flow of an ideal gas around a fixed circular cylinder with unit radius. The far-field flow is aligned with the x-axis, constant, and homogeneous: $\mathbf{v}_\infty = u_\infty \mathbf{e}_x$. In the limit of vanishing Mach number, $M_\infty \to 0$, the flow field is given by the classical potential flow solution. To conform to the study by van der Vegt and van der Ven[6], we chose the free-stream Mach number as $M_\infty = 0.38$. At the inflow, we use far-field boundary conditions for the mass flux, stagnation enthalpy $h^{tot} = h + \frac{1}{2}v_r^2$, and pressure: $\rho_R \mathbf{v}_R = \rho_\infty \mathbf{v}_\infty$, $h_R^{tot} = h_\infty^{tot}$, $p_R = p_L$, respectively, with $(\cdot)_L$ denoting the flow state on the inside of the flow domain, while $(\cdot)_R$ is the boundary data. Corresponding outflow conditions are $\rho_R \mathbf{v}_R = \rho_L \mathbf{v}_L$, $s_R = s_L$, $p_R = p_\infty$, with the specific enthalpy s. Due to the absence of viscosity in the Euler equations, a slip flow condition is applied on the cylinder surface: $\mathbf{v} \cdot \mathbf{n} = 0$. This is enforced by setting the momentum in the boundary data for the HLLC flux to $\rho_R \mathbf{v}_R = \rho_L \mathbf{v}_L - 2\rho_L(\mathbf{v}_L \cdot \mathbf{n})\mathbf{n}$.

We indicate the error of the numerical method using the total pressure loss π, defined as

$$\pi = 1 - \frac{p}{p_\infty}\left(\frac{1 + \frac{1}{2}(\gamma-1)M^2}{1 + \frac{1}{2}(\gamma-1)M_\infty^2}\right)^{\frac{\gamma}{\gamma-1}} . \tag{14}$$

The pressure loss should be zero everywhere, deviations occur only due to the discrete approximation, which leads to entropy production at the surface of the cylinder. The resulting error according to a computation with conservation variables on a mesh with 48×32 elements is plotted in Figure 1. Note that in all results we plot the *discontinuous* solution without an averaging per element or per node. Figure 2 shows the total pressure loss along the cylinder surface for entropy variables using two different orders of the spatial representation (since this is a steady state case, we use $n_t = 0$ throughout). Results for different variable sets are hardly distinguishable in this test, so we do not present a comparison.

4.2 Oblique shock

We consider an inviscid supersonic flow at $M = 2$ on the unit square. The inflow is at an angle of 10° with the lower boundary of the domain, which is a wall (with slip condition). Consequently a shock forms at an angle of 29.3°, cf. Hauke and Hughes[1]. The density field including

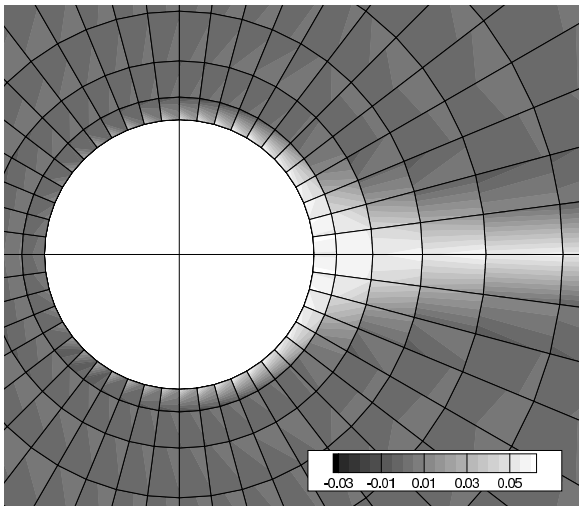

Figure 1: Total pressure loss close to the cylinder. Computed on a 48×32 mesh using conservation variables U.

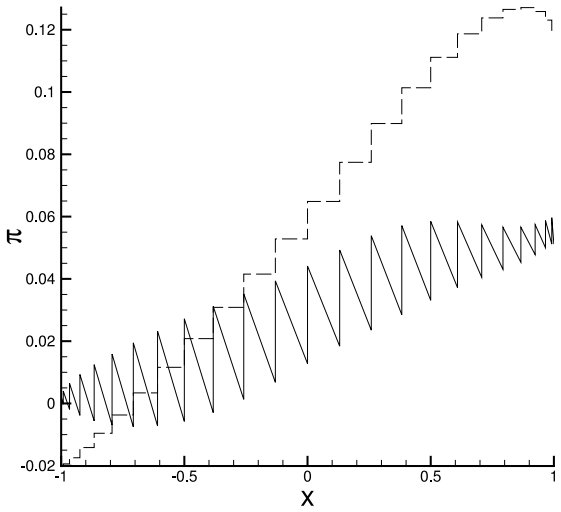

Figure 2: Comparison of the total pressure loss computed using V-variables on a 48×32. Dashed line: $n_s = 0$ (constant representation of V_h per element), solid line: $n_s = 1$ (bilinear representation of V_h).

the shock is plotted in Figure 3. Note that we are not using any form of discontinuity capturing or stabilization, hence the over- and undershoot close to the shock. In this way it is also reasonable to compare the results for different sets of variables, cf. Figure 4. Minor differences can be observed. Presumably these arise due to amplification by the different transformations between the pressure primitive or entropy variables and the plotted variable ρ. On the other hand, the overall location of the shock is correctly represented, in contrast to formulations that use non-conservative formulations.

5 CONCLUSIONS & OUTLOOK

We have described the extension of a discontinuous Galerkin finite element discretization of the Euler equations to allow using different sets of variables. In particular we have considered entropy variables, because of their theoretically and practically appealing properties. This choice allows to treat compressible and incompressible flows with the same numerical method. We have presented numerical examples for the compressible case.

The extension of the numerical method to include the diffusive terms of the Navier-Stokes equations is under way using the discretization described by Klaij et al.[7]. The goal is to combine the presented scheme with an interface tracking method[8] to compute multiphase flows with compressible and incompressible media.

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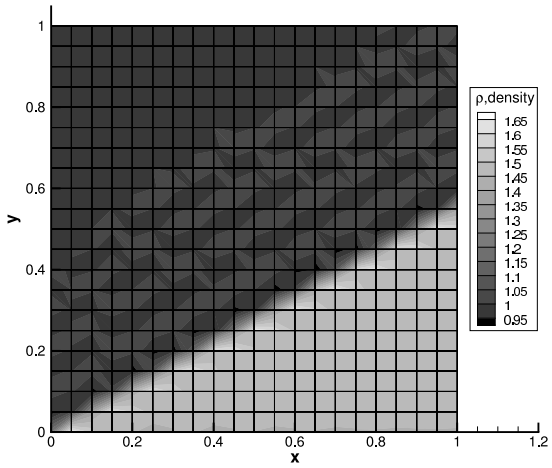

Figure 3: Oblique shock test case; density field computed on a 20×20 mesh using entropy variables.

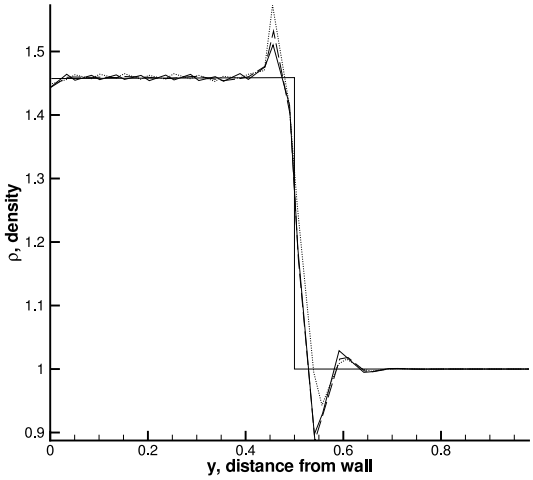

Figure 4: Comparison of the computed density fields along $x = 0.9$. Solid line: U, dashed: Y, dotted: V-variables. All computations carried out on a 20×20 mesh with $n_s = 1$ and no stabilization added to expose the differences between the different variable sets.

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