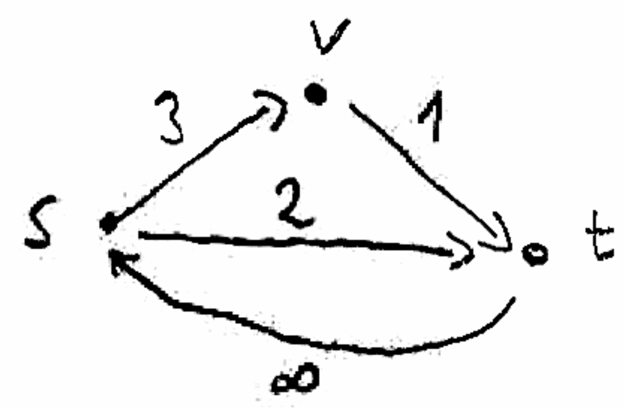
An Example for the Primal-Dual Pair Max-Flow and Min-Cut

Max-Flow Problem: (with arc capacities)



problem: Maximize flow along arc (t,s)

The primal LP: (P) $\max\ c\,x$

$$\text{s.t. } Ax = 0 \quad \leftarrow \text{flow balance}$$

$$Ex \leq u \quad \leftarrow \text{capacities}$$

$$x \geq 0,$$

where $c = (0,0,0,1) \;\hat{=}\;$ maximize flow along (t,s).

So this gives the following on our example:

$$\max\ (0,0,0,1) \cdot \begin{pmatrix} x_{sv} \\ x_{st} \\ x_{vt} \\ x_{ts} \end{pmatrix}$$

$$\text{s.t.}\quad \begin{array}{c} \\ s \to \\ v \to \\ t \to \end{array} \underbrace{\begin{array}{cccc} (s,v) & (s,t) & (v,t) & (t,s) \\ \hline 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}}_{A} \cdot \begin{pmatrix} x_{sv} \\ x_{st} \\ x_{vt} \\ x_{ts} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{c} \pi_s \\ \pi_v \\ \pi_t \end{array}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{E} \cdot \begin{pmatrix} x_{sv} \\ x_{st} \\ x_{vt} \\ x_{ts} \end{pmatrix} \leq \begin{pmatrix} 3 \\ 2 \\ 1 \\ \infty \end{pmatrix} \quad \begin{array}{c} \alpha_{sv} \\ \alpha_{st} \\ \alpha_{vt} \\ \alpha_{ts} \end{array}$$

$$x \geq 0$$

dual var.'s (one per constraint)

Now remember that the dual is the following LP

(D) $\min\ (0,u) \cdot (\pi, \alpha)$

$$\text{s.t. } [A^t, E^t] \begin{pmatrix} \pi \\ \alpha \end{pmatrix} \geq c$$

$$\alpha \geq 0$$

which becomes the following on our example:

$$\min\ (3,2,1,\infty) \cdot \begin{pmatrix} \alpha_{sv} \\ \alpha_{st} \\ \alpha_{vt} \\ \alpha_{ts} \end{pmatrix}$$

← all π's have coefficients 0

$$\text{s.t.}\quad \begin{array}{c} \\ (s,v) \\ (s,t) \\ (v,t) \\ (t,s) \end{array} \begin{array}{ccc|cccc} s & v & t & & & & \\ \hline 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \begin{pmatrix} \pi_s \\ \pi_v \\ \pi_t \\ \alpha_{sv} \\ \alpha_{st} \\ \alpha_{vt} \\ \alpha_{ts} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\quad}_{A^t} \qquad \underbrace{\quad}_{E^t = E}$$

$$\alpha \geq 0$$

so that the constraints are:

$$\alpha_{sv} + \pi(s) - \pi(v) \geq 0$$

$$\alpha_{st} + \pi(s) - \pi(t) \geq 0$$

$$\alpha_{vt} + \pi(v) - \pi(t) \geq 0$$

$$\alpha_{ts} + \pi(t) - \pi(s) \geq 1$$

and setting $\pi(s) = 0$ and $\alpha_{ts} = 0$, this is

$$\alpha_{sv} \geq \pi(v)$$

$$\alpha_{st} \geq \pi(t)$$

$$\alpha_{vt} \geq \pi(t) - \pi(v)$$

$$\pi(t) \geq 1$$