Discrete Optimization 2010

Hand-in Assignments (due Nov. 15, 2010)

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1. (4 points) The problem PRIMES is the following decision problem: Given $n \in \mathbb{N}$, is n a prime? (A prime is a number that is divisible only by 1 and itself.) There exists a divsion algorithm, let us call it $\mathcal{D}(a, n)$, that tells us in time $O((\log n)^2)$, if a is a divisor of n or not (for any integer $a \leq n$). Prove one of the following claims.

   - PRIMES $\in \mathcal{P}$
   - PRIMES $\in \mathcal{NP}$
   - PRIMES $\in$ co-$\mathcal{NP}$

2. (4 points) For some combinatorial optimization problem

$$\begin{aligned} &\min\ c^t x \\ &\text{s.t. } Ax = b \\ &\qquad x \in X\,, \end{aligned}$$

   consider the Lagrangian relaxation

$$L(\lambda) = \min_{x \in X} L(\lambda, x) = \min_{x \in X} (c^t x - \lambda^t (Ax - b))\,.$$

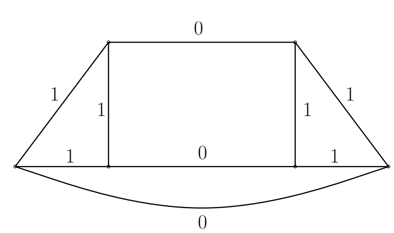
   Suppose for given λ that x^* attains the the minimum in $\min_{x \in X} L(\lambda, x)$. Show that $b - Ax^*$ is a subgradient of L at λ, that is, show that $L(\lambda') \leq L(\lambda) + (b - Ax^*)^t(\lambda' - \lambda)$ for all λ'.

   (Hint: Use the definition of $L(\cdot)$ and optimality of x^* for $L(\lambda, x)$ for the given λ.)

3. (4 points) Consider the following integer linear programming formulation for the TSP.

$$\begin{aligned} \min\ & \sum_{e \in E} c_e x_e & & \\ \text{s.t. } & \sum_{e \in E} x_e = n & & \\ & \sum_{e \in \delta(v)} x_e = 2 & & v \in V \\ & \sum_{e \in \delta(S)} x_e \geq 2 & & \emptyset \subset S \subset V \\ & x \in \{0,1\}^n & & \end{aligned}$$

   You are to show that the Langrangian dual (after Lagrangian relaxation of degree constraints $\sum_{e \in \delta(v)} x_e = 2$ as done in Lecture 8) has an "optimality gap". To show that, consider the following TSP instance (see sheet 16 from Lecture 8). Now show that there



is always a solution to the Lagrangian relaxation with value $L(\lambda) \leq 3$, no matter what the dual variables λ are. You may assume v_1 is the leftmost vertex. It then follows that $LD \leq 3$, while the optimum solution for the TSP problem is obviously 4, hence the optimality gap is at least 4/3.

4. (not for delivery) We defined instances of the TSP problem as follows. Given is a complete undirected graph $G = (V, E)$ and integer edge lengths $c_e \geq 0$, $e \in E$. A tour (also called Hamiltonian cycle) is a cycle in G visiting each node $v \in V$ exactly once.

   Show that there exists a Turing reduction from TSP-OPT to TSP-DEC, that is, if the decision problem TSP-DEC ('is there a tour of length $\leq k$?') can be solved in polynomial time, then the optimization problem TSP-OPT ('find a tour of minimal total length') can be solved in polynomial time.

References

[1] Cook, W. J., W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, Wiley-Interscience, 1998.