Discrete Optimization 2010

Hand-in Assignments (due Nov. 1, 2010)

Marc Uetz (`m.uetz@utwente.nl`), Applied Mathematics, University of Twente

Chapters 5.1 and 5.2 of [1] are provided as additional reading about matching theory. In particular, 5.2 describes also Edmonds' famous matching algorithm (blossom shrinking). Also, Chapter 6 of [1] contains plenty of material about polyhedral aspects of the matching problem.

1. (2 points) Show that the node-edge incidence matrix of an undirected graph $G = (V, E)$ is totally unimodular if and only if G is bipartite (assume a graph without loops, that is, no edegs $\{v, v\}$).

2. (4 points) Linear Programs with constraint matrix $A \in \{0,1\}^{m\times n}$ and consecutive one's in either columns or rows can be efficiently solved via minimum-cost flow techniques.

   In this exercise, consider the following linear programming (LP) problem with consecutive one's in the rows of the constraint matrix A.

$$\begin{aligned} \text{min.} \quad & c^t x \\ s.t. \quad & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 5 \\ 12 \\ 10 \\ 6 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

   Show how this LP can be solved as minimum cost flow problem, and decide if the system is feasible.

   (Hint: First add a redundant constraint 0x=0 to the set of constraints, and then use elementary row operations to obtain a minimum cost flow problem.)

3. (5 points) Given an undirected connected graph $G = (V, E)$ with n nodes and m edges. An *edge cover* is a subset C of the edges of the graph such that each node $v \in V$ is incident to at least one edge $e \in C$ (i.e., a set of edges that "cover" all the nodes of the graph). Denote by $\alpha(G)$ the size of a *minimum* cardinality edge cover of G, and by $\mu(G)$ the size of a *maximum* cardinality matching of G. Show that $\mu(G) + \alpha(G) = n$.

   (Hint: From any maximum mating M, construct an edge cover to show $\mu(G) + \alpha(G) \leq n$. From any minimum edge cover C, construct a matching to show $\mu(G) + \alpha(G) \geq n$.)

4. (5 points) Let M be a (inclusion) maximal matching, and M^* be an arbitrary matching. Show that $|M| \geq \frac{1}{2}|M^*|$. (In particular, the claim is true if M^* is a maximum cardinality matching for G.)

References

[1] Cook, W. J., W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, Wiley-Interscience, 1998.