Discrete Optimization 2010

Hand-in Assignments (due Oct. 25, 2010)

Marc Uetz (m.uetz@utwente.nl), Applied Mathematics, University of Twente

Again, as a reminder, Chapter 12 of [2] is provided as an introduction (or refreshment) of linear programming duality. Chapter 6 of [1] contains material about polyhedra, total unimodularity, and more. Please read 6.1, 6.2, 6.4, and 6.5 (You may skip 6.3 and the digressions on the matching problem, but of course, you may also read these parts).

1. (4 points) For the minimum cost flow problem, we consider the primal dual pair of linear programs as discussed in the lecture. That is,

$$\begin{aligned} \min\ & c^t \cdot x \\ \text{s.t.}\ & Ax = b \\ & -Ex \geq -u \\ & x \geq 0 \end{aligned}$$

and the corresponding dual

$$\begin{aligned} \max\ & (b^t, -u^t) \cdot \begin{pmatrix} \pi \\ \alpha \end{pmatrix} \\ \text{s.t.}\ & [A^t, -E] \begin{pmatrix} \pi \\ \alpha \end{pmatrix} \leq c \\ & \alpha \geq 0 \end{aligned}$$

Consider a flow x and node potentials π upon termination of the successive shortest path algorithm, and as in the lecture, define a solution of the dual (π, α) via

$$\alpha_{ij} = \max\{0, -c^{\pi}_{ij}\}\,,$$

where $c^{\pi}_{ij} = c_{ij} - \pi(i) + \pi(j)$, for all arcs $(i,j) \in A$. We already argued that (π, α) is a feasible solution for the dual, because $\alpha \geq 0$ and because the dual constraints are exactly $\alpha_{ij} \geq -c^{\pi}_{ij}$, which is true by definition of α.

Show that x and (π, α) are an optimal primal dual pair, by showing that the two complementary slackness conditions are fulfilled.

2. (4 points) Professor Mae B. Wrong has the following conjectures. Prove, or disprove by giving a counterexample.

   (a) Matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular if and only if the matrix $\begin{bmatrix} A \\ -A \end{bmatrix}$ is totally unimodular.

   (b) Matrices $A, B \in \mathbb{Z}^{m\times n}$ are both totally unimodular if and only if the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is totally unimodular.

(see other page)

3. (6 points) Let P with $\emptyset \neq P \subseteq \mathbb{R}^n$ be a polyhedron, and suppose that $P \subseteq \{x \in \mathbb{R}^n \mid x \geq 0\}$. Show that P is *pointed*, that is, P has a vertex.

   (Hint: By definition, any polyhdron can be written $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for appropriate A and b. Consider an optimization problem with constraints $Ax \leq b$.)

4. (4 points) For a given network (G, u) with arc costs $c_{ij} \geq 0$, consider the standard minimum-cost (s,t)-flow problem, with the only additional feature that any arc that is actually used (that is, flow value > 0) incurs a *fixed cost* $f_{ij} > 0$. So

$$\text{the cost of flow } x_{ij} \text{ along arc } (i,j) = \begin{cases} f_{ij} + c_{ij}x_{ij} & \text{if } x_{ij} > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

   This problem is known as the *fixed charge network flow* problem. Give an integer linear programming formulation for this problem, Please specify what the decision variable are, and what they mean, and (briefly!) explain each of the constraints that you use.

References

[1] Cook, W. J., W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, Wiley-Interscience, 1998.

[2] V. V. Vazirani, *Approximation Algorithms*, Springer, 2001