Semi-infinite Programming: An Introduction
“preliminary version”
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Abstract

A semi-infinite programming problem is an optimization problem in which finitely many variables appear in infinitely many constraints. This model naturally arises in an abundant number of applications in different fields of mathematics, economics and engineering. The present paper which intends to make a compromise between an introduction and a survey, treats the theoretical basis and numerical methods of the field.

1 Introduction

1.1 Problem formulation

A semi-infinite program (SIP) is an optimization problem in finitely many variables \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) on a feasible set described by infinitely many constraints:

\[
P: \quad \min_x f(x) \quad \text{s.t.} \quad g(x, y) \geq 0 \quad \forall y \in Y ,
\]

where \( Y \) is an infinite index set. \( \mathcal{F} \) will denote the feasible set, \( v = \inf\{ f(x) \mid x \in \mathcal{F} \} \) the optimal value and \( S = \{ \overline{x} \in \mathcal{F} \mid f(\overline{x}) = v \} \) the set of minimizers of the problem. We assume that \( Y \subset \mathbb{R}^m \) is compact, \( g(x, y) \) is continuous on \( \mathbb{R}^n \times Y \) and for any fixed \( y \in Y \) the gradient \( \nabla_x g(x, y) \) (wrt. \( x \)) exists and is continuous on \( \mathbb{R}^n \times Y \).

We also will consider generalizations of SIP, where the index set \( Y = Y(x) \) is allowed to be dependent on \( x \) (GSIP for short),

\[
\min_x f(x) \quad \text{s.t.} \quad g(x, y) \geq 0 \quad \forall y \in Y(x) .
\]
During the last five decades the field of Semi-infinite Programming has known a tremendous development. More than 1000 articles have been published on the theory, numerical methods and applications of SIP. Originally the research on SIP was strongly related to (Chebyshev) approximation, see e.g., [12], [8]. As excellent review articles on the field we refer to Polak [25] and Hettich/Kortanek [9]. Since a first contribution [13] the generalized SIP problem (2) became a topic of intensive research (see e.g., [22] and [32]). For an extensive treatment of linear semi-infinite problems we refer to the book by Goberna/Lopez [10].

**Remark 1** For shortness and a clearer presentation we omit additional equality constraints $h_i(x) = 0$ in the formulation of SIP. It is not difficult to generalize (under appropriate assumptions on $h_i$) all results in this paper to the more general situation.

The paper is organized as follows. The next section presents some illustrative applications of SIP and GSIP. Section 3 discusses the structure of the feasible set of finite- and semi-infinite problems. Section 4 derives first order necessary and sufficient optimality conditions. Linear semi-infinite and linear semidefinite programming is treated in Section 5. The following Section 6 presents second order optimality conditions based on the so-called reduction approach. In Section 7 different numerical methods for solving SIP are surveyed. Finally, the last section discusses the relation between SIP and bilevel programming and mathematical programs with equilibrium constraints. The appendix gives some additional definitions facts from optimization.

## 2 Applications

In [25], [9] and [10] many applications of SIP are mentioned in different fields such as (Chebyshev) approximation, robotics, boundary- and eigen value problems from Mathematical Physics, engineering design, optimal control, transportation problems, fuzzy sets, cooperative games, robust optimization and statistics. From this large list of applications we have chosen some illustrative examples.

**Chebyshev Approximation:** Let be given a function $f(y) \in C(\mathbb{R}^m, \mathbb{R})$ and a space of approximating functions $p(x, y), p \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$, parameterized by $x \in \mathbb{R}^n$. We want to approximate $f$ by functions $p(x, \cdot)$ in the max-norm (Chebyshev-norm) $\|f\|_\infty = \max_{y \in Y} |f(y)|$ on a compact set $Y \subset \mathbb{R}^m$. To minimize the approximation error $\epsilon = \|f - p\|_\infty$, leads to the problem:

$$\min_{x, \epsilon} \epsilon \quad \text{s. t.} \quad g^\pm(x, y) := \pm(f(y) - p(x, y)) \leq \epsilon \quad \text{for all} \quad y \in Y.$$  \hspace{1cm} (3)

This is a semi-infinite problem.

The so-called reverse Chebyshev problem consists of fixing the approximation error $\epsilon$ and making the region $Y$ as large as possible (see [16] for such problems). Suppose, the set $Y = Y(d)$ is parameterized by $d \in \mathbb{R}^k$ and $v(d)$ denotes the volume of $Y(d)$.
(e.g. $Y(d) = \prod_{i=1}^{k} [-d_i, d_i]$). The reverse Chebyshev problem then leads to the GSIP ($\epsilon$ fixed).

$$\max_{d,x} v(d) \quad \text{s.t.} \quad g^\pm(x, y) := \pm(f(y) - p(x, y)) \leq \epsilon \quad \text{for all} \quad y \in Y(d). \quad (4)$$

where the index set $Y(d)$ depends on the variable $d$. We refer the reader also to [16].

**Mathematical Physics.** The so-called defect minimization approach leads to semi-infinite programming models for solving problems from Mathematical Physics which is different from the common finite element and finite difference approaches. We give an example.

**Shape optimization problem:** Consider the following Boundary-value problem, **BVP:** Given $G_0 \subset \mathbb{R}^m$ ($G_0$ a simply connected open region with smooth boundary $\partial G_0$, (closure $\overline{G_0}$)) and $k > 0$. Find a function $u \in C^2(\overline{G_0}, \mathbb{R})$ such that with the Laplacian

$$\Delta u = u_{yy} + \ldots + u_{y_my_m};$$

$$u(y) = k, \quad \forall y \in G_0$$

$$u(y) = 0, \quad \forall y \in \partial G_0$$

By choosing a linear space of appropriate trial functions $u_j \in C^2(\mathbb{R}^m, \mathbb{R})$,

$$S = \{u(x, y) = \sum_{i=1}^{n-1} x_iu_i(y)\}$$

this BVP can approximately be solved via the following SIP:

$$\min_{\epsilon, x} \epsilon \quad \text{s.t.} \quad \pm(\Delta u(x, y) - k) \leq \epsilon, \quad y \in G_0$$

$$\pm u(x, y) \leq \epsilon, \quad y \in \partial G_0$$

In [6] the related but more complicated so-called Shape Optimization Problem has been considered theoretically.

**SOP:** Find a (simply connected) region $G \in \mathbb{R}^m$ with normalized volume $\mu(G) = 1$ and a function $u \in C^2(\overline{G}, \mathbb{R})$ which solves with a given objective function $F(G, u)$ the problem

$$\min_{\mu(G) = 1, u} F(G, u) \quad \text{s.t.} \quad \Delta u(y) = k, \quad \forall y \in G$$

$$u(y) = 0, \quad \forall y \in \partial G$$

This is a problem with variable region $G$ and can be solved approximately via the following GSIP-problem:

Choose some appropriate set of regions $G(z)$ depending on a parameter $z \in \mathbb{R}^p$ and
satisfying $\mu(G(z)) = 1$ for all $z$. Fix some small error bound $\epsilon > 0$. Then we solve with trial functions $u(x, y) \in \mathcal{S}$ the program

$$
\min_{z, x} F(G(z), u(x, \cdot)) \quad \text{s.t.} \quad \pm \left( \Delta u(x, y) - k \right) \leq \epsilon, \quad \forall y \in G(z)
$$

$$
\pm u(x, y) \leq \epsilon, \quad \forall y \in \partial G(z)
$$

For further contributions to the theory and numerical results of this approach we refer e.g. to [31], [9].

**Robotics.** Control problems in robotics can often be modeled as a semi-infinite problem. We refer to the Ex.2.1 in [9] and the many references therein (cf. also [11]). As another example we discuss the so-called Maneuverability problem:

Let $\Theta = \Theta(\tau) \in \mathbb{R}^m$ denote the position of the so-called tool center point of the robot (in robot coordinates) at time $\tau$. Let $\dot{\Theta}, \ddot{\Theta}$ be the corresponding velocities, accelerations (derivatives w.r.t. $\tau$). The dynamical equation has (often) the form

$$
g(\Theta, \dot{\Theta}, \ddot{\Theta}) := A(\Theta)\ddot{\Theta} + F(\Theta, \dot{\Theta}) = K,$$

with (external) forces $K \in \mathbb{R}^m$. Here, $A(\Theta)$ is the inertia matrix and $F$ describes the friction, gravity, centrifugal forces, etc. The forces $K$ are bounded by

$$
K^- \leq K \leq K^+.
$$

For fixed $\Theta$, $\dot{\Theta}$, the set of feasible (possible) accelerations is given by

$$
Z(\Theta, \dot{\Theta}) = \{ \ddot{\Theta} | K^- \leq g(\Theta, \dot{\Theta}, \ddot{\Theta}) \leq K^+ \}.
$$

Note that, since $g$ is linear in $\ddot{\Theta}$, for fixed $(\Theta, \dot{\Theta})$, the set $Z(\Theta, \dot{\Theta})$ is convex (intersection of half-spaces). Let now be given an ‘operating region’ $Q$, e.g.

$$
Q = \{ (\Theta, \dot{\Theta}) \in \mathbb{R}^{2m} | (\Theta^-, \dot{\Theta}^-) \leq (\Theta, \dot{\Theta}) \leq (\Theta^+, \dot{\Theta}^+) \}
$$

with bounds $(\Theta^-, \dot{\Theta}^-)$ and $(\Theta^+, \dot{\Theta}^+)$. Then, the set of feasible accelerations $\ddot{\Theta}$ (accelerations which can be realized in every point $(\Theta, \dot{\Theta}) \in Q$) becomes

$$
Z_0 = \bigcap_{(\Theta, \dot{\Theta}) \in Q} Z(\Theta, \dot{\Theta}) = \{ \ddot{\Theta} | K^- \leq g(\Theta, \dot{\Theta}, \ddot{\Theta}) \leq K^+, \text{ for all } (\Theta, \dot{\Theta}) \in Q \}.
$$

The set $Z_0$ is convex (as an intersection of the convex sets $Z$). For the steering of the robot one has to check whether a desired acceleration $\ddot{\Theta}$ is possible, i.e. whether $\ddot{\Theta} \in Z_0$. Often, this check takes to much time due to the complicated description of $Z_0$. Then, one is interested in a simple body $Y$ (e.g. a ball) as large as possible, which is contained in $Z_0$. Instead of the test $\ddot{\Theta} \in Z_0$ one performs the (quicker) check $\ddot{\Theta} \in Y$. Suppose the body $Y(d)$ depends on the parameter $d \in \mathbb{R}^q$ and $v(d)$ is the volume of $Y(d)$. Then, to maximize the volume of the body gives the following GSIP called the maneuverability problem,

$$
\max_d v(d) \quad \text{s.t.} \quad K^- \leq g(\Theta, \dot{\Theta}, \ddot{\Theta}) \leq K^+, \text{ for all } (\Theta, \dot{\Theta}) \in Q, \ddot{\Theta} \in Y(d).
$$

(5)
**Geometry.** Semi-infinite problems can be interpreted in a geometrical setting. Given a family of sets $S(x) \subset \mathbb{R}^p$ depending on $x \in \mathbb{R}^n$, we wish to find a ‘body’ $Y$ of a certain class and a value $x$ such that $Y$ is contained in $S(x)$ and $Y$ is as large as possible.

The mathematical formulation is as follows. Suppose $S(x)$ is defined by

$$S(x) = \{ y \in \mathbb{R}^p \mid g(x, y, t) \geq 0, \text{ for all } t \in Q \}$$

where $Q$ is a given compact set in $\mathbb{R}^s$ and $g \in C^2(\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^s, \mathbb{R})$. Let the body $Y(d) \subset \mathbb{R}^p$ be parameterized by $d \in \mathbb{R}^q$ with $v(d)$, a measure for the size of $Y(d)$ (e.g. the volume). To maximize $v(d)$ for $Y(d) \subset S(x)$ then becomes:

$$\max_{d,x} v(d) \text{ s.t. } g(x, y, t) \geq 0 \text{ for all } y \in Y(d), t \in Q. \quad (6)$$

For the case that the set $S$ is fixed this problem is known as design centering problem.

**Optimization under uncertainty (Robust optimization).** As a concrete situation we consider a linear program

$$\min_x c^T x \text{ s.t. } a^T x - b_j \geq 0, \forall j \in J,$$

$J$ a finite index set. Often in the model the data $a_j$ and $b_j$ are not known exactly. It is only known that the vectors $(a_j, b_j)$ may vary in a set $Y_j \subset \mathbb{R}^{n+1}$. In a “pessimistic way” we now can restrict the problem to such $x$ which are feasible for all possible data vectors leading to a SIP

$$\min_x c^T x \text{ s.t. } a^T x - b \geq 0, \forall (a, b) \in Y := \bigcup_{j \in J} Y_j.$$

In the next example we discuss such a “robust optimization” model in economics. For more details we refer to [29], [35], [3], [23] and [1].

**Economics.** We consider the following Portfolio problem. We wish to invest $K$ euros into $n$ shares, say for a period of one year. We invest $x_i$ euros in share $i$ and expect at the end of the period a return of $y_i$ euros per 1 euro investment in share $i$.

Our goal is to maximize the portfolio value $v = y^T x$ after a year, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. The problem is that the value $y$ is not known in advance. However often models from economics suggest that the vector $y$ will vary in some compact subset $Y$ of $\mathbb{R}^n$. In this case we are led to solve the linear SIP:

$$\max_{v,x} v \text{ s.t. } y^T x - v \geq 0 \forall y \in Y$$

$$\sum_i x_i = K, \quad x \geq 0.$$

We refer to [30] for details and numerical experiments.

**Different other applications** There are other interesting applications in game theory (see [27]) in data envelopment analysis (see [18]) and in probability theory (see [5]).
3 Constraints and constraint qualifications

3.1 Constraints

In this section we discuss the structure of the feasible sets \( \mathcal{F} \) of finite and semi-infinite optimization problems.

\( \mathcal{F} \) in linear programming (LP): The feasible set is defined by finitely many linear constraints

\[
\mathcal{F} = \{ x \mid a_j^T x \geq b_j, \; j \in J \}
\]

with a finite index set \( J = \{1, \ldots, m\} \) and \( a_j \in \mathbb{R}^n, \; b_j \in \mathbb{R} \).

\( \mathcal{F} \) in semi-infinite linear programming (LSIP): The feasible set is defined by infinitely many linear constraints

\[
\mathcal{F} = \{ x \mid a_y^T x \geq b_y, \; y \in Y \}
\]

with an infinite index set \( Y \subset \mathbb{R}^m \) and functions \( y \to a_y \in \mathbb{R}^n, \; y \to b_y \in \mathbb{R} \).

Any such set \( \mathcal{F} \) is closed and convex but also the converse can be proven with an appropriate "separation theorem".

**Lemma 1** A subset \( \mathcal{F} \subset \mathbb{R}^n \) is convex and closed iff it can be defined in the form (7).

**Proof.** To prove that the closed convex set \( \mathcal{F} \) can be written in the form (7) let us chose a point \( y \notin \mathcal{F} \) arbitrarily. In view of the Separation Theorem (e.g., [7, Th.10.1]) the point \( y \notin \mathcal{F} \) can be separated from the closed convex set \( \mathcal{F} \) by a separating halfspace \( H_y^\geq = \{ x \mid a_y^T x \geq b_y \}, \) i.e.,

\[
a_y^T x \geq b_y > a_y^T y \quad \forall x \in \mathcal{F}.
\]

So, the set \( \mathcal{F} \) can be written as \( \mathcal{F} = \bigcap_{y \notin \mathcal{F}} H_y^\geq = \{ x \mid a_y^T x \geq b_y, \; \forall y \notin \mathcal{F} \} \).

\( \mathcal{F} \) in finite programming (FP): The feasible set is defined by finitely many (nonlinear) constraints

\[
\mathcal{F} = \{ x \mid g_j(x) \geq 0, \; j \in J \}
\]

where \( J = \{1, \ldots, m\} \) is finite and \( g_j : \mathbb{R}^n \to \mathbb{R} \) are continuous.

\( \mathcal{F} \) in semi-finite programming (SIP): Feasible set defined by infinitely many (nonlinear) constraints

\[
\mathcal{F} = \{ x \mid g(x, y) \geq 0, \; y \in Y \}
\]

with infinite index set \( Y \subset \mathbb{R}^m \) and continuous \( g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \).

\( \mathcal{F} \) in generalized semi-finite programming (GSIP):

\[
\mathcal{F} = \{ x \mid g(x, y) \geq 0, \; y \in Y(x) \}
\]
with variable index set $Y(x)$ defined by a set valued mapping $Y : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and continuous $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$.

The topological structure of the feasible sets $\mathcal{F}$ in FP and SIP are the same.

**Ex. 1** The feasible sets $\mathcal{F}$ in FP and SIP are closed.

The feasible set in GSIP need not be closed.

**Ex. 2** Consider the problem

$$\min_{x \in \mathbb{R}} x \quad \text{s.t.} \quad x \geq 1 - y, \quad \forall y \in Y(x) = \{ y \in \mathbb{R} \mid 0 \leq y, \ y \leq -x \}$$

Obviously, the feasible set consists of the open interval $(0, \infty)$ ($Y(x) = \emptyset$ means that $x$ is feasible). A minimizer of the program does not exist. The problem here is that the index set $Y(x) = [0, -x]$ for $x \leq 0$, $Y(x) = \emptyset$ for $x > 0$, is not continuous at $x = 0$.

If $Y(x)$ is continuous then, non-closedness of $\mathcal{F}$ is excluded.

**Lemma 2** Suppose the mapping $Y : K \rightarrow \mathbb{R}^m$ is continuous on a compact set $K \subset \mathbb{R}^n$. Then the feasible set $\mathcal{F} \cap K$ of GSIP is compact (in particular closed).

### 3.2 Constraint qualifications (CQ)

Throughout the subsection we assume that $g_j$ and $g$ are $C^1$-functions.

**CQ in FP:** For $\bar{x} \in \mathcal{F}$ (see (8) we introduce the active index set $J_0(\bar{x}) = \{ j \in J \mid g_j(\bar{x}) = 0 \}$. Locally near $\bar{x}$ the feasible set is defined by the constraints $g_j \geq 0$, $j \in J_0(\bar{x})$. The Linear Independence Constraint Qualification (LICQ) is said to hold at $\bar{x}$ if the vectors

$$\nabla g_j(\bar{x}), \ j \in J_0(\bar{x}),$$

are linearly independent.

Under LICQ at $\bar{x} \in \mathcal{F}$ locally (up to a diffeomorphism) the feasible has a simple geometrical form.

**Lemma 3** Let at $\bar{x} \in \mathcal{F}$ the condition LICQ hold with say $J_0(\bar{x}) = \{1, \ldots, p\}$ ($p \leq n$). Then in a (sufficiently small) neighborhood $U$ of $\bar{x}$ “up to a local diffeomorphism” the set $\mathcal{F} \cap U$ is given by

$$\begin{align*}
x \in U \quad \text{s.t.} \quad x_j & \geq 0, \quad j = 1, \ldots, p.
\end{align*}$$

**Proof.** By LICQ the gradients $\nabla g_j(\bar{x}), \ j = 1, \ldots, p$ are linearly independent and we can complete these vectors to a basis of $\mathbb{R}^n$ by adding vectors $v_j, \ j = p + 1, \ldots, n$. Now we define the transformation $y = T(x)$ by

$$\begin{align*}
y_j &= g_j(x), \quad i = 1, \ldots, p, \\
y_j &= v_j^T(x - \bar{x}), \quad i > p.
\end{align*}$$

By construction, the Jacobian $\nabla T(\bar{x})$ is regular. So $T$ defines locally a diffeomorphism. This means that there exists $\varepsilon > 0$ and neighborhoods $B_\varepsilon(\bar{x})$ of $\bar{x}$ and $U_\varepsilon(\bar{y}) :=
Let the feasible set $\mathcal{F}$ of a FP be compact and suppose LICQ holds at any point $x \in \mathcal{F}$. Then for any (small) $C^1$-perturbation $\tilde{g}_j$ of the functions $g_j$ the perturbed feasible set $\tilde{\mathcal{F}} = \{x \mid \tilde{g}_j(x) \geq 0, \ j \in J\}$ is diffeomorphic to $\mathcal{F}$.

We now present a “bad example”: $\mathcal{F}$ is described by the inequalities in $\mathbb{R}^3$

$$g_1(x) := x_3 \geq 0, \quad g_2(x) := x_3 + x_1^2 + x_2^2 \leq 0.$$ 

Obviously $\mathcal{F} = \{0\}$ has (only) dimension 0, and any small perturbation $\tilde{g}(x) := x_3 - \epsilon \geq 0$ ($\epsilon > 0$) yields an empty feasible set.

The following constraint qualification (which is weaker than LICQ, see Ex. 3) excludes such a bad behavior. The Mangasarian Fromovitz Constraint Qualification (MFCQ) is said to hold at $\overline{x}$ if there exists a vector $d \in \mathbb{R}^n$ such that

$$\nabla g_j(\overline{x}) d > 0, \quad j \in J_0(\overline{x}).$$

EX. 3 If LICQ holds at $\overline{x} \in F$, then also MFCQ is satisfied.

EX. 4 Let MFCQ hold at $\overline{x} \in F$ with the vector $d$. Then for any $\tau > 0$ small enough the points $\overline{x} + \tau d$ are interior points of $\mathcal{F}$.

Structural stability is equivalent with MFCQ as is shown in [17]. (We again present the result informally.)

Let the feasible set $\mathcal{F}$ of a FP be compact. Then for any (small) $C^1$-perturbation $\tilde{g}_j$ of the functions $g_j$ the perturbed feasible set $\tilde{\mathcal{F}} = \{x \mid \tilde{g}_j(x) \geq 0, \ j \in J\}$ is (Lipschitz-) homeomorphic to $\mathcal{F}$ if and only if MFCQ is satisfied at any point $x \in \mathcal{F}$. 


CQ in SIP: For \( \bar{x} \in F \) (see (9) we define the active index set \( Y_0(\bar{x}) = \{ y \in Y \mid g(\bar{x}, y) = 0 \} \).

The Linear Independence Constraint Qualification (LICQ) is said to hold at \( \bar{x} \in F \) if the vectors
\[
\nabla_x g(\bar{x}, y), \ y \in Y_0(\bar{x}),
\]
form a linearly independent set.

Note that if LICQ holds at \( \bar{x} \) the active set \( Y_0(\bar{x}) \) cannot contain more than \( n \) points.

The Mangasarian Fromovitz Constraint Qualification (MFCQ) is said to hold at \( \bar{x} \) if there exists a vector \( d \in \mathbb{R}^n \) such that
\[
\nabla_x g(\bar{x}, y)d > 0, \ \forall y \in Y_0(\bar{x}).
\]

The statements of Ex. 3, 4 remain true in the SIP case as well as the the structural stability result (see [20] for a precise formulation and a proof):

\[
\text{Let the feasible set } F \text{ of a SIP be compact. Then for any (small) } C^1\text{-perturbation } \tilde{g} \text{ of the function } g \text{ the perturbed feasible set } \tilde{F} = \{ x \mid \tilde{g}(x, y) \geq 0, \ y \in Y \} \text{ is (Lipschitz-) homeomorphic to } F \text{ if and only if MFCQ is satisfied at any point } x \in F.\]

4 First order optimality conditions

In this section, first order optimality conditions are derived for the SIP problem \( P \) in (1). We assume \( f, g \in C^1 \).

A feasible point \( \bar{x} \in F \) is called a local minimizer of SIP if there is some \( \varepsilon > 0 \) such that
\[
f(x) - f(\bar{x}) \geq 0 \quad \text{for all } x \in F \text{ with } \|x - \bar{x}\| < \varepsilon. \quad (11)
\]

The minimizer \( \bar{x} \) is said to be global if this relation holds for any \( \varepsilon > 0 \). We call \( \bar{x} \in F \) a strict local minimizer of order \( p > 0 \) if there exist some \( q > 0 \) and \( \varepsilon > 0 \) such that
\[
f(x) - f(\bar{x}) \geq q \|x - \bar{x}\|^p \quad \text{for all } x \in F \text{ with } \|x - \bar{x}\| < \varepsilon. \quad (12)
\]

It is not difficult to see that near a point \( \bar{x} \in F \), where the active index set \( Y_0(\bar{x}) = \{ y \in Y \mid g(\bar{x}, y) = 0 \} \) is empty, the SIP problem represents a common unconstrained minimization problem. So throughout the paper we assume that for any (candidate) minimizer of \( P \) the condition \( Y_0(\bar{x}) \neq \emptyset \) is satisfied.

**Lemma 4** [Primal necessary optimality condition] Let \( \bar{x} \in F \) be a local minimizer of \( P \). Then there cannot exist a strictly feasible descent direction \( d \), i.e., a vector \( d \in \mathbb{R}^n \) satisfying the relations
\[
\nabla f(\bar{x})d < 0, \ \nabla_x g(\bar{x}, y)d > 0, \ \forall y \in Y_0(\bar{x}).
\]

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Proof. Let \( d \) be a strictly feasible descent direction. Then \( f(\bar{x} + \tau d) < f(\bar{x}) \) holds for small \( \tau > 0 \) (recall \( f \in C^1 \)). We claim that there exists some \( \tau_0 > 0 \) with the property

\[
g(\bar{x} + \tau d, y) > 0 \quad \text{for all} \quad 0 < \tau \leq \tau_0 \quad \text{and} \quad y \in Y
\]

showing that \( \bar{x} \) cannot be a local minimizer. Suppose that the claim is false. Then to each \( k \geq 1 \) there exists some \( 0 < \tau_k < 1/k \) and some \( y_k \in Y \) such that \( g(\bar{x} + \tau_k d, y_k) \leq 0 \). Since \( Y \) is compact, there must exist some convergent subsequence \( (\tau_{k_i}, y_{k_i}) \) such that \( \tau_{k_i} \rightarrow 0 \) and \( y_{k_i} \rightarrow y^* \in Y \). The continuity of \( g \) then yields \( g(\bar{x} + \tau_{k_i} d, y_{k_i}) \rightarrow g(\bar{x}, y^*) \), which implies \( g(\bar{x}, y^*) = 0 \) and \( y^* \in Y_0(\bar{x}) \). On the other hand, the Mean Value Theorem provides us with numbers \( 0 < \kappa_{k_i} < \tau_{k_i} \) such that

\[
0 \geq g(\bar{x} + \tau_{k_i} d, y_{k_i}) - g(\bar{x}, y_{k_i}) = \tau_{k_i} \nabla_x g(\bar{x} + \kappa_{k_i} d, y_{k_i}) d
\]

and hence \( \nabla_x g(\bar{x} + \kappa_{k_i} d, y_{k_i}) d \leq 0 \). So the continuity of \( \nabla_x g \) entails

\[
\nabla_x g(\bar{x}, y^*) d = \lim_{i \rightarrow \infty} \nabla_x g(\bar{x} + \kappa_{k_i} d, y_{k_i}) d \leq 0,
\]

which contradicts the hypothesis on \( d \).

\[\square\]

**Theorem 1.** [First order sufficient condition] Let \( \bar{x} \) be feasible for \( P \). Suppose that there does not exist a vector \( 0 \neq d \in \mathbb{R}^n \) satisfying

\[
\nabla f(\bar{x}) d \leq 0, \quad \nabla_x g(\bar{x}, y) d \geq 0, \quad \forall y \in Y_0(\bar{x})\).
\]

Then \( \bar{x} \) is a strict local minimizer of SIP of order \( p = 1 \).

Proof. If \( \bar{x} \) is an isolated feasible point then the result is trivially fulfilled. If not then, it is not difficult to see that the set \( K(\bar{x}) := \{d \in \mathbb{R}^n \mid \|d\| = 1, \nabla_x g(\bar{x}, y) d \geq 0, \forall y \in Y_0(\bar{x})\} \) is nonempty. (Consider with a sequence of \( x_k \in \mathcal{F}, x_k \rightarrow \bar{x} \), an accumulation point \( d \) of \( (x_k - \bar{x})/\|x_k - \bar{x}\| \)). Since \( K(\bar{x}) \) is also compact, the value

\[
c_1 = \min_{d \in K(\bar{x})} \nabla f(\bar{x}) d
\]

is attained by some \( d' \in K(\bar{x}) \). By assumption, \( c_1 = \nabla f(\bar{x}) d' > 0 \) holds. Fix any \( 0 < c < c_1 \). We claim that there is some \( \varepsilon > 0 \) with the property

\[
f(x) - f(\bar{x}) \geq c \|x - \bar{x}\| \quad \text{for all} \quad x \in \mathcal{F}, \quad \|x - \bar{x}\| < \varepsilon.
\]

Suppose to the contrary that this claim is false and there is an infinite sequence of feasible points \( x_k \rightarrow \bar{x} \) with the property \( f(x_k) - f(\bar{x}) < c \|x_k - \bar{x}\| \). We write \( x_k = \bar{x} + \tau_k d_k \) with \( \tau_k > 0 \) and \( d_k \in S = \{d \in \mathbb{R}^n \mid \|d\| = 1\} \). Then, \( x_k \rightarrow \bar{x} \) implies \( \tau_k \rightarrow 0 \). Moreover, the compactness of \( S \) ensures that some subsequence of \( (d_k) \) converges. Without loss of generality, we thus assume \( d_k \rightarrow \bar{d}, \bar{d} \in S \).
The differentiability assumption together with the feasibility condition \( g(x_k, y) \geq 0 \) and the property \( g(\bar{x}, y) = 0 \) implies

\[
\begin{align*}
  c |\tau_k| & > f(x_k) - f(\bar{x}) = \tau_k \nabla f(\bar{x}) d_k + o(|\tau_k|), \\
  0 & \leq g(x_k, y) - g(\bar{x}, y) = \tau_k \nabla g(\bar{x}, y) d_k + o(|\tau_k|), \quad y \in Y_0(\bar{x}).
\end{align*}
\]

Divide all these relations by \( \tau_k \). Letting \( k \to \infty \) (or \( k \to 0 \)), we conclude that \( \nabla g(\bar{x}, y) d \geq 0 \), i.e., \( d \in K(\bar{x}) \) and \( \nabla f(\bar{x}) d \leq c \). This contradicts our choice of \( c < c_1 \). So (13) must be true.

\[ \square \]

**REMARK.** The assumptions of Theorem 1 are rather strong and can be expected to hold only in special cases (in Chebyshev approximation cf. e.g. [12]). It is not difficult to see that the assumptions imply that the set of gradients \( \{ \nabla x g(\bar{x}, y) \mid y \in Y_0(\bar{x}) \} \) contain a basis of \( \mathbb{R}^n \). So in particular \( |Y_0(\bar{x})| \geq n \).

More general sufficient optimality conditions need second order information (cf. Section 6 below).

We now derive the famous *Fritz John* (FJ) and the *Karush-Kuhn-Tucker* (KKT) optimality conditions.

**THEOREM 2** [Dual Necessary Optimality Conditions] Let \( \bar{x} \) be a local minimizer of \( P \). Then the following holds.

(a) There exist multipliers \( \mu_0, \mu_1, \ldots, \mu_{k+1} \geq 0 \) and indices \( y_1, \ldots, y_{k+1} \in Y_0(\bar{x}) \), \( k \leq n \), such that \( \sum_{j=0}^{k+1} \mu_j = 1 \) and

\[
\mu_0 \nabla f(\bar{x}) - \sum_{j=1}^{k+1} \mu_j \nabla x g(\bar{x}, y_j) = 0. \tag{14}
\]

(b) If \( MFCQ \) holds at \( \bar{x} \), then there exist multipliers \( \mu_1, \ldots, \mu_k \geq 0 \) and indices \( y_1, \ldots, y_k \in Y_0(\bar{x}) \), \( k \leq n \), such that

\[
\nabla f(\bar{x}) - \sum_{j=1}^{k} \mu_j \nabla x g(\bar{x}, y_j) = 0. \tag{KKT-condition} \tag{15}
\]

**Proof.** Consider the set \( S = \{ \nabla f(\bar{x}) \} \cup \{ -\nabla x g(\bar{x}, y) \mid y \in Y_0(\bar{x}) \} \subseteq \mathbb{R}^n \). Since \( \bar{x} \) is a local minimizer of \( P \), there is no strictly feasible descent direction \( d \) at \( \bar{x} \) (cf. Lemma 4). This means that

there is no \( d \in \mathbb{R}^n \) with \( d^T s < 0 \) for all \( s \in S \).

Now \( Y_0(\bar{x}) \) is compact. Therefore (by continuity of \( \nabla x g(\bar{x}, \cdot) \)) also \( S \) is compact. By Lemma 9 it follows \( 0 \in \text{conv} \ S \). In view of Caratheodory’s Lemma 7, 0 is a convex combination of at most \( n+1 \) elements of \( S \), i.e.,

\[
\sum_{j=0}^{k} \mu_j s_j = 0 \quad s_j \in S, \quad \mu_j \geq 0, \quad \sum_{j=0}^{k} \mu_j = 1 \quad \text{with } k \leq n, \tag{16}
\]

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which implies (a).

Now assume that \( d \in \mathbb{R}^n \) is a strictly feasible direction at \( \bar{x} \) i.e., MFCQ holds. For statement (b) it suffices to show: \( \mu_0 \neq 0 \) in the representation (a) (as division by \( \mu_0 > 0 \) in (14) yields a representation of type (15)). Suppose to the contrary that \( \mu_0 = 0 \) is true, and multiply (14) with \( d \). Since \( \mu_j > 0 \) holds for at least one \( j \geq 1 \), we obtain the contradiction

\[
0 > -\sum_{j=1}^{k+1} \mu_j \nabla g(\bar{x}, y_j)d = 0^T d = 0.
\]

\[ \square \]

**Remark 2** Note that the results of this and the next section in particular remain true for a finite program. In fact, given a finite program (see (8), we only have to chose a finite set \( Y = \{y_1, \ldots, y_m\} \) and to identify \( g(x, y_j) := g_j(x), j = 1, \ldots, m \).

**Ex. 5** Assume that \( \bar{x} \in \mathcal{F} \) is a KKT-point such that (15) holds with \( k = n \) linear independent vectors \( \nabla g(\bar{x}, y_j), j = 1, \ldots, n, \) (LICQ) and \( \bar{\mu}_j \geq 0 \). Show that then the sufficient conditions of Theorem 1 are satisfied.

## 5 Convex and linear semi-infinite programs.

### 5.1 Convex SIP

The semi-infinite program is called *convex* if the objective function \( f(x) \) is convex and, for every index \( y \in Y \), the constraint function \( g_y(x) = g(x, y) \) is concave (i.e., \(-g_y(x) \) is convex).

**Ex. 6** It is not difficult to show that for a convex SIP the feasible set is convex and that any local minimizer \( \bar{x} \) is a global minimizer. Moreover, (if \( g \in C^1 \)) the existence of a MFCQ vector \( d \) at a feasible point \( \bar{x} \) is equivalent with the Slater condition: There exists a feasible point \( \hat{x} \) such that \( g(\hat{x}, y) > 0 \) for all \( y \in Y \).

Recall that a local minimizer of a convex program is actually a global one (see Ex. 6). As in finite programming, the KKT-conditions are sufficient for optimality.

**Theorem 3** Assume that the SIP problem \( P \) is convex. If \( \bar{x} \) is a feasible point that satisfies the Kuhn-Tucker condition (15), then \( \bar{x} \) is a (global) minimizer of \( P \).

**Proof.** By the convexity assumption, we have for every feasible \( x \) and \( \bar{y} \in Y_0(\bar{x}) \) (cf. Lemma 8)

\[
f(x) - f(\bar{x}) \geq \nabla f(\bar{x})(x - \bar{x})
\]

\[
0 \leq g(x, \bar{y}) = g(x, \bar{y}) - g(\bar{x}, \bar{y}) \leq \nabla g(\bar{x}, \bar{y})(x - \bar{x}).
\]
Hence, if there are multipliers \( \mu_j \geq 0 \) and index points \( y_j \in Y_0(\bar{x}) \) such that \( \nabla f(\bar{x}) = \sum_{j=1}^{k} \mu_j \nabla x g(\bar{x}, y_j), \) we conclude

\[
f(x) - f(\bar{x}) \geq \nabla f(\bar{x})(x - \bar{x}) = \sum_{j=1}^{k} \mu_j \nabla x g(\bar{x}, y_j)(x - \bar{x}) \geq 0.
\]

\( \square \)

### 5.2 Linear SIP

An important special case of (convex) SIP is given by the *linear semi-infinite problem* (LSIP), where the objective function \( f \) and the function \( g \) are linear in \( x \):

\[
P: \quad \min \limits_{x} \ c^T x \quad \text{s.t.} \quad a^T y x \geq b_y \quad \forall y \in Y. \tag{17}
\]

For an intensive treatment of LSIP we refer to the monograph [10]. Here we only derive strong duality results.

Recall that any convex optimization problem

\[
\min \limits_{x} \ c^T x \quad \text{s.t.} \quad x \in \mathcal{F}, \quad \mathcal{F} \subset \mathbb{R}^n \quad \text{a closed convex set},
\]

can be written as a LSIP. LSIP is said to be continuous, if \( Y \) is compact and the functions \( y \to a_y = a(y), \ y \to b_y = b(y) \) are continuous on \( Y \).

**Ex. 7** Let \( P \) be a continuous LSIP. Show that the function \( \varphi(x) := \min_{y \in Y} \{ a(y)^T x - b(y) \} \) is concave. So LSIP can be written as a convex program: \( \min \limits_{x} \ c^T x \) s.t. \( \varphi(x) \geq 0 \).

Assume from now on that LSIP is continuous. We also write (17) formally in the familiar form

\[
(P) \quad \min \limits_{x} \ c^T x \quad \text{s.t.} \quad Ax \geq b,
\]

where \( A \) is a matrix with infinitely many rows \( a^T_y, \ y \in Y \) and \( b \) is a vector with infinitely many components \( b_y \). We now define the (so-called Haar) *dual* of (P) as

\[
(D) \quad \max \limits_{u} \ b^T u \quad \text{s.t.} \quad A^T u = c, \quad u \geq 0,
\]

with the following understanding: \( u = (u_y) \) is *dual feasible* if \( u_y \geq 0, \ y \in Y \) and \( u_y \geq 0 \) for only *finitely many* \( y \in Y \). So \( b^T u = \sum_{y} b_y u_y \) is actually a finite sum and so is \( A^T u = \sum_{y} a_y u_y \). Note that, by definition,

\[
(D) \text{ is feasible} \iff c \in \text{cone} \{ a_y \mid y \in Y \}. \tag{18}
\]

The optimal objective function values of (P) and (D) will be denoted by \( v_P \) and \( v_D \).
REMARK. Recall from Caratheodory’s Lemma 7 that the sum $c = \sum_y a_y u_y$ can be expressed as sums with at most $n$ non-zero coefficients $u_y$. As in LP, if $x \in \mathbb{R}^n$ and $u = (u_y)$ are primal resp. dual feasible, then

$$c^T x - u^T b = u^T (Ax - b) = \sum_y u_y (a_y^T x - b_y) \geq 0.$$ 

**THEOREM 4** (Weak duality, complementary slackness) If $x$ and $u$ are primal resp. dual feasible, then $c^T x \geq b^T u$. If $c^T = b^T u$ then $x$ resp. $u$ are optimal with $v_P = v_D$.

We emphasize that in contrast to ordinary (finite) linear programs, linear semi-infinite problems do not necessarily have the strong duality property unless additional constraint qualifications hold (cf. Ex. 8). A further notable difference to finite linear programming is the fact that the existence of primal and dual feasible solutions need not imply the existence of optimal solutions (cf. Ex. 9). To assure e.g. the existence of an optimal solution of (P) we may assume strict feasibility of the dual program.

The **Slater constraint qualification** for (P) assumes the existence of a strictly feasible primal solution (**Slater point**, see also Ex. 6):

(SC$_P$) There exists some $\hat{x} \in \mathbb{R}^n$ with $A\hat{x} > b$.

We also strengthen the dual feasibility condition (18) to the dual **Slater condition**:

(SC$_D$) $c \in \text{int cone } \{a_y | y \in Y\}$.

One can show that a feasible LSIP has an minimal solution if the condition SC$_D$ is fulfilled. This leads to the following strong duality result (see e.g., [7] for a proof).

**THEOREM 5** (Strong duality) Assume that the Slater conditions (SC$_P$) and (SC$_D$) are satisfied, i.e., (P) and (D) are strictly feasible. Then optimal primal and dual solutions $\bar{x}$ and $\bar{u}$ exist, and $v_P = v_D$ holds. Moreover in this case $\bar{x}$ is an optimal solution of (P) if and only if $\bar{x}$ is a KKT-point, i.e., there exist a dually feasible $\bar{u}$ satisfying $\bar{u}_y, j \geq 0$ for $y_j \in Y_0(\bar{x}), \ j = 1, \ldots, k$, $k \leq n$, and $\bar{u}_y = 0$ otherwise, such that

$$c = \sum_{j=1}^k \bar{u}_y a_{y_j}.$$ 

In particular $\bar{u}^T (A\bar{x} - b) = 0$.

**Ex. 8** Let $\kappa > 0$ be fixed. Show $v_P = 0$ and $v_D = \kappa$ and that neither the primal nor the dual Slater condition is satisfied for the linear (SIP)

$$\min -x_2 \quad \text{s.t.} \quad -y^2 x_1 - yx_2 \geq 0, \quad y \in Y = [0, 1] \quad \text{and} \quad x_2 \leq \kappa.$$
**Ex. 9** Show that \( F = \{ (x_1, x_2) \mid x_2 \geq e^{x_1} \text{ for } x_1 \leq 0 \text{ and } x_2 \geq x_1 + 1 \text{ for } x_1 \geq 0 \} \) is the feasible set of the (LSIP)

\[
(P) \quad \min x_2 \quad \text{s.t.} \quad -yx_1 + x_2 \geq y(1 - \ln y), \quad y \in Y = [0, 1].
\]

Show furthermore: \( D \) is feasible and \( P \) satisfies the primal Slater condition but does not have an optimal solution. (Consequently, \( SC_D \) cannot be satisfied).

### 5.3 Linear semidefinite programs

A semidefinite program (SDP) is of the type

\[
(P) : \min_{x \in \mathbb{R}^n} \ c^T x \quad \text{s.t.} \quad \sum_{i=1}^{n} A_i x_i - B \succeq 0 \quad (\text{positive semidefinite}),
\]

where \( B, A_1, \ldots, A_n \in S^{k \times k} \) are given \( k \times k \) matrices. \( S^{k \times k} \) denotes the set of (real) symmetric \( k \times k \) matrices. By the following trick a semidefinite program can be transformed into an LSIP. Recall that for \( S = (s_{ij}) \in S^{k \times k} \) the relation

\[
S \succeq 0 \iff y^T Sy = \sum_{i,j} s_{ij} y_i y_j \geq 0 \quad \forall y \in Y := \{ y \in \mathbb{R}^k \mid \|y\| = 1 \}
\]

holds. So we may state (19) equivalently as

\[
\min_{x} \ c^T x \quad \text{s.t.} \quad a(y)^T x - b(y) \geq 0 \quad \forall y \in Y
\]

with

\[
a(y)^T = (y^T A_1 y, \ldots, y^T A_n y) \quad \text{and} \quad b(y) = y^T B y.
\]

Let us denote the inner product of \((k \times k)\)-matrices \( C = (c_{ij}) \) and \( D = (d_{ij}) \) by \( C \circ D = \sum_{i,j=1}^{k} c_{ij} d_{ij} \). Then it is not difficult to see that the LSIP dual of (21) can be written in the form (see [7])

\[
(D) : \max_{U} \ B \circ U \quad \text{s.t.} \quad A_i \circ U = c_i, \ i = 1, \ldots, n, \quad U \succeq 0.
\]

The primal Slater condition takes the form:

\[
(SC_P) \quad \text{There exists } \hat{x} \in \mathbb{R}^n \quad \text{s.t.} \quad \sum A_i \hat{x}_i - B \succ 0,
\]

where \( S \succ 0 \) indicates that \( S \) is a positive definite matrix. Moreover it is not difficult to show (cf. [7]) that the dual slater condition can be written as

\[
(SC_D) \quad \text{there exists } U \succ 0 \text{ such that } A_i \circ U = c_i, \ i = 1, \ldots, n.
\]

So in principle, optimality and duality results for SDP can directly be deduced from the corresponding results in LSIP. We obtain in this way (see [7] for a proof)
**Corollary 1 (Strong duality)** Assume that the Slater conditions (SC\(_P\)) and (SC\(_D\)) are satisfied, i.e., \((P)\) and \((D)\) are strictly feasible. Then optimal primal and dual solutions \(\bar{x}\) and \(\bar{U}\) exist, and \(v_P = v_D\) holds. Moreover in this case \(\bar{x}\) is an optimal solution of \((P)\) if and only if there exist a dually feasible \(\bar{U}\) satisfying \(\bar{U} \cdot (\sum_i A_i \bar{x}_i - B) = 0\).

**Numerical methods for SDP:** From the numerical viewpoint, semidefinite programming plays a special role in LSIP. Generally we cannot expect that LSIP can be solved efficiently (i.e., in polynomial time with respect to the input data). However, as LP problems, also semidefinite programs can be solved efficiently e.g. by the interior point method or the ellipsoid method. We refer to [37] for an extensive treatment of this issue.

Here however, we will try to motivate the fact that SDP can be solved efficiently from the viewpoint of the ellipsoid method. Firstly, we emphasize that it is not difficult to see that a convex program can be solved efficiently if the corresponding feasibility problem can be solved efficiently. To find a feasible point \(\bar{x} \in \mathcal{F}\), the ellipsoid method proceeds as follows.

**Ellipsoid Method**

**INIT:** Start with a (sufficiently large) ellipsoid \(E_0\) so that \(\mathcal{F} \cap E_0 \neq \emptyset\).

**ITER:** Given the ellipsoid \(E_j = \{x \mid (x - x_j)^T A_j (x - x_j) \leq 1\}, A_j \succ 0\), such that \(\mathcal{F} \cap E_j = \mathcal{F} \cap E_0\)

(1) if \(x_j \in \mathcal{F}\) STOP

(2) OTHERWISE: compute a separating halfspace \(H_j^\geq\) such that \(\mathcal{F} \subseteq H_j^\geq, x_j \notin H_j^\geq\).

and find (the smallest) ellipsoid \(E_{j+1}\) with \(\text{vol } E_{j+1} \leq q \text{ vol } E_j \) \((0 < q < 1)\) containing \(E_j \cap H_j^\geq\) and thus \(\mathcal{F} \cap E_0\).

So the Ellipsoid Method generates ellipsoids \(E_0, E_1, \ldots\) with the property

\[
\text{vol } (\mathcal{F} \cap E_0) \leq \text{vol } E_j \leq q^j \text{ vol } E_0 \to 0.
\]

Hence, if \(0 < \text{vol } (\mathcal{F} \cap E_0)\), it is clear that the algorithm will find a point \(x_j \in \mathcal{F}\) after a finite number of iterations.

The overall ellipsoid method can be shown to be polynomial if the feasibility test \(\bar{x} \in \mathcal{F}\) in step (1) and the construction of a separating halfspace in step (2) can be done in polynomial time. Let \(G(x) := \sum_{i=1}^n A_i x_i - B \in \mathbb{S}^{k \times k}\).

**Lemma 5** The feasibility test \(G(\bar{x}) \geq 0\) and the construction of a separating halfspace can be done in time \(O(k^3)\).
Proof. It is well-known that by applying a modification of the Gauss algorithm to \( \overline{G} = G(x) \) we can compute, in time \( O(k^3) \), a decomposition

\[
Q \overline{G} Q^T = D \quad \text{with } Q \text{ regular and } D = \text{diag} (d_1, \ldots, d_k),
\]
such that \( \overline{G} \succeq 0 \) if and only if \( d_i \geq 0, \forall i \). So if \( d_i \geq 0, \forall i \), the point \( x \) is feasible. If not say \( d_1 < 0 \), than by defining \( a_1 = Q^T e_1 \) (\( e_1 \) the unit vector) it follows for all feasible \( x \), i.e., \( G(x) \succeq 0 \):

\[
a_1^T G(x) a_1 \geq 0 > d_1 = e_1^T De_1 = e_1^T Q \overline{G} Q^T e_1 = a_1^T \overline{G} a_1 .
\]

So the linear inequality \( a_1^T G(x) a_1 = \sum_{i=1}^n (a_1^T A_i a_1) x_i - a_1^T B a_1 \geq 0 \) defines a separating halfspace.

\( \square \)

6 Second order optimality conditions

We now derive second order optimality conditions for semi-infinite problems (1) and (2) by applying the so-called reduction approach which goes back to Wetterling [36].

Let \( \overline{x} \in \mathcal{F} \) be a feasible point of SIP or GSIP. We assume that \( f, g \in C^2 \) and that the index set \( Y(x) \) is defined as the solution set of inequalities with functions \( v_l \in C^2(\mathbb{R}^{n \times m}, \mathbb{R}) \):

\[
Y(x) = \{ y \in \mathbb{R}^m \mid v_l(x, y) \geq 0, \ l \in L \} , \quad L := \{1, \ldots, q\} .
\]

The following observation is crucial for this approach. By definition, any active point \( \overline{y} \) from \( Y_0(\overline{x}) \) is a (global) minimizer of the following parametric optimization problem

\[
Q(\overline{x}) : \quad \min_y g(\overline{x}, y) \quad \text{s.t. } v_l(\overline{x}, y) \geq 0, \ l \in L , \quad (23)
\]

the so-called lower level problem, depending on \( \overline{x} \) as a parameter. So (under some CQ) any point \( \overline{y} \in Y_0(\overline{x}) \) must satisfy the KKT-conditions for the finite program \( Q(\overline{x}) \),

\[
\nabla_y L^{(\overline{x}, \overline{y})}(\overline{x}, \overline{y}, \overline{y}) = 0 , \quad (24)
\]

with a multiplier vector \( 0 \leq \overline{\gamma} \in \mathbb{R}^{|L_0(\overline{x}, \overline{y})|} \) where \( L_0(\overline{x}, \overline{y}) = \{ l \in L \mid v_l(\overline{x}, \overline{y}) = 0 \} \) is the active index set and \( L^{(\overline{x}, \overline{y})} \) the Lagrange function of \( Q(x) \) at \( \overline{x} \in \mathcal{F}, \overline{y} \in Y_0(\overline{x}) \),

\[
L^{(\overline{x}, \overline{y})}(x, y, \gamma) = g(x, y) - \sum_{l \in L_0(\overline{x}, \overline{y})} \gamma_l v_l(x, y) .
\]

We will assume, that the following strong regularity conditions (of finite optimization) are satisfied for all points in \( Y_0(\overline{x}) \).

\( \text{A}_{\text{red}}: \quad \text{For any } \overline{y} \in Y_0(\overline{x}) \ \text{the following holds (implying that } \overline{y} \text{ is a strict (local) minimizer of } Q(\overline{x}), \text{ see e.g. [7, Th.12.6]} : \)
1. LICQ: \( \nabla_y v_l(\bar{x}, \bar{y}) \), \( l \in L_0(\bar{x}, \bar{y}) \) are linearly independent.

2. (Under LICQ there are unique multipliers \( 0 \leq \bar{y} \in \mathbb{R}^{\vert L_0(\bar{x}, \bar{y}) \vert} \) satisfying (24)). We assume: \( \bar{y}_l > 0 \), \( l \in L_0(\bar{x}, \bar{y}) \) (strict complementary slackness).

3. The second order condition (SOC): With \( \bar{y} \) in 2.,
   \[ \eta^T \nabla_y^2 \mathcal{L}(\bar{x}, \bar{y}, \bar{y}) \eta > 0, \quad \text{for all } \eta \in T(\bar{x}, \bar{y}) \setminus \{0\} \]
   where \( T(\bar{x}, \bar{y}) = \{ \eta \in \mathbb{R}^m \mid \nabla_y v_l(\bar{x}, \bar{y}) \eta = 0, \ l \in L_0(\bar{x}, \bar{y}) \} \).

By using techniques from parametric optimization, under this assumption, the following can be proven.

\textbf{Theorem 6} Let at the feasible point \( \bar{x} \) for GSIP the condition \( A_{\text{red}} \) be satisfied. Then:

(a) The active index set is finite, \( Y_0(\bar{x}) = \{\bar{y}_1, \ldots, \bar{y}_r\} \), and there exist neighborhoods \( U_{\bar{x}} \) of \( \bar{x} \) and \( V_{\bar{y}_j} \) of \( \bar{y}_j \) and continuous mappings
   \[ y_j : U_{\bar{x}} \to V_{\bar{y}_j}, \quad y_j(\bar{x}) = \bar{y}_j \quad \text{and} \quad \gamma_j : U_{\bar{x}} \to \mathbb{R}^{\vert L_0(\bar{x}, \bar{y}) \vert}, \quad \gamma_j(\bar{x}) = \bar{y}_j \]
   such that for every \( x \in U_{\bar{x}} \) the value \( y_j(x) \) is the unique local minimizer of \( Q(x) \) in \( V_{\bar{y}_j} \) with corresponding Lagrange multiplier vector \( \gamma_j(x) \).

(b) With the functions in (a) the following finite reduction holds: \( x \in U_{\bar{x}} \cap \mathcal{F} \) is a local solution of GSIP if and only if \( \bar{x} \) is a local solution of the so-called reduced problem
   \[ P_{\text{red}}(\bar{x}) : \quad \min_{x \in U_{\bar{x}}} f(x) \quad \text{s.t.} \quad G_j(x) := g(x, y_j(x)) \geq 0, \quad j = 1, \ldots, r. \quad (26) \]

(c) The functions \( G_j(x) = g(x, y_j(x)) \) are \( C^2 \)-functions in \( U_{\bar{x}} \) with derivatives
   \[
   \begin{align*}
   \nabla G_j(x) &= \nabla_x \mathcal{L}(x, y_j(x), \gamma_j(x)), \\
   \nabla^2 G_j(x) &= \nabla_x^2 \mathcal{L}(x, y_j(x), \gamma_j(x)) - \nabla^T y_j(x) \nabla_y^2 \mathcal{L}(x, y_j(x), \gamma_j(x)) \nabla y_j(x) \\
   &\quad - \sum_{l \in L_0(\bar{x}, \bar{y})} \nabla^T [\gamma_j]_l(x) \nabla_x v_l(x, y_j(x)) + \nabla^T v_l(x, y_j(x)) \nabla [\gamma_j]_l(x).
   \end{align*}
   \]

\textit{Proof.} (See [32] for details). Choose \( j \in \{1, \ldots, r\} \) and put for brevity \( (\bar{y}, \bar{y}) = (\bar{y}_j, \bar{y}_j) \). Assume \( L_0(\bar{x}, \bar{y}) = \{1, \ldots, p\} \) and define \( v := (v_1, \ldots, v_p) \). Consider the following system of Karush-Kuhn-Tucker equations for \( Q(x) \), near \( (\bar{x}, \bar{y}, \bar{y}) \), with Lagrange function \( \mathcal{L} = \mathcal{L}(x, y, \gamma) \):

\[ F(x, y, \gamma) := \frac{\nabla_x \mathcal{L}(x, y, \gamma)}{v(x, y)} = 0 \]

Under assumption LICQ and SOC of \( A_{\text{red}} \) the Jacobian \( \nabla_{(y, \gamma)} F(\bar{x}, \bar{y}, \bar{y}) \) can be shown to be regular. The Implicit Function Theorem applied to (27) then yields the result (a) and (b). The derivative of \( y(x) (= y_j(x)) \) can be calculated by differentiating
the identity \( F(x, y(x), \gamma(x)) = 0 \). We obtain for the second relation (the arguments \( x, y(x), \gamma(x) \) omitted)

\[
\nabla_x v + \nabla_y v \nabla y = 0. \quad (28)
\]

For \( G(x) = g(x, y(x)) \) we find using \( \nabla_x g \nabla y = \gamma^T \nabla_x v \nabla y = -\gamma^T \nabla_x v \) (cf. (27), (28))

\[
\nabla G(x) = \nabla_x g(x, y(x)) + \nabla_y g(x, y(x)) \nabla y(x) \\
= \nabla_x g(x, y(x)) - \gamma^T(x) \nabla_x v(x) = \nabla_x \tilde{L}(x, y(x), \gamma(x)) \quad (29)
\]

and in a similar way the formula for \( \nabla^2 G \).

By this theorem, locally near \( \tilde{x} \), the problem GSIP is equivalent to \( P_{\text{red}}(\tilde{x}) \) in (26), which is a common finite optimization problem. Thus, the standard optimality conditions of finite optimization can be applied to obtain optimality conditions for the semi-infinite problem \( P \). To that end we define the cone of critical directions

\[
C(\tilde{x}) = \{ \xi \in \mathbb{R}^n \mid \nabla f(\tilde{x})\xi \leq 0, \ \nabla G_j(\tilde{x})\xi \geq 0, \ j = 1, \cdots, r \}
\]

and the Lagrange function (of the upper level)

\[
\tilde{L}(x, \mu) = \mu_0 f(x) - \sum_{j=1}^r \mu_j G_j(x) .
\]

We now state the optimality condition of FJ-type. For more details, necessary conditions and sufficient conditions under weaker assumptions, see [15] and [22].

**Theorem 7** Suppose, the assumption A\(_{\text{red}}\) is satisfied for \( \tilde{x} \in \mathcal{F} \) such that by Theorem 6, in a neighborhood \( U \) of \( \tilde{x} \), the problem GSIP can be locally reduced to \( P_{\text{red}}(\tilde{x}) \) according to (26). Then the following holds.

a. Suppose, \( \tilde{x} \) is a local minimizer of GSIP. Then, to any \( \xi \in C(\tilde{x}) \) there exists a multiplier \( \tilde{\mu} \geq 0 \) such that (with \( \nabla_x \tilde{L}, \nabla^2_x \tilde{L} \) given below)

\[
\nabla_x \tilde{L}(\tilde{x}, \tilde{\mu}) = 0 \quad \text{and} \quad \xi^T \nabla^2_x \tilde{L}(\tilde{x}, \tilde{\mu}) \xi \geq 0 .
\]

b. Suppose, for any \( \xi \in C(\tilde{x}) \setminus \{0\} \), that there exists a multiplier \( \tilde{\mu} \geq 0 \) such that

\[
\nabla_x \tilde{L}(\tilde{x}, \tilde{\mu}) = 0 \quad \text{and} \quad \xi^T \nabla^2_x \tilde{L}(\tilde{x}, \tilde{\mu}) \xi > 0 .
\]

Then \( \tilde{x} \) is a (strict) local minimizer of GSIP. The expressions for \( \nabla_x \tilde{L}(\tilde{x}, \tilde{\mu}) \) and \( \nabla^2_x \tilde{L}(\tilde{x}, \tilde{\mu}) \) read:

\[
\nabla_x \tilde{L}(\tilde{x}, \tilde{\mu}) = \mu_0 \nabla f(\tilde{x}) - \sum_{j=1}^r \tilde{\mu}_j \nabla_x g(\tilde{x}, \tilde{y}_j) + \sum_{j=1}^r \tilde{\mu}_j \left( \sum_{l \in L_0(\tilde{x}, \tilde{y}_j)} [\nabla_{y_j}]_l \nabla_x v_l(\tilde{x}, \tilde{y}_j) \right)
\]

\[
\nabla^2_x \tilde{L}(\tilde{x}, \tilde{\mu}) = \mu_0 \nabla^2 f(\tilde{x}) - \sum_{j=1}^r \tilde{\mu}_j \nabla_x^2 g(\tilde{x}, \tilde{y}_j) + \sum_{j=1}^r \tilde{\mu}_j \nabla^T y_j(\tilde{x}) \nabla^2 y_j L(\tilde{x}, \tilde{y}_j)(\tilde{x}, \tilde{y}_j, \tilde{y}_j) \nabla y_j(\tilde{x})
\]

\[
+ \sum_{j=1}^r \tilde{\mu}_j \left( [\nabla_{y_j}]_l \nabla_x^2 v_l(\tilde{x}, \tilde{y}_j) + \nabla^T [\gamma_j]_l(\tilde{x}) \nabla_x v_l(\tilde{x}, \tilde{y}_j) + \nabla^T y_j(\tilde{x}, \tilde{y}_j) \nabla [\gamma_j]_l(\tilde{x}) \right)
\]

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The first and second terms in $\nabla_{x} \widetilde{L}(\bar{x}, \bar{\mu})$ and $\nabla^{2}_{x} \widetilde{L}(\bar{x}, \bar{\mu})$ are present in finite optimization, the third term in $\nabla^{2}_{x} \widetilde{L}(\bar{x}, \bar{\mu})$ is the additional term for SIP and the third term in $\nabla_{x} \widetilde{L}(\bar{x}, \bar{\mu})$ and the fourth term in $\nabla^{2}_{x} \widetilde{L}(\bar{x}, \bar{\mu})$ are typical for GSIP, containing the dependence of $v_{l}$ (and $Y$) on $x$.

Proof. The formulas for $\nabla_{x} \widetilde{L}$ and $\nabla^{2}_{x} \widetilde{L}$ follow immediately by using the formulas for $\nabla G_{j}$, $\nabla^{2} G_{j}$ in Theorem 6.

Remark 3 We say that at $\bar{x} \in \mathcal{F}$ the condition LICQ is satisfied if for the functions in Theorem 6:

\[(\text{LICQ}) \quad \nabla G_{j}(\bar{x}) = \nabla_{x} \widetilde{L}(\bar{x}, \bar{\mu}_{j}), \quad \bar{y}_{j} \in Y_{0}(\bar{x}) \text{ are linearly independent.}\]

Note that under a constraint qualification at $\bar{x} \in \mathcal{F}$ the multiplier $\mu_{0}$ in the optimality conditions of Theorem 7 can be set to $\mu_{0} = 1$.

7 Numerical methods

7.1 Primal methods

We discuss the so-called method of feasible directions for the SIP problem (1). The idea is to move from a current (non-optimal) feasible point $x_{k}$ to the next point $x_{k+1} \approx x_{k} + t_{k}d_{k}$ in such a way that $x_{k+1}$ remains feasible and has a smaller objective value. The simplest choice would be to move along a (strictly) feasible descent direction $d$ defined by:

\[
\nabla f(\bar{x})d < 0, \quad \nabla_{x} g(\bar{x}, y)d > 0, \quad \forall y \in Y_{0}(\bar{x}).
\]

Feasible Direction Method (for FP, Zoutendijk)

INIT: Choose a starting point $x_{0} \in \mathcal{F}$
ITER: WHILE $x_{k}$ is not a Fritz John point DO
BEGIN
Choose a strictly feasible descent direction $d_{k}$.
Determine a solution $t_{k}$ for the problem
\[ (*) \quad \min \{ f(x_{k} + td_{k}) \mid t > 0, \ x_{k} + td_{k} \in \mathcal{F} \} \]
Set $x_{k+1} = x_{k} + t_{k}d_{k}$.
END
Let us consider the following “stable” way to obtain a strictly feasible descent direction \( d \) that takes all constraints into account. For FP this is the method suggested by Topkis and Veinott. We solve the linear SIP:

\[
\begin{align*}
\min_{x, z} & \quad \nabla f(x_k) d \\ 
\text{s.t.} & \quad \nabla g(x_k, y) d + g(x_k, y) \geq -z \\
& \quad y \in Y \\
& \quad \pm d_i \leq 1 \\
& \quad i = 1, \ldots, n.
\end{align*}
\]

(31)

**Remark.** Note that \((d, z) = (0, 0)\) is always feasible for (31) and if (31) has an optimal value \( z = 0 \) then (according to the proof of the next theorem) no strictly feasible descent direction \( d \) (cf, (30)) exists, i.e. (by Gordan’s Lemma 9), \( x_k \) satisfies the FJ necessary optimality conditions (14).

**Theorem 8** Assume that the Feasible Direction Method generates the points \( x_k \) and computes the \( d_k \) as solutions of (31). Suppose further that a subsequence \((x_s)\) converges to \( \bar{x} \), \( S \subseteq \mathbb{N} \) an infinite subset of indices \( s \). Then \( \bar{x} \) is a Fritz John point.

**Proof.** Since \( \mathcal{F} \) is closed, the limit \( \bar{x} \) of the points \( x_s \in \mathcal{F} \) also belongs to \( \mathcal{F} \), i.e., \( \bar{x} \) is feasible. Suppose that the Theorem is false and \( \bar{x} \) is not a Fritz John point. Then by Gordans’ Lemma 9 there exists a (strictly feasible descent) direction \( d \) (see (30)). With the same arguments as in the proof of Lemma 4 we find that there is some \( \tau_0 > 0 \) such that

\[ g(\bar{x}, y) + \tau \nabla g(\bar{x}, y) d > 0, \quad \forall 0 < \tau < \tau_0, \quad y \in Y. \]

So by choosing \( \tau > 0 \) such that \( \tau |d_i| \leq 1, \forall i \), we have shown that there exists \( z < 0 \) and a vector \( d(= \tau d) \) with \( |d_i| \leq 1 \) satisfying

\[ \nabla f(\bar{x}) d < 2z \quad \text{and} \quad g(\bar{x}, y) + \nabla g(\bar{x}, y) d > -2z, \quad \forall y \in Y. \]

The continuity of the gradients \( \nabla f(x) \) and \( \nabla g(x, y) \) and the compactness of \( Y \) now implies that also

\[ \nabla f(x_s) d_s < z \quad \text{and} \quad g(x_s, y) + \nabla g(x_s, y)d_s > -z, \quad \forall y \in Y \]

must hold if \( s \) is sufficiently large. So \((d, z)\) is feasible for (31), which in turn implies \( z \geq z_s \) for the optimal solution \((d_s, z_s)\) of (31) at \( x_s \). In particular, we note

\[ \nabla f(x_s) d_s < z \quad \text{and} \quad g(x_s, y) + \nabla g(x_s, y)d_s > -z, \quad \forall y \in Y. \]

Let \( \delta > 0 \) be such that \( ||\nabla f(x) - \nabla f(\bar{x})|| < |z|/3n \) holds whenever \( \|x - \bar{x}\| < \delta \). Then \( \|d_s\| \leq n \) together with the inequality of Cauchy-Schwarz yields

\[ |(\nabla f(x) - \nabla f(\bar{x})) d_s| \leq ||\nabla f(x) - \nabla f(\bar{x})|| \cdot \|d_s\| < |z|/3. \]

So, if \( \|x - \bar{x}\| < \delta \) and \( s \) is large

\[ \nabla f(x) d_s \leq \nabla f(\bar{x}) d_s + |z|/3 \leq \nabla f(x_s) d_s + 2|z|/3 \leq z/3. \]
Applying the same reasoning to $\nabla_x g(x, y)$, we therefore conclude
\[
g(x_s, y) + \nabla_x g(x, y) d_s \geq -z/3 \quad \forall y \in Y.
\]
We are interested in points $x$ of the form $x = x_s + t d_s$, $0 < t < 2n$. If $s$ is so large that $\|x - \bar{x}\| < \delta/2$, then $\|x - \bar{x}\| < \delta$. Moreover, the Mean Value Theorem guarantees some $0 < \tau < t$ that exhibits
\[
f(x_s + td_s) = f(x_s) + t\nabla f(x_s + \tau d_s) d_s < f(x_s) + tz/3.
\]
Similarly, we find for any $y \in Y$
\[
g(x_s + td_s, y) = g(x_s, y) + t\nabla_x g(x_s + \tau y d_s, y) d_s
\]
\[
= (1 - t)g(x_s, y) + t[g(x_s, y) + \nabla_x g(x_s + \tau y d_s, y) d_s]
\]
\[
\geq (1 - t)g(x_s, y) - tz/3 > 0
\]
(use the feasibility $g(x_s, y) \geq 0$), which tells us $x_s + td_s \in F$. So, the minimization step (*) in the algorithm produces some $t_s$ ensuring a decrement of at least
\[
f(x_{s+1}) - f(x_s) = f(x_s + t_s d_s) - f(x_s) \leq -\frac{\delta}{2n} \cdot \|z\|/3 = -\delta \|z\|/6n
\]
for all sufficiently large $s$. Hence, $f(x_s) \to -\infty$ as $s \to \infty$ contradicting our assumption that $x_s \to \bar{x}$ and thus $f(x_s) \to f(\bar{x})$. \qed

**Remark 4** The following variant of (31) is more efficient (and the results of Theorem 8 remain true): Chose some (small) $\varepsilon > 0$ and replace the constraints $\nabla_x g(x_k, y) d + g(x_k, y) \geq -z$, $\forall y \in Y$ by
\[
\nabla_x g(x_k, y) d + g(x_k, y) \geq -z \quad \forall y \in Y_k(x_k) := \{ y \in Y \mid g(x_k, y) \leq \min_{y \in Y} g(x_k, y) + \varepsilon \}
\]
Another possibility is to use the reduction approach in Section 6 and to solve the inequalities:
\[
G_j(x_k) + \nabla G_j(x_k) d \geq -z, \forall j
\]
where $G_j(x) = g(x, y_j(x))$ (see (26)).

A foundation of descent methods based on nonsmooth analysis is to be found in the review article of Polak [25].
7.2 KKT-approach, genericity and the SQP-method

In this subsection we discuss the so-called dual or KKT-approach for solving SIP and GSIP. These methods are directly based on the system of Karush-Kuhn-Tucker necessary optimality conditions.

**FP case:** Consider a finite program (FP)

\[
\min_x f(x) \quad \text{s.t.} \quad g_j(x) \geq 0, \quad j \in J = \{1, \ldots, m\}.
\]

Recall that under the constraint qualification LICQ (see Section 3.2) at a local minimizer x necessarily the KKT-equations

\[
\nabla_x L(x, \bar{\mu}) := \nabla f(x) - \sum_{j \in J_0(\bar{x})} \bar{\mu}_j \nabla g_j(x) = 0,
\]

\[
g_j(x) = 0, \quad j \in J_0(\bar{x})
\]

must hold with (unique) multipliers \( \bar{\mu}_j \geq 0, \quad j \in J_0(\bar{x}) \), where \( L \) denotes the Lagrange function at \( \bar{x} \), \( L(x, \mu) = f(x) - \sum_{j \in J_0(\bar{x})} \mu_j g_j(x) \). By putting \( G = [g_j, \quad j \in J_0(\bar{x})] \), the Jacobian of the system (32) at a solution \( (\bar{x}, \bar{\mu}) \) reads:

\[
J(\bar{x}, \bar{\mu}) = \begin{pmatrix}
\nabla^2 L(\bar{x}, \bar{\mu}) & -\nabla^T G(\bar{x}) \\

\nabla G(\bar{x}) & 0
\end{pmatrix}.
\]

The system (32) represents a system of \( n + |J_0(\bar{x})| \) equations in equally many unknowns \( x_i, \mu_j \). So we may apply the Newton method to solve this system (32). It is well-known that the Newton method (locally) has quadratic convergence if the Jacobian \( J(\bar{x}, \bar{\mu}) \) is regular at a solution \( (\bar{x}, \bar{\mu}) \) (cf. e.g. [7]). We now discuss the question whether in the “general” (generic) case this regularity condition is satisfied. We introduce the

**Problem set**

\( \mathcal{P} = \{ P = (f, g_1, \ldots, g_m) \} \equiv C^\infty(\mathbb{R}^n, \mathbb{R})^{1+m} \).

The function space \( C^\infty(\mathbb{R}^n, \mathbb{R}) \) is assumed to be endowed with the so-called strong Whitney topology (see [21]). By a generic subset of a set \( \mathcal{P} \) we mean a subset which is dense and open in \( \mathcal{P} \). The following result states that the Newton method is generically applicable.

**Theorem 9** (Jongen/Jonker/Twilt [21]) There is an open and dense subset \( \mathcal{P}_0 \subset \mathcal{P} \) such that for all finite programs \( P \in \mathcal{P}_0 \), LICQ holds at each feasible point \( x \) and at each solution \( (\bar{x}, \bar{\mu}) \) of (32) the Jacobian \( J(\bar{x}, \bar{\mu}) \) is regular.

**Remark 5** It is interesting to note that the common second order sufficient optimality conditions for FP at a local minimizer \( \bar{x} \) imply that the Jacobian \( J(\bar{x}, \bar{\mu}) \) is regular (see e.g. [7, Ex.12,30]).

**SQP-method** The KKT-approach, to solve (32) by Newton-type iterations is a conceptual idea for the computation of a solution of a FP numerically. The **SQP-method**
presents one of the (most efficient) practical variants of this approach. In this method in each iteration a “descent direction” \( d_k \) is computed by solving a quadratic program (see e.g., [7] for details and convergence results). Under appropriate assumptions the SQP-method is superlinear convergent. A step of this method proceeds as follows:

**SQP-method for FP**

**step k:** given \( x_k, B_k > 0 \) and \( r > 0 \) large enough.

1. compute a solution \( s_k \) of the quadratic program:
   
   \[
   \min_{s \in \mathbb{R}^n} \nabla f(x_k)s + \frac{1}{2} s^T B_k s \quad \text{s.t.} \quad g_j(x_k) + \nabla g_j(x_k)s \geq 0, \quad j \in J.
   \]

2. perform a line search with the merit-function \( M_r := f(x_k) - r \sum_j \min\{0, g_j(x_k)\} \)
   
   \[
   \min_{t > 0} M_r(x_k + ts_k) \quad \text{with a solution} \quad t_k \quad \text{and put} \quad x_{k+1} = x_k + t_k s_k
   \]

**SIP and GSIP case:** Let \( \bar{x} \) be a local minimizer of a SIP or GSIP. Recall from Section 6 that under LICQ (see Remark 3 and \( A_{red} \)) the minimizer \( \bar{x} \) necessarily satisfies the following complete system of optimality conditions with the active index set \( Y_0(\bar{x}) = \{\bar{y}_1, \ldots, \bar{y}_r\} \):

\[
\nabla f(x) - \sum_{j=1}^r \mu_j \nabla_x \left( g(x, y_j) - \sum_{l \in L_0(\bar{x}, \bar{y}_j)} [y_j]_l \nabla y_l(x, y_j) \right) = 0 \quad (33)
\]

\[
\forall j : \quad g(x, y_j) = 0
\]

and for all \( y_j, \quad j = 1, \ldots, r : \)

\[
\nabla y g(x, y_j) - \sum_{l \in L_0(x, y_j)} [y_j]_l \nabla y l(x, y_j) = 0 \quad (34)
\]

\[
\forall l \in L_0(x, y_j) : \quad v_l(x, y_j) = 0
\]

As in the FP case, this system has \( K = n + r(m + 1) + \sum_{j=1}^r |L_0(\bar{x}, \bar{y}_j)| \) equations and equally many unknowns \( x, \mu_j, y_j, y_j' \). So again, Newton-type methods may be applied to solve GSIP numerically (see [33]). As ’natural’ regularity conditions we assume

**RC:** For each \( x \in \mathcal{F} \) with any choice of solutions \( y_j \) of (34) for each \( y_j \in Y_0(x) \) the following is true: If there is a solution \( \mu \) of the relations (33) then

1. LICQ holds: \( \nabla_x \mathcal{L}^{(x,y)}(x, y_j, y_j') \), \( y_j \in Y_0(x) \) are linearly independent.

2. For any \( y_j \in Y_0(x) \) the assumption \( A_{red} \) holds (see Section 6).

**Remark 6** Note that for common SIP-problems the system simplifies. Since the functions \( v_l \) do not depend on \( x \), in this case, in the first equations the sum over \( [y_j]_l \nabla y l(x, y_j) \) vanishes.
For common SIP the following genericity result has been proven, which gives the theoretical basis for the Kuhn-Tucker methods for solving SIP:

**Theorem 10** (Jongen/Zwier [19]) The set \( P = \{ (f, g, v_1, l \in L) \} \equiv C^\infty(\mathbb{R}^n, \mathbb{R}) \times C^\infty(\mathbb{R}^{n+m}, \mathbb{R})^{1+|I|} \) of all \( C^\infty \)-SIP problems contains an open and dense subset \( P_0 \subset P \) such that for all programs \( P \in P_0 \) the regularity condition RC is satisfied.

Unfortunately the situation is more complicated for GSIP where a genericity result as in Theorem 10 is not true. A counterexample is provided by the next

**Ex. 10**

\[
\min_{x \in \mathbb{R}^2} x_1^2 + (x_2 - 1)^2 \quad \text{s.t.} \quad y - x_2 \geq 0, \quad \forall y \in Y(x)
\]

with index set \( Y(x) = \{ y \in \mathbb{R} \mid y - x_1 \geq 0, \; y + x_1 \geq 0 \} \). The set \( Y(x) \) behaves Lipschitz continuous but is not smooth at points \((0, x_2)\). The feasible set reads \( F = \{ x \in \mathbb{R}^2 \mid |x_1| \geq x_2 \} \) with a re-entrant corner point at \( x = (0, 0) \). The solutions of the problem are the points \( \bar{x} = (\pm 1/2, 1/2) \). Obviously the problem is equivalent with the program in disjunctive form (not intersection but union of half spaces):

\[
\min_{x \in \mathbb{R}^2} x_1^2 + (x_2 - 1)^2 \quad \text{s.t.} \quad x \in \{ x \mid x_2 \leq x_1 \} \cup \{ x \mid x_2 \leq -x_1 \}
\]

Near the re-entrant corner point \( \bar{x} = (0, 0) \) of \( F \) with corresponding active index point \( \bar{y} = 0 \) the system of optimality conditions can be written as:

\[
2 \begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \gamma_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \gamma_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
y - x_2 = 0
\]

\[
1 - (\gamma_1 + \gamma_2) = 0
\]

\[
y - x_1 = 0
\]

\[
y + x_1 = 0
\]

The vector \( (\bar{x}, \bar{y}, \bar{\mu}, \bar{\nu}) = (0, 0, 0, 1, 1/2, 1/2) \) is a solution of this system. However the condition LICQ of \( A_{red} \) is not fulfilled.

On the other hand the Jacobian of the system can be shown to be regular at this solution \( (\bar{x}, \bar{y}, \bar{\mu}, \bar{\nu}) \), \textit{i.e.}, this point is an attraction point for the Newton method. But \( \bar{x} \) is not a critical point in the common sense since there is a feasible direction \( d = (\pm 1, 1) \) of descent: \( \nabla f(\bar{x})d = -2 < 0 \) (so \( \bar{x} \) is not a candidate for a solution).

In [26] it is conjectured that the genericity result will at least hold at all local solutions of GSIP (a proof is given for the linear case).

**Remark.** Conceptually there are two strategies for solving the above system (33-34):

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• Compute an approximate solution \( \tilde{x} \) of GSIP (for example by a Discretization method; see the next section). Use these data as starting values for a (Quasi-)Newton iteration for solving (33). This strategy has the disadvantage that often the approximate solution does not yet produce data within the region of attraction of the Newton method. Therefore the next approach is preferable especially for problems of medium/large size.

• Use a Quasi-Newton method (such as the SQP-method) in step (2) of the conceptual “method based on the reduction” below.

For many implementational details and references we refer to [9].

**Method for GSIP based on reduction (SQP-method):** To obtain a practical method for solving GSIP problems we may try to solve the locally reduced problem \( P_{\text{red}}(x) \) described in Section 6. We give a conceptual description (see [9, Section 7.3] for more details).

**Method based on the reduction:**

Step \( k \): Given \( x_k \) (not necessarily feasible)

1. Determine the local minima \( y_1, \ldots, y_{r_k} \) of \( Q(x_k) \).

2. Apply \( N_k \) steps (of a finite programming algorithm) to the locally reduced problem (cf. (26))

\[
P_{\text{red}}(x_k) : \min_x f(x) \quad \text{s.t. } G_j(x) := g(x, y_j(x)) \geq 0, \quad j = 1, \ldots, r_k,
\]

leading to iterates \( x_{k,i}, i = 1, \ldots, N_k \).

3. Put \( x_{k+1} = x_{k,N_k} \) and \( k = k + 1 \).

The iteration in sub-step 2 can be done by performing 'SQP-steps' applied to the Karush-Kuhn-Tucker system of \( P_{\text{red}}(x_k) \). For a discussion of such a method combining globally convergence and locally superlinear convergence we refer to [9].

### 7.3 Discretization methods

In a discretization method we choose finite subsets \( Y' \) of \( Y \), and instead of \( P \equiv P(Y) \) we solve the finite programs

\[
P(Y') : \min f(x) \quad \text{s.t. } g(x, y) \geq 0, \quad \forall y \in Y'.
\]

Let \( v(Y') \), \( F(Y') \) and \( S(Y') \) denote the the minimal value, the feasible set and the set of (global) minimizers of \( P(Y') \) respectively.
We introduce the Hausdorff distance (meshsize) \( \rho(Y') \) between \( Y' \) and \( Y \) by

\[
\rho(Y') := \max_{y \in Y} \text{dist} (y, Y') \quad \text{where} \quad \text{dist} (y, Y') = \min_{y' \in Y'} \| y - y' \| .
\]

The following relation is trivial but important:

\[
Y_2 \subset Y_1 \Rightarrow \mathcal{F} (Y_1) \subset \mathcal{F} (Y_2) \quad \text{and} \quad v(Y_2) \leq v(Y_1) . \tag{35}
\]

We consider different discretization concepts: \( P = P(Y) \) is called \textit{finitely reducable} if there is a finite set \( Y' \subset Y \) such that \( v(Y') = v(Y) \).

The SIP is called \textit{weakly discretizable} if there exists a sequence of discretizations \( Y_k \) such that

\[
v(Y_k) \to v(Y) .
\]

\( P(Y) \) is said to be \textit{discretizable} if for each sequence of finite grids \( Y_k \subset Y \) satisfying \( \rho(Y_k) \to 0 \) (for \( k \) large enough) there exist solutions \( \bar{x}_k \) of \( P(Y_k) \) and for each sequence of solutions the relation

\[
dist(\bar{x}_k, S(Y)) \to 0 \quad \text{and} \quad v(Y_k) \to v(Y)
\]

holds. We also introduce a local concept. Given a local minimizer \( \bar{x} \) of \( P(Y) \) the SIP is called locally discretizable at \( \bar{x} \) if the discretizability relation holds locally, i.e. if the problem \( P^l(Y) \),

\[
P^l(Y) : \min_{x \in U_\bar{x}} f(x) \quad \text{s.t.} \quad g(x, y) \geq 0 \quad \forall y \in Y ,
\]

obtained as the restriction of \( P(Y) \) to an open neighborhood \( U_\bar{x} \) of \( \bar{x} \), is discretizable. From a numerical viewpoint only the concept of discretizability is useful.

**Finite reducability and weak discretizability.** From the definition we immediately deduce

\( P \) is finitely reducable \( \Rightarrow \) \( P \) is weakly discretizable.

We begin with two negative examples.

**Ex. 11** Let us consider the nonconvex SIP problem

\[
P : \min -x_2 \quad \text{s.t.} \quad (x_1 - y)^2 + x_2 \geq 0 , \quad y \in Y = [0, 1] , \quad 0 \leq x_1 \leq 1 .
\]

Obviously \( \mathcal{F} = \{ (x_1, x_2) \mid 0 \leq x_1 \leq 1 \text{ and } x_2 \geq 0 \} , \) \( v(Y) = 0 \) and \( S = \{ (\bar{x}_1, \bar{x}_2) \mid 0 \leq \bar{x}_1 \leq 1 \text{ and } \bar{x}_2 = 0 \} \). On the other hand, whichever grid \( Y' \) we take, it is evident that \( v(Y') < 0 \) and \( S(Y') \) is a (nonempty) finite set. So \( P \) is not finitely reducable but it is discretizable (as is not difficult to see).

**Ex. 12** The problem of Ex 9 is not weakly discretizable. In this example \( v(Y') = -\infty \), for every finite grid \( Y' \subset Y \), whereas \( v(Y) = 0 \).
We now consider LSIP problems. Define for $Y' \subset Y$ the pair of primal-dual problems:

\[
P(Y') : \min \ c^T x \quad \text{s.t.} \quad a_y x - b_y \geq 0, \quad \forall y \in Y' \\
D(Y') : \max \sum_{y \in Y'} u_y b_y \quad \text{s.t.} \quad \sum_{y \in Y'} u_y a_y = c, \quad u_y \geq 0.
\]

Let $v(Y')$, $v_D(Y')$ and $\mathcal{F}(Y')$, $\mathcal{F}_D(Y')$ denote the minimal values and feasible sets of the problems $P(Y')$, $D(Y')$ respectively. Instead of $\sum_{y \in Y'} u_y b_y$ for $u \in \mathcal{F}_D(Y')$, as in Section 5.2, we will write $b^T u$.

From the LP theory it follows (see [7]) that for any finite $Y'$ (with $\mathcal{F}(Y') \neq \emptyset$ or $\mathcal{F}_D(Y') \neq \emptyset$) the relation

\[
v_D(Y') = v(Y') \leq v_D(Y) \leq v(Y)
\]

holds. Recall that a finitely reducable LSIP is in particular weakly discretizable. For the case $v(Y) = -\infty$, the problem $P$ is trivially finitely reducable.

**Theorem 11** Let $P(Y)$ be feasible and bounded. Then

$P(Y)$ is finitely reducable $\iff$ $D(Y)$ is solvable and $P(Y)$ is weakly discretizable.

*Proof.* $\Rightarrow$: Let $Y' \subset Y$ be finite with $v(Y) = v(Y')$, $-\infty < v(Y) < \infty$. The LP theory entales that the problem $D(Y')$ has an optimal solution $u' \in \mathcal{F}_D(Y') \subset \mathcal{F}_D(Y)$. Using $v(Y) = v(Y')$, we find from (36) that $v_D(Y') = v_D(Y)$ must hold, i.e., $u'$ is an optimal solution of $D(Y)$. Recall that a finitely reducable problem is always weakly discretizable.

$\Leftarrow$: Assume that we have given a sequence of grids $Y_k$ in $Y$ such that $v(Y_k) \to v(Y)$ ($-\infty < v(Y) < \infty$) and $u'$ is a maximizer of $D(Y)$. With the finite set $Y' = \text{supp } u'$ (the support of $u'$) the solution $u'$ is also a maximizer of $D(Y')$ and it follows (see (36))

\[
v(Y_k) = v_D(Y_k) \leq v_D(Y') = v(Y') = v_D(Y) \leq v(Y).
\]

In view of $v(Y_k) \to v(Y)$ we find $v(Y) = v(Y')$.

$\square$

**Theorem 12** Let $P(Y)$ be feasible. Then

$P(Y)$ is weakly discretizable $\iff v(Y) = v_D(Y)$ (no duality gap)

*Proof.* $\Rightarrow$: Assume that $Y_k$ is a sequence of grids in $Y$ such that $v(Y_k) \to v(Y)$ ($< \infty$). By (36) we find

\[
v(Y_k) = v_D(Y_k) \leq v_D(Y) \leq v(Y)
\]

and by taking the limit $k \to \infty$ also $v(Y) = v_D(Y)$.

$\Leftarrow$: If $v(Y) = v_D(Y) = -\infty$, i.e., $\mathcal{F}_D(Y) = \emptyset$, $P(Y)$ is finitely reducable and thus
weakly discretizable. In the other case, there exist sequences of feasible solutions $x_k \in \mathcal{F}(Y)$ and $u_k \in \mathcal{F}_D(Y)$ such that

$$b^T u_k \rightarrow v_D(Y) = v(Y) \leftarrow c^T x_k. \quad (37)$$

With the finite sets $Y_k = \text{supp} u_k$ from (36) we deduce

$$b^T u_k = v_D(Y_k) = v(Y_k) \leq v(Y) \leq c^T x_k$$

and (37) implies $v(Y_k) \rightarrow v(Y)$. □

Note that by the last theorems weak discretizability is directly related to duality and finite reducability to the solvability of the dual. A sufficient condition for the existence of an optimal solution of the dual is (for example) the fact that the Slater condition holds.

**Discretizability.** The next example illustrates the difference between weak discretizability and discretizability.

**Example 1** Consider the linear SIP (with some fixed $\varepsilon > 0$):

$$\min x_1 \text{ s.t. } x_1 \cos y + x_2 \sin y \geq 1, \; y \in Y := \begin{cases} [\pi, 3\pi/2] & \text{case A} \\ [\pi - \varepsilon, 3\pi/2] & \text{case B} \end{cases}$$

The minimizer of $P(Y)$ is $\overline{x} = (-1, 0)$.

**Case A:** The problem is weakly discretizable but not discretizable. For a grid $Y'$ containing $y = \pi$ we have $v(Y') = v(Y)$. On the other hand for any $Y'$ not containing $\pi$ the value is unbounded, $v(Y') = -\infty$.

**Case B:** The problem is discretizable as is easily shown. Note that in case B the condition $c \in \text{int} \; \text{cone} \{a(y) \mid y \in Y\}$ is satisfied but not in case A (cf. also Theorem 14).

The following algorithm is based on the concept of discretizability.

**Algorithm 1** *(Conceptual discretization method)*

**Step k:**

i. Compute a solution $x_k$ of $P(Y_k)$.

ii. Stop, if $x_k$ is feasible within a fixed accuracy $\alpha > 0$, i.e. $g(x_k, y) \geq -\alpha$, $y \in Y$. Otherwise, select a finer discretization $Y_{k+1} \subset Y$.

Under a compactness assumption on the feasible sets we obtain a general convergence result for this method.

**Theorem 13** *Let the sequence of discretizations $Y_k$ satisfy

$$Y_0 \subset Y_k \quad \forall k \geq 1 \; \text{and} \; \rho(Y_k) \rightarrow 0 \quad \text{for} \; k \rightarrow \infty.$$

Suppose $\mathcal{F}(Y_0)$ is compact. Then $P(Y)$ is discretizable, i.e., the problems $P(Y_k)$ have solutions $x_k$ and each such sequence of solutions satisfies $\text{dist}(x_k, S(Y)) \rightarrow 0$. 


Proof. By assumption and using $Y_0 \subset Y_k \subset Y$ the feasible sets $\mathcal{F}(Y)$, $\mathcal{F}(Y_k)$, of $P(Y)$, $P(Y_k)$ respectively, are compact and satisfy $\mathcal{F}(Y) \subset \mathcal{F}(Y_k) \subset \mathcal{F}(Y_0)$, $k \in \mathbb{N}$. Consequently, solutions $x_k$ of $P(Y_k)$ exist. Suppose now that a sequence of such solutions does not satisfy $\text{dist}(x_k, S(Y)) \to 0$. Then there exist $\varepsilon > 0$ and a subsequence $x_{k_\nu}$ such that

$$\text{dist}(x_{k_\nu}, S(Y)) \geq \varepsilon > 0 \quad \forall \nu.$$ 

Since $x_{k_\nu} \in \mathcal{F}(Y_0)$ we can select a convergent subsequence. Without restriction we can assume $x_{k_\nu} \to \bar{x}$, $\nu \to \infty$. In view of $\mathcal{F}(Y) \subset \mathcal{F}(Y_k)$ the relation $f(x_{k_\nu}) \leq \nu(Y)$ holds and thus by continuity of $f$ we find

$$f(\bar{x}) \leq \nu(Y).$$

We no show that $\bar{x} \in S(Y)$ in contradiction to our assumption. To do so it suffices to prove that $\bar{x} \in \mathcal{F}(Y)$. Let $\bar{y} \in Y$ be given arbitrarily. Since $\rho(Y_{k_\nu}) \to 0$ for $\nu \to \infty$ we can choose $y_{k_\nu} \in Y_{k_\nu}$, such that $y_{k_\nu} \to \bar{y}$. In view of $g(x_{k_\nu}, y_{k_\nu}) \geq 0$, by taking the limit $\nu \to \infty$, it follows $g(\bar{x}, \bar{y}) \geq 0$, i.e. $\bar{x} \in \mathcal{F}(Y)$.

The following result on discretizability of LSIP is contained in [12, p. 70-75].

**Theorem 14** Let the LSIP problem $P(Y)$ be feasible and assume that the condition $c \in \text{int cone } \{a(y) \mid y \in Y\}$ holds. Then $S(Y)$ is non-empty and bounded, so a solution of $P(Y)$ exists. Moreover $P(Y)$ is discretizable.

We now consider discretizability for nonlinear semi-infinite problems. Here we have to make use of the local concept. The following can be easily proven (e.g. in [34])

**Lemma 6** Let be given a sequence of grids $Y_k \subset Y$ with $\rho_k := \rho(Y_k) \to 0$.

(a) Let $x_k$ be points in $\mathcal{F}(Y_k) \cap K$, where $K$ is a compact subset of $\mathbb{R}^n$. Then there exists $c > 0$ such that for all $\rho_k > 0$ small enough

$$g(x_k, y) \geq -c \rho_k \quad \forall y \in Y.$$ 

(b) Let MFCQ be satisfied at $\bar{x}$ with the vector $d$ (cf. (10)). Then there exist numbers $\tau > 0$, $\varepsilon_1 > 0$ such that for small $\rho_k$ the points $x + \tau \rho_k d$ are feasible for $P(Y)$ for all $x_k \in \mathcal{F}(Y_k)$ with $\|x_k - \bar{x}\| < \varepsilon_1$.

**Theorem 15** Let $\bar{x}$ be a local minimizer of $P(Y)$ of order $p$. Suppose MFCQ holds at $\bar{x}$. Then $P$ is locally discretizable at $\bar{x}$. More precisely, there is some $\sigma > 0$ such that for any sequence of grids $Y_k \subset Y$ with $\rho(Y_k) \to 0$ and any sequence of solutions $x_k$ of the locally restricted problem $P^d(Y_k)$ (see the definition of discretizability) the following relation holds:

$$\|x_k - \bar{x}\| \leq \sigma \rho(Y_k)^{1/p}. \quad (38)$$
Proof. Consider the SIP restricted to the closed ball cl $B_k(\bar{x})$ with small $\kappa$ chosen such that $\kappa < \varepsilon, \varepsilon_1$ (with $\varepsilon$ in (12) and $\varepsilon_1$ in Lemma 6):

$$P^I(Y_k) : \min f(x) \quad \text{s.t.} \quad x \in \mathcal{F}(Y_k) \cap \text{cl} B_k(\bar{x}).$$

Obviously, since $\bar{x} \in \mathcal{F}(Y_k)$ and $\mathcal{F}(Y_k) \cap \text{cl} B_k(\bar{x})$ is compact (and nonempty), a solution $x_k^I$ exists. Note that $\bar{x}$ is the unique (global) minimizer of $P^I(Y)$. Put $\rho_k := \rho(Y_k)$ and consider any sequence of solutions $x_k^I$ of $P^I(Y_k)$. In view of $\mathcal{F}(Y) \subset \mathcal{F}(Y_k)$ and $x_k^I + \tau \rho_k d \in \mathcal{F}(Y) \cap \text{cl} B_k(\bar{x})$ (for large $k$, see Lemma 6(b)) we find

$$f(x_k^I) \leq f(\bar{x}) \leq f(x_k^I + \tau \rho_k d) \ .$$

Since $\bar{x}$ is a minimizer of order $p$ (see 12) it follows

$$||x_k^I + \tau \rho_k d - \bar{x}||^p \leq \frac{1}{q} \left( f(x_k^I + \tau \rho_k d) - f(\bar{x}) \right) \leq \frac{1}{q} \left( f(x_k^I + \tau \rho_k d) - f(x_k^I) \right) = O(\rho_k) .$$

Finally, the triangle inequality yields

$$||x_k^I - \bar{x}|| \leq ||x_k^I + \tau \rho_k d - \bar{x}|| + ||\tau \rho_k d|| = O(\rho_k^{1/p}) \ . \quad (39)$$

In particular $||x_k^I - \bar{x}|| < \kappa$ (for large $k$) such that $x_k^I$ are (global) minimizers of the problem $P^I(Y_k)$ restricted to the open neighborhood $B_k(\bar{x})$.

Remark 7 The result of Theorem 15 remains true for the global minimization problem $P$ if the sets $\mathcal{F}(Y_k)$ and $\mathcal{F}(Y)$ are restricted to a compact subset $K \subset \mathbb{R}^n$.

Under additional assumptions on the quality of the discretizations $Y_k$ one can prove a faster convergence than in (38) (see [34]).

7.4 Exchange method

We also outline the exchange method which is often more efficient than a pure discretization method. This method can be seen as a compromise between the discretization method in Section 7.3 and the continuous reduction approach in Section 7.2.

Algorithm 3 (Conceptual exchange method)

Step k: Given a discretization $Y_k \subset Y$ and a fixed, small value $\alpha > 0$.

i. Compute a solution $x_k$ of SIP($Y_k$).

ii. Compute local solutions $y_k^i, \ i = 1, \ldots, i_k \ (i_k \geq 1)$ of $Q(x_k)$ (cf. (23)) such that one of them, say $y_k^1$, is a global solution, i.e., $g(x_k, y_k^1) = \max_{y \in \mathcal{F}} g(x_k, y)$
iii. Stop, if $g(x_k, y^1_k) \geq -\alpha$, with a solution $\bar{x} \approx x_k$. Otherwise, update

$$Y_{k+1} = Y_k \cup \{y^i_k, \quad i = 1, \ldots, i_k\}.$$ \hfill (40)

**Theorem 16** Suppose that the (starting) feasible set $\mathcal{F}(Y_0)$ is compact. Then, the exchange method (with $\alpha = 0$) either stops with a solution $\bar{x} = x_{k_0}$ of $P(Y)$ or the sequence $\{x_k\}$ of solutions of $P(Y_k)$ satisfies $\text{dist}(x_k, S(Y)) \to 0$.

**Proof.** We consider the case that the algorithm does not stop with a minimizer of $P(Y)$. As in the proof of Theorem 13, by our assumptions, a solution $x_k$ of $P(Y_k)$ exists, $x_k \in \mathcal{F}(Y_0)$ and with the subsequence $x_{k_0} \to \bar{x}$ we find

$$f(\bar{x}) \leq v(Y).$$

Again we have to show $\bar{x} \in \mathcal{F}$ or equivalently $\varphi(\bar{x}) \geq 0$ for the value function $\varphi(x)$ of $Q(x)$. In view of $\varphi(x_k) = g(x_k, y^1_k)$ (see Algorithm 2, step ii) we can write

$$\varphi(\bar{x}) = \varphi(x_k) + \varphi(\bar{x}) - \varphi(x_k) = g(x_k, y^1_k) + \varphi(\bar{x}) - \varphi(x_k).$$

Since $y^1_k \in Y_{k+1}$ we have $g(x_{k+1}, y^1_k) \leq 0$ and by continuity of $g$ and $\varphi$ we find

$$\varphi(\bar{x}) \geq (g(x_k, y^1_k) - g(x_{k+1}, y^1_k)) + (\varphi(\bar{x}) - \varphi(x_k)) \to 0 \quad \text{for} \quad k \to \infty.$$

We refer to the review paper [9] for more details on this approach.

### 8 Bilevel problems and Mathematical programs with equilibrium constraints

Recall the GSIP problems (2). Bilevel problems are of the form

$$\begin{align*}
\text{BL:} \quad \min_{\lambda, y} f(x, y) \quad \text{s.t.} \quad h(x, y) &\geq 0 \\
\text{and } y &\text{ is a solution of } Q(x) : \quad \min_{t} F(x, t) \quad \text{s.t.} \quad t \in Y(x)
\end{align*}$$ \hfill (41)

We also consider mathematical programs with equilibrium constraints

$$\begin{align*}
\text{MPEC:} \quad \min_{\lambda, y} f(x, y) \quad \text{s.t.} \quad h(x, y) &\geq 0 \\
y &\in Y(x) \\
\quad g(x, y, t) &\geq 0 \quad \forall t \in Y(x).
\end{align*}$$ \hfill (42)

BL and MPEC problems both represent important classes of optimization problems which have been investigated in a large number of papers and books (see e.g., [2], [24] and the references therein).
8.1 Comparison between the structure of the problems

In this subsection we discuss the relations between the three problem classes GSIP, BL and MPEC. Obviously, GSIP is a special case of MPEC. But also BL can be written in MPEC form as follows.

**BL → MPEC:** If we write the second constraint in BL equivalently as

\[ y \in Y(x) \quad \text{and} \quad F(x, t) - F(x, y) \geq 0 \quad \forall t \in Y(x) \]

the BL problem becomes

\[
\begin{align*}
\text{BL} : & \quad \min_{x, y} f(x, y) \quad \text{s.t.} \quad h(x, y) \geq 0 \\
& \quad y \in Y(x) \\
& \quad F(x, t) - F(x, y) \geq 0 \quad \forall t \in Y(x).
\end{align*}
\] (43)

**Remark.** Note however that there is a subtle difference in the interpretation of the constraint

\[ g(x, y) \geq 0 \quad \forall y \in Y(x) \quad \text{or} \quad g(x, y, t) \geq 0 \quad \forall t \in Y(x). \] (44)

In the case that \( Y(x) \) is empty, for BL and MPEC because of the additional condition \( y \in Y(x) \), no feasible point \((x, y)\) exists (for this \(x\)). For the GSIP problem however an empty index set \( Y(x) \) means that there are no constraints and such points \(x\) are feasible.

**GSIP → BL:** We assume \( Y(x) \neq \emptyset, \forall x \), and recall the (lower level) problem (see (23))

\[
Q(x) : \quad \min_t g(x, t) \quad \text{s.t.} \quad t \in Y(x),
\] (45)

depending on the parameter \(x\). Then (assuming that \(Q(x)\) is solvable) we can write

\[ g(x, t) \geq 0 \quad \forall t \in Y(x) \quad \iff \quad g(x, y) \geq 0 \quad \text{and} \quad y \text{ solves } Q(x). \]

So GSIP takes the BL form:

\[
\begin{align*}
\text{GSIP} : & \quad \min_{x, y} f(x, y) \quad \text{s.t.} \quad g(x, y) \geq 0 \\
& \quad \text{and} \quad y \text{ is a solution of } Q(x) : \quad \min_t g(x, t) \quad \text{s.t.} \quad t \in Y(x).
\end{align*}
\] (46)

This problem is a BL program with the special property that the objective function \(f\) does not depend on \(y\) and that the constraint function \(g\) in the first level and the lower level objective function coincide.

**MPEC → BL:** By applying the same trick as before we consider the programs

\[
Q(x, y) : \quad \min_t g(x, y, t) \quad \text{s.t.} \quad t \in Y(x),
\] (47)
depending on the parameter \((x, y)\) and find
\[
g(x, y, t) \geq 0 \quad \forall t \in Y(x) \quad \iff \quad g(x, y, t) \geq 0 \quad \text{and} \quad t \quad \text{solves} \quad Q(x, y) .
\]
Consequently MPEC turns into a problem of BL type (with \(x\) replaced by \((x, y)\)):
\[
\text{MPEC} : \quad \min_{x, y, t} f(x, y) \quad \text{s.t.} \quad h(x, y) \geq 0
\]
\[
y \in Y(x)
\]
\[
g(x, y, t) \geq 0
\]
\[
\text{and} \quad t \quad \text{is a solution of} \quad Q(x, y) : \quad \min_u g(x, y, u) \quad \text{s.t.} \quad u \in Y(x)
\]
\[
(48)
\]
Special case \(g(x, y, y) = 0\) : Under the extra condition
\[
g(x, y, y) = 0 , \quad \forall y
\]
\[
(49)
\]
in MPEC we can 'eliminate' the \(t\) variable as follows. In view of (49 ) we find
\[
y \in Y(x) \quad \text{and} \quad g(x, y, u) \geq 0 \quad \forall u \in Y(x) \quad \iff \quad y \quad \text{solves} \quad Q(x, y)
\]
and MPEC is equivalent with
\[
\text{MPEC} : \quad \min_{x, y} f(x, y) \quad \text{s.t.} \quad h(x, y) \geq 0
\]
\[
y \in Y(x)
\]
\[
g(x, y, u) \geq 0 \quad \forall u \in Y(x)
\]
\[
(50)
\]
Remark. Note that now the problem MPEC in (50) has a more complicated structure than BL, since in MPEC the lower level problem \(Q(x, y)\) depends on the variable \(y\) which should at the same time solve \(Q(x, y)\).

For a comparison from the structural and generic viewpoint between BL and GSIP we refer to [26] and between BL and MPEC to [4].

To solve these problems numerically it is convenient to reformulate the problems as nonlinear programs. For shortness we only consider (the most complicated case) MPEC. To do so we assume that the sets \(Y(x)\) are given explicitly in the form
\[
Y(x) = \{y \in \mathbb{R}^m \mid v(x, y) \geq 0\} \quad \text{with a} \quad C^1\text{-function} \quad v : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^q .
\]
Let also \(g\) be from \(C^1\). Then, if \(t\) is a solution of \(Q(x, y)\) which satisfies some constraint qualification (CQ) then \(t\) must necessarily satisfy the Kuhn-Tucker conditions:
\[
\nabla_t g(x, y, t) - \lambda^T \nabla_v v(x, t) = 0
\]
\[
\lambda^T v(x, t) = 0
\]
\[
(51)
\]
with some multiplier $0 \leq \lambda \in \mathbb{R}^m$. So we can consider the following relaxation of MPEC.

$$\min_{x, y, t} f(x, y) \quad \text{s.t.}$$

$$g(x, y, t) \geq 0$$

$$\nabla_t g(x, y, t) - \lambda^T \nabla_t v(x, y) = 0$$

$$\lambda^T v(x, y) = 0$$

$$\lambda, h(x, y), v(x, y), v(x, t) \geq 0$$

(52)

The program (52) is a relaxation of the MPEC (48) in the sense that (under CQ) the feasible set of the MPEC is contained in the feasible set of (52). In particular, any solution $(x, y, t)$ of (52) with the property that $t$ is a minimizer of $Q(x, y)$, must also be a solution of the original MPEC.

### 8.2 Convexity conditions for $Q(x, y)$

Let us now consider the special case that $Q(x, y)$ represents a convex problem, i.e., for any fixed $x$ and $y$ the function $g(x, y, t)$ is convex in $t$, and for any fixed $x$, the functions $- v_i(x, t)$ are convex in $t$. Then, it is well-known that the Kuhn-Tucker conditions (51) are sufficient for $t$ to be a solution of $Q(x, y)$. So in this case any solution $(x, y, t)$ of (52) provides a solution of the MPEC. If moreover CQ is satisfied for $Q(x, y)$, then (52) is equivalent with the original MPEC.

In the form (52), an MPEC is transformed into a nonlinear program with complementarity constraints (see e.g. [28]). In this form the problems can be solved numerically, for instance by some interior point method approach. For GSIP problems this has been done successfully in [30].

### 9 Appendix

This section contains some definitions and auxiliary results. A set $C$ is called convex if $C$ contains with any $x, y \in C$ also the whole line segment $[x, y] = \{(1 - \lambda)x + \lambda y | 0 \leq \lambda \leq 1\}$. For an arbitrary set $A \subset \mathbb{R}^n$ we define its convex cone by

$$\text{cone } A = \{a = \sum_{j=1}^{k} \lambda_j a_j | k \geq 1, \ a_j \in A, \ \lambda_j \geq 0\}.$$ 

and its convex hull by

$$\text{conv } A = \{a = \sum_{j=1}^{k} \lambda_j a_j | k \geq 1, \ a_j \in A, \ \lambda_j \geq 0, \ \sum_{j=1}^{k} \lambda_j = 1\}.$$ 

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**Lemma 7** (Carathéodory) For $A \subset \mathbb{R}^n$, each $a \in \text{conv} A$ can be represented as a convex combination of (at most) $n+1$ vectors: $a = \sum_{j=1}^{n+1} \lambda_j a_j$, and each element $a \in \text{cone} A$ as a conic combination of (at most) $n$ vectors: $a = \sum_{j=1}^{n} \lambda_j a_j$

A function $f : C \to \mathbb{R}$ defined on a convex set $C \subset \mathbb{R}^n$ is called convex if for all $x, y \in C$ and $0 \leq \lambda \leq 1$,

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

**Lemma 8** A differentiable function $f : C \to \mathbb{R}$ ($C \subset \mathbb{R}^n$, open and convex) is convex on $C$ if and only if for any $x, y \in C$: $f(x) \geq f(x) + \nabla f(y)(x - y)$.

**Lemma 9** [Generalized Gordan Lemma] Let $A \subset \mathbb{R}^n$ be a compact set. Then exactly one of the following alternatives is true.

(i) $0 \in \text{conv} A$.

(ii) There exists some $d \in \mathbb{R}^n$ such that $a^T d < 0$ for all $a \in A$.

**References**


