Examination: Continuous Optimization

3TU- and LNMB-course, Utrecht December 22, 2009, 13.00-16.00

Ex. 1

(a) Given $a \in \mathbb{R}^n$, show that the matrix aa^T is positive semidefinite.

(b) Show that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only $A \bullet C \geq 0$ holds for all positive semidefinite matrices C.
(*Here, for symmetric matrices, $A \bullet C$ denotes the "inner product", $A \bullet C = \sum_{i,j} a_{ij}c_{ij}$*)

Ex. 2 Consider the convex problem

$$(CO) \qquad \min f(x) \quad \text{s.t.} \quad g_j(x) \leq 0,\ j = 1,\dots,m,\ x \in \mathbb{R}^n$$

with convex functions $f, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$. Suppose a feasible point $\overline{x}$ satisfies the KKT-conditions (Karush-Kuhn-Tucker) with a multiplier vector $\overline{y} \geq 0$.
(a) Show that $(\overline{x}, \overline{y})$ is a saddle point for the Lagrangian function $L(x, y)$ of (CO).
(b) Show also that $(\overline{x}, \overline{y})$ is a solution of the Wolfe-Dual (WD) of (CO).

Ex. 3 Let $f_i : C \to \mathbb{R},\ i \in I := \{1,\dots,m\}$ be convex functions on the convex compact set $C \subset \mathbb{R}^n$. Define for $x \in C$ the function f by:

$$f(x) = \min\left\{ \sum_{i=1}^m \lambda_i f_i(x_i) \;\middle|\; x = \sum_{i=1}^m \lambda_i x_i,\ \sum_{i=1}^m \lambda_i = 1;\ x_i \in C, \lambda_i \geq 0\ \forall i \in I \right\}$$

Show that $f(x)$ is the greatest convex function $g(x)$ such that $g(x) \leq f_i(x)\ \forall x \in C$ and $\forall i \in I$, in the following way:

(a) With the epigraphs epi (f_i) we consider the set $F :=$ conv $\{$epi $(f_i),\ i = 1,\dots,m\}$. Show now that $F =$ epi (f) holds and conclude that f is convex on C.
(*By definition, conv*$\{$epi $(f_i),\ i = 1,\dots,m\}$ *is the set*

$$\{\sum_{i=1}^m \lambda_i z_i \mid \sum_{i=1}^m \lambda_i = 1;\ \lambda_i \geq 0\ , z_i \in \text{epi } (f_i), i \in I\},$$

i.e. the set F is the smallest convex set containing all sets epi $(f_i),\ i = 1,\dots,m$.)

(b) Show that $f(x) \leq f_i(x) \forall x \in C$ and $\forall i \in I$ holds and that f is the greatest convex function with this property (*i.e.*, for all convex functions $g(x)$ such that $g(x) \leq f_i(x) \forall x \in C$ and $\forall i \in I$, we have $g(x) \leq f(x),\ \forall x \in C$).

Ex.4 We consider the unconstrained minimization problem: $\min_{x \in \mathbb{R}^n} \; f(x)$ with $f \in C^1(\mathbb{R}^n, \mathbb{R})$.

(a) Let H be a positive definite (real) $n \times n$ matrix and let $\nabla f(x_k) \neq 0$. Show that $d_k = -H\nabla f(x_k)$ is a descent direction for f in x_k.

(b) Given a (non-positive definite) symmetric (real) $n \times n$ matrix A, show that there is a number σ_0 such that for all $\sigma > \sigma_0$ the matrices $A + \sigma I$ are positive definite.
(Here, I is the unit matrix). *Hint: Consider the number* $\min_{\|x\|=1} x^T A x$

Ex. 5 Consider the unconstrained minimization problem with the quadratic function $q(x) = \frac{1}{2}x^T A x + b^T x$, where A is a positive definite matrix.
Determine the (global) minimizer $\overline{x}$ of q and show that for any starting point x_0 the Newton iteration computes the minimizer $\overline{x}$ in one step.

Ex. 6 Consider the problem (in connection with the design of a cylindrical can with height h, radius r and volume at least 2π such that the total surface area is minimal):

$$(P): \quad \min \; f(h,r) := 2\pi(r^2 + rh) \quad \text{s.t.} \; -\pi r^2 h \leq -2\pi, \quad (\text{and } h > 0, r > 0)$$

(a) Compute a (the) solution $(\overline{h}, \overline{r})$ of the KKT conditions of (P). Show that (P) is not a convex optimization problem.

(b) Show that the solution $(\overline{h}, \overline{r})$ in (a) is a local minimizer. Why is it the unique global solution?
Hint: Use the sufficient optimality conditions

Ex. 7 We consider the constrained program

$$(P) \qquad \min \; f(x) \quad \text{s.t.} \quad x \in \mathcal{F} \qquad \text{where} \qquad \mathcal{F} := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in J\}$$

with $f, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$ and $J = \{1, \dots, m\}$.
Let d_k be a strictly feasible descent direction in $x_k \in \mathcal{F}$, *i.e.*,

$$\nabla f(x_k)^T d_k < 0 \,, \qquad \nabla g_j(x_k)^T d_k < 0, \;\; \forall j \in J_{x_k}$$

holds. Show that for any $t > 0$, t small enough, we have:

$$f(x_k + td_k) < f(x_k) \quad \text{and} \quad x_k + td_k \in \mathcal{F}$$

Points: 36+4 =40

1 a : 2	2 a : 3	3 a : 3	4 a : 2	5 : 4	6 a : 4
b : 3	b : 3	b : 3	b : 3		b : 2
7 : 4					

A copy of the lecture-sheets may be used during the examination.
Good luck!