Examination: Continuous Optimization

3TU- and LNMB-course, Utrecht December 22, 2008,   13.00-16.00

Ex. 1

(a) Show that with matrices $A, B \in \mathbb{R}^{n\times n}$ and columns a_j of A and rows b_j^T of B the relation holds: $A \cdot B = \sum_{j=1}^n a_j b_j^T$

(b) Show that a matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only if it is of a form $A = \sum_{j=1}^n b_j b_j^T$ (with $b_j \in \mathbb{R}^n$).

Ex. 2 Consider with $0 \neq c \in \mathbb{R}^n$ the program:

$$(P) \qquad \min_{x\in\mathbb{R}^n} c^T x \quad \text{s.t.} \quad x^T x \le 1 \; .$$

(a) Show that $\overline{x} = -\frac{c}{\|c\|}$ is the minimizer of (P) with minimum value $v(P) = -\|c\|$. ($\|x\|$ means here the Euclidian norm.)

(b) Compute the solution $\overline{y}$ of the Lagrangean dual (D) of (P). Show in this way that for the optimal values strong duality holds, i.e., $v(D) = v(P)$.

Ex. 3 Let be given a general convex program:

$$(CO) \qquad \min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t.} \quad g_j(x) \le 0 \quad \forall j \in J$$

and define with the Lagrangean function $L(x, y)$ for $y \ge 0$ the function $\psi(y) := \inf_x L(x, y)$.

(a) Considering the Wolfe-Dual (WD), show that for any feasible point $(\overline{x}, \overline{y})$ of (WD) we have:

$$L(\overline{x}, \overline{y}) = \psi(\overline{y}) \; .$$

(b) Conclude from (a) that for the optimal values $v(WD)$ of the Wolfe-Dual (WD) and $v(D)$ of the Lagrangean-dual (D) of (CO) the relation holds:

$$v(WD) \le v(D)$$

Ex. 4 Let $f : C \to \mathbb{R}$, be a strictly convex function on the convex set $C \subset \mathbb{R}^n$, *i.e.*, for all $x, y \in C,\ x \neq y$, and $0 < \lambda < 1$ we have

$$f((1-\lambda)x + \lambda y) < (1-\lambda) f(x) + \lambda f(y)$$

Show: If $\overline{x} \in C$ is a local minimizer of f then $\overline{x}$ is the unique global minimizer of f on C.

Ex. 5 Let be given the quadratic function $q(x) = \frac{1}{2}x^T Ax + b^T x$, with positive definite matrix A. Let $d_k \neq 0$ be a descent direction for q in x_k. Show that the minimizer t_k of the line-minimization problem $\min_{t \geq 0} q(x_k + td_k)$ is given by

$$t_k = -\frac{(Ax_k + b)^T d_k}{d_k^T A d_k}$$

Ex. 6 Consider the problem:

$$\min\ (x_1 + 1)^2 + x_2^2 \quad \text{s.t.} \quad \begin{array}{r} x_2 - x_1^3 \leq 0 \\ -x_1 - x_2 \leq 0 \end{array}$$

(a) Show that for $\overline{x} = (0, 0)$ the Karush-Kuhn-Tucker conditions are satisfied.

(b) Show that $\overline{x} = (0, 0)$ is a strict local minimizer of order $p = 1$.
Is $\overline{x}$ also the (unique) global minimizer?

Ex. 7 We consider the problem

$$(P) \qquad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad x \in \mathcal{F}$$

where $\mathcal{F} \subset \mathbb{R}^n$ is a compact convex set.

(a) Show that the function $g(x, y) := \|x - y\|$ is convex in the (whole) variable $z = (x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$.

(b) Show that

$$h(x) := \min_{y \in \mathcal{F}} \|x - y\|$$

is a convex function of x on $\mathbb{R}^n$.
(*HINT: For x_1, x_2 consider minimizers y_1, y_2 of $h(x_1), h(x_2)$.*)

(c) Conclude that problem (P) can equivalentely be written as

$$(P) \qquad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) \leq 0\ .$$

Points:

1 a : 2 b : 3	2 a : 2 b : 3	3 a : 2 b : 2	4 : 5	5 : 4	6 a : 3 b : 3
7 a : 2 b : 3 c : 2					

Points: 36+4 =40

A copy of the lecture-sheets may be used during the examination.
Good luck!