Extra proofs of some Results.

Version: 19-10-2011 (Georg Still)

Th.1.19 [*Krein–Milman Theorem*]
Let $\mathcal{C} \subset \mathbb{R}^n$ be a compact convex set. Then $\mathcal{C}$ is the convex hull of its extreme points.

Proof By using

- L.1.18 and the following
- Th.2.23 Let $\emptyset \neq U \subset \mathbb{R}^n$ be convex and $w \notin U$. Then there exists a separating hyperplane $H = \{x \mid a^T x = \alpha\}$, $0 \neq a \in \mathbb{R}^n, \alpha \in \mathbb{R}$ such that

$$a^T w \geq \alpha \geq a^T x \quad \forall x \in U$$

and $\alpha > a^T u_0$ for some $u_0 \in U$.

Proof. By induction on $k := \dim C$ $(= \dim \text{aff}\,(C))$
k=1: Then $C = \{c\}$, a singleton, and the result is true.

$\boldsymbol{k-1 \to k}$: Define
$K := \text{conv}\,\{\overline{x} \mid \overline{x}$ is extreme point of C$\} \subset C$. Then $\emptyset \neq K$, convex. We show $K = C$ by contradiction.

Assume that K is strictly smaller than C, i.e., there exists a point $y \in C \setminus K$. By the separation theorem, Th. 2.23, there exist $H = \{x \mid a^T x = \alpha\}$, $(0 \neq a)$, $x_0 \in K$ such that

$$(\star) \qquad a^T y \geq \alpha \geq a^T x \;\forall x \in K, \;\; \alpha > a^T x_0 \,.$$

Now consider $m := \max_{x \in C} a^T x$ (shift the hyperplane H) and define $H' = \{x \mid a^T x = m\}$,

$$C' := C \cap H' \,.$$

C' is nonempty, convex and compact. We consider two cases.
Case, $a^T y = m$, i.e., $y \in C'$: By

$a^T y \geq \alpha > a^T x_0$ it follows $x_0 \in C \setminus H'$ and thus $\dim C' < \dim C$.

By applying the induction argument (to C') the element $y \in C'$ can be written as

$$y = \sum_{i=1}^{k} \lambda_i x^i, \ \lambda_i \geq 0, \ \sum_{i=1}^{k} \lambda_i = 1, \ x^i \text{ extreme points in } C'.$$

But as we shall show below:

$$(\star\star) \quad \tilde{x} \text{ is extreme point of } C' \Rightarrow \tilde{x} \text{ is extreme point of } C.$$

Consequently, y is a convex combination of extreme points in C contradicting $y \notin K$.
<u>case, $a^T y < m$, i.e., $y \notin C'$:</u> By L 1.18, C' has an extreme point $\tilde{x}$ and by $(\star\star)$ $\tilde{x} \in K$. But (see $(\star)$)

$$m = a^T \tilde{x} > a^T y \geq a^T x \ \ \forall x \in K$$

yields a contradiction.
<u>We finally show $(\star\star)$:</u> Suppose, $\tilde{x} \in C'$ is not an extreme point of C.

Then $\tilde{x}$ can be written as $\tilde{x} = \lambda x^1 + (1-\lambda)x^2, 0 < \lambda < 1$, $x^1 \neq x^2 \in C$. We find (using $a^T x^i \leq m$)

$$m = a^T \tilde{x} = \lambda a^T x^1 + (1-\lambda) a^T x^2 \leq m$$

and then $a^T x^1 = a^T x^2 = m$, so that $x^1, x^2 \in C'$. Consequently $\tilde{x}$ cannot be an extreme point of C'. $\square$

Th.2.23 Let $\emptyset \neq U \subset \mathbb{R}^n$ be convex and $w \notin U$. Then there exists a separating hyperplane $H = \{x \mid a^T x = \alpha\}$, $0 \neq a \in \mathbb{R}^n, \alpha \in \mathbb{R}$ such that

$$a^T w \geq \alpha \geq a^T x \quad \forall x \in U$$

and $\alpha > a^T u_0$ for some $u_0 \in U$.

Proof: Case $w \notin \partial U$: (∂U, *boundary*). Then by [FKS,Th.10.1] it holds for some $a \neq 0, \alpha$:

$$a^T w > \alpha \geq a^T x \quad \forall x \in U$$

Case $w \in \partial U$: Then by [FKS,Th.10.2] it holds for some $a \neq 0, \alpha$:

$$a^T w = \alpha \geq a^T x \quad \forall x \in U$$

It remains to show: (*in this case* $w \in \partial U$)

$$(\star) \qquad \alpha > a^T u_0 \text{ for some } u_0 \in U \,.$$

We reduce the construction to $\mathrm{aff}\, U$. Note that $w \in \partial U$ implies $w \in \mathrm{aff}\, U$ (*closed*). The proof is in three steps:

1. Construct $u_0 \in \mathrm{rint}\, U$: **With (fixed) $\overline{u} \in U$ the affine hull of U is given by $\mathrm{aff}\, U = \overline{u} + V$ with**

$$V = \left\{ \sum_{j=1}^{m} \mu_j (x_j - \overline{u}), \mu_j \in \mathbb{R}, x_j \in U \right\}, v_j := x_j - \overline{u} \text{ lin. indep.}$$

Define $u_0 := \frac{1}{m+1}(\overline{u} + \sum_j x_j) \in U$. Then for any k and any $0 < \varepsilon < \frac{1}{m+1}$ we have $u_0 \pm \varepsilon(x_k - \overline{u}) \in U$. This follows by (*convex combination*)

$$u_0 \pm \varepsilon(x_k - \overline{u}) = (\frac{1}{m+1} \mp \varepsilon)\overline{u} + (\frac{1}{m+1} \pm \varepsilon)x_k + \sum_{j \neq k} \frac{1}{m+1} x_j$$

Thus, there must be some $\delta > 0$ such that

$$u_0 \pm v \in U \quad \forall v \in V, \|v\| < \delta \, .$$

This implies $u_0 \in U^0$

2. Transformation from "$\mathbb{R}^n$ to $\mathbb{R}^m$ ": Consider the transformation $x : \mathbb{R}^m \to \mathbb{R}^n$

$$(\star\star) \qquad x := \overline{u} + \sum_j y_j v_j = \overline{u} + By, \quad B := (v_1, \ldots, v_m)$$

This transformation defines a bijection between

$$x \in U \leftrightarrow y \in U_y \subset \mathbb{R}^m$$

with $u_0 \in \mathrm{rint}\, U \leftrightarrow y_0 \in \mathrm{int}\, U_y \subset \mathbb{R}^m$ and $w \in \partial U \leftrightarrow w_0 \in \partial U_y$. One can show thet U_y is convex. Now in $\mathbb{R}^m$, w_0 and U_0 can be separated, i.e. with $b \neq 0, \beta$ (see [FKS, Th10.2]):

$$b^T w_0 = \beta \geq b^T y \ \ \forall y \in U_y.$$

Using $y_0 + \rho b \in U_y$ for some $\rho > 0$ we obtain

$$\beta \geq b^T(y_0 + \rho b) = b^T y_0 + \rho ||b||^2 \quad \text{or} \quad \beta > b^T y_0 \ .$$

Then with $a := B(B^T B)^{-1} b \in \mathbb{R}^n$ and $\alpha := 2a^T \overline{u} + \beta$ the relation $(\star)$ is satisfied.