6. Relaxations of integer programs

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Contents: We consider LP-, Lagrangian-, semidefinite programming- and completely positive programming relaxations.

- For some results we refer to [FKS].

Rough idea:

- Replace (relax) an NP-hard integer program:

$$(IP) \qquad \min_x f(x) \qquad \text{s.t.} \qquad x \in \mathcal{F}(IP)$$

- by an (easier) continuous convex program:

$$(R) \qquad \min_x f(x) \qquad \text{s.t.} \qquad x \in \mathcal{F}(R)$$

LP- and (*partial*) Lagrange relaxation

We consider the *integer program*: (with $Q \succeq 0$)

$$(IP) \qquad \min_x \; x^T Q x + 2c^T x \qquad \text{s.t.}$$
$$Ax \le b, \; Bx \le d,$$
$$\boxed{x \in \{0,1\}^n}$$

with min-value $v(IP)$ and feasible set $\mathcal{F}(IP)$.

LP relaxation: *Only relax $x_i \in \{0,1\}$ to $x_i \in [0,1]$.*

$$(LP) \qquad \min_x \; x^T Q x + 2c^T x \qquad \text{s.t.}$$
$$Ax \le b, \; Bx \le d,$$
$$\boxed{x \in [0,1]^n}$$

Rem. *Since $\mathcal{F}(IP) \subset \mathcal{F}(LP)$ it follows* $v(LP) \le v(IP)$.

(Partial) Lagrangian relaxation (*Relax the "difficult" constraint $Ax \leq b$.*): **For $y \geq 0$ we define the function**

$$\boldsymbol{L(y) = \min_{x} \quad \{x^T Qx + 2c^T x + y^T(Ax - b) \\ \mid Bx \leq d,\ x \in \{0,1\}^n\}}$$

and the Lagrangian-dual: $\boxed{(D_p) \qquad \max_{y \geq 0} L(y)}$

Ex.6.1 (*"Weak duality"*) **For any $y \geq 0$ we have $L(y) \leq v(IP)$. Thus $v(D_p) \leq v(IP)$.**

PROOF. $y \geq 0,\ Ax - b \leq 0$ *implies* $y^T(Ax - b) \leq 0$. *So with* $f(x) := x^T Qx + 2c^T x$

$$\begin{aligned} v(IP) &= \min_x\{f(x) \mid Ax \leq b, Bx \leq d, x \in \{0,1\}^n\} \\ &\geq \min_x\{f(x) + y^T(Ax - b) \mid Bx \leq d, x \in \{0,1\}^n\} \\ &= L(y) \end{aligned}$$

By taking $\max_y$ *we obtain* $v(D_p) \leq v(IP)$.

By applying the strong duality result for CO (Th.2 in Chap 2) we obtian:

Th.6.1 **Let $Q \succeq 0$ and let the system $Ax \le b,\ Bx \le d,\ x \in [0,1]^n$ be feasible. Then**

$$v(LP) \le v(D_p) \le v(IP)$$

PROOF. We prove $v(D_p) \ge v(LP)$: With $f(x) := x^T Qx + 2c^T x$

$$\begin{aligned} v(D_p) &= \max_{y \ge 0} \min_x \{f(x) + y^T(Ax - b) \mid Bx \le d, \mathbf{x} \in \{\mathbf{0},\mathbf{1}\}^n\} \\ &\ge \max_{y \ge 0} \underbrace{\min_x \{f(x) + y^T(Ax - b) \mid Bx \le d, \mathbf{0} \le \mathbf{x} \le \mathbf{1}\}}_{(\star)} \end{aligned}$$

We apply "strong duality" to (the primal problem) $(\star)$ *with L-multipliers:*
$-x \le 0 \leftrightarrow \mu,\ x - 1 \le 0 \leftrightarrow \lambda,\ Bx - d \le 0 \leftrightarrow u$;
Slater holds: $Bx - d \le 0,\ 0 \le x \le 1$ *is feasible.*

and find that the last right-hand-side equals

$$
\begin{aligned}
v(D_p) &\geq \textit{r-h-s} \\
&= \max_{y\geq 0}\big(\max_{\mu,\lambda,u\geq 0}\min_{x}\big)\Big\{f(x) + y^T(Ax-b) + \\
&\qquad\qquad u^T(Bx-d) - \mu^T x + \lambda^T(x-1)\Big\} \\
&= \min_{x}\max_{y,\mu,\lambda,u\geq 0}\Big\{f(x) + y^T(Ax-b) + \\
&\qquad\qquad u^T(Bx-d) - \mu^T x + \lambda^T(x-1)\Big\} \\
&= \min_{x}\{f(x) \mid Ax-b\leq 0,\ Bx-d\leq 0,\ 0\leq x\leq 1\} \\
&= v(LP)
\end{aligned}
$$

Here for the second "$=$" we applied strong duality "back" (wrt. all constraints). The Slater condition for this program holds since by assumption there is a feasible point satisfying all (linear) constraints $Ax \leq b,\ Bx \leq d,\ 0 \leq x \leq 1$.

Notation:

- $\mathcal{S}^n = \{A \in \mathbb{R}^{n \times n} \mid A = A^T\}$ (*symmetric matrices*)
- $\mathcal{S}^n_+ = \{A \in \mathcal{S}^n \mid A \succeq 0\}$ (*pos. semidef. matrices*)

We consider the *quadratic program*

$$(P_q) \qquad \min \; q_0(x) \quad \text{s.t.} \quad q_j(x) \leq 0 \quad \forall j \in J$$

with $J = \{1, \ldots, m\}$, $q_j(x) = x^T Q_j x + 2b_j^T x + c_j$. ($Q_j \in \mathcal{S}^n$)

<u>Lifting procedure:</u> Introduce $\boxed{X = xx^T}$ and use

- $x^T Q x = Q \bullet xx^T$
- $x^T Q x + 2b^T x + c = \begin{pmatrix} c & b^T \\ b & Q \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix}$
- $X = xx^T \quad \Leftrightarrow \quad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^T$

In the variables x, X: (P_q) is equivalent with

$$(P'_q) \qquad \min_{x,X} \begin{pmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \qquad \text{s.t.}$$

$$\begin{pmatrix} c_j & b_j^T \\ b_j & Q_j \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \leq 0 \quad \forall j \in J$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^T \qquad (\star)$$

Remark. (P'_q) has a linear objective, linear inequalities but a quadratic equality constraint $(\star)$ (nonconvex!).

Semidefinite Relaxation By relaxing $(\star)$ to $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$ we obtain the (*convex*) ***SDP-relaxation*** of (P'_q):

$$(P_{SDP}) \qquad \min_{x,X} \begin{pmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \qquad \text{s.t.}$$

$$\begin{pmatrix} c_j & b_j^T \\ b_j & Q_j \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \leq 0 \quad \forall j \in J$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$$

Ex.6.2 Show that the feasible set $\mathcal{F}(P_{SDP})$ of (P_{SDP}) is convex and (P_{SDP}) is a convex program. Show further:
$X - xx^T \succeq 0 \Leftrightarrow \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$.

Ex.6.3 (P_{SDP}) is a relaxation of (P'_q), *i.e.*, $\mathcal{F}(P'_q) \subset \mathcal{F}(P_{SDP})$ and thus $\underline{v(P_{SDP}) \le v(P'_q) = v(P_q)}$

Ex.6.4 Let be given a quadratic constraint (*with* $Q \succeq 0$):

$$x^T Q x + 2b^T x + c \le 0 \qquad (\star)$$

(a) Show: If $X - xx^T \succeq 0$ holds then

$$x^T Q x + 2b^T x + c \le Q \bullet X + 2b^T x + c$$

(b) Show: x satisfies $(\star)$ $\Leftrightarrow$ with some X we have: $X - xx^T \succeq 0$ and $Q \bullet X + 2b^T x + c \le 0$

As an implication of Ex. 6.4 it follows:

Cor. 6.1 Let (P_q) be convex (i.e., $Q_j \succeq 0$). Then the SDP relaxation (P_{SDP}) is exact: $\boxed{v(SDP) = v(P_q)}$

The Lagrangian dual of (P_q): By defining for $u \geq 0$ the Lagrange-min-function wrt. (P_q),

$$L(u) := \min_x \{q_0(x) + \sum_j u_j q_j(x)\}$$

the Lagrangian dual of (P_q) becomes

$$(D) \qquad \max_{u \geq 0} \ L(u)$$

Rem. 6.1 (D) can be shown to be the semidefinite dual max–problem of the primal (P_{SDP}) (see also [FKS, p195,321]). So by weak SDP-duality: $v(D) \leq v(P_{SDP})$.

Summarizing: $\boxed{v(D) \leq v(P_{SDP}) \leq v(P_q)}$

Rem. The same holds for equality constraints $q_i(x) = 0$. Here, the L-multiplier $u \in \mathbb{R}^m$ is not constrained in $L(u)$.

Semidefinite relaxation of integer programs

<u>**Basic idea:**</u> *Write the 0-1 condition* $x_i \in \{0, 1\}$ *equivalently as quadratic constraint* $\boxed{x_i^2 - x_i = 0}$

We consider the 0-1 program:

$$(IP) \qquad \min_{x} \; x^T Q x + 2c^T x \qquad \text{s.t.}$$
$$\begin{aligned} Ax &\le b \\ x_i^2 - x_i &= 0 \;\; \forall i \end{aligned}$$

Using again $x^T Q x = Q \bullet xx^T$, $X = xx^T$, *the condition* $x_i^2 = x_i$ *becomes* $\underline{X_{ii}} = x_i^2 = \underline{x_i}$ *and the SDP-relaxation of* (IP) *reads:*

$$(P_{SDP}) \qquad \min_{x,X} \; Q \bullet X + 2c^T x \qquad \text{s.t.}$$
$$\begin{aligned} Ax &\le b \\ X_{ii} - x_i &= 0 \;\; \forall i \\ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} &\succeq 0 \end{aligned}$$

As in the general quadratic case we consider the (full) L-dualization:

$$(D) \qquad \max_{y \geq 0, u} L(y, u)$$

with dual function

$$L(y,u) = \min_x \{ \quad x^T Q x + 2c^T x + y^T (Ax - b) + \sum_i u_i (x_i^2 - x_i) \}$$

Again, (D) is the SDP dual of (P_{SDP}) so that:

$$(\star) \qquad v(D) \leq v(P_{SDP}) \leq v(IP)$$

Case that Q is positive semidefinite

LP-relaxation (*Note:* $x_i \in [0,1] \Leftrightarrow x_i^2 - x_i \le 0$)

$$(LP) \qquad \min_x \; x^T Q x + 2c^T x \quad \text{s.t.}$$
$$Ax \le b$$
$$(x_i \in [0,1] \Leftrightarrow) \; x_i^2 - x_i \le 0 \;\; \forall i$$

By applying strong duality we can prove

Th. 6.2 **Let $Q \succeq 0$ and assume that (LP) has a Slater point $\overline{x}$, *i.e.*, $A\overline{x} \le b$, $\overline{x}_i^2 - \overline{x}_i < 0$. Then $v(LP) \le v(D)$.**

PROOF: Letting $f(x) := x^T Q x + 2c^T x$ *and*
$L(x, y, u) = f(x) + y^T(Ax - b) + \sum_i u_i(x_i^2 - x_i)$ *we find:*

$$
\begin{aligned}
v(D) &= \max_{y\geq 0,\ \boxed{u\in\mathbb{R}^n}} \min_x L(x,y,u) \\
&\geq \max_{y\geq 0,\ \boxed{u\geq 0}} \min_x L(x,y,u) \\
\textit{apply str. dual.} \rightarrow \quad &= \min_x \max_{y\geq 0, u\geq 0} L(x,y,u) \\
&= \min_x \{ f(x) \mid Ax-b\leq 0,\ x_i^2 - x_i \leq 0 \forall i \} \\
&= v(LP)
\end{aligned}
$$

Summarising in this case:

$$\boldsymbol{v(LP) \leq v(D) \leq v(P_{SDP}) \leq v(IP)}$$

6.3 Linear 0-1 programs

We consider as special case the 0-1 linear program:

$$(IP) \qquad \min_x \; c^T x \qquad \text{s.t.}$$
$$\begin{aligned} Ax &\le b \\ x_i^2 - x_i &= 0 \;\; \forall i \; (\Leftrightarrow x_i \in \{0, 1\}) \end{aligned}$$

<u>SDP-relaxation:</u> $(P_{SDP}) \qquad \min_{x,X} \; c^T x \qquad \text{s.t.}$

$$\begin{aligned} Ax &\le b \\ X_{ii} - x_i &= 0 \;\; \forall i \\ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} &\succeq 0 \end{aligned}$$

<u>(full) L-dualization:</u> $(D) \qquad \max_{y \ge 0, u} L(y, u)$

with dual function

$$L(y, u) = \min_x \left\{ c^T x + y^T (Ax - b) + \sum_i u_i (x_i^2 - x_i) \right\}$$

LP-relaxation:

$$(LP) \qquad \min_x \ c^T x \qquad \text{s.t.}$$
$$Ax \le b$$
$$x_i \in [0,1] \quad \forall i$$

Remark: Obviously in (P_{SDP}) the elements $X_{ij},\ i \neq j$ of X do not play any role so that $X_{ii} = x_i$ gives

$$X \succeq xx^T \Leftrightarrow X_{ii} \ge x_i^2 \Leftrightarrow x_i \ge x_i^2 \Leftrightarrow x_i \in [0,1]$$

So (P_{SDP}) is equivalent to the LP-relaxation, *i.e.*, $v(LP) = v(P_{SDP})$ and thus by Rem. 6.1, p9:

Summarizing: For the linear IP we have (*without any assumptions*):

$$v(LP) = v(D) = v(P_{SDP}) \le v(IP)$$

Dualize only $Ax \le b$: *We also consider the partial Lagrange dualization:* **for $y \ge 0$ we define** *(the partial)* **L-function**

$$\begin{aligned} L_1(y) &:= \min_{x_i \in \{0,1\}} \{c^T x + y^T (Ax - b)\} \\ &= \min_{x_i \in \{0,1\}} \{-b^T y + x^T (A^T y + c)\} \\ &= -b^T y + \sum_i \min\{0, (A^T y + c)_i\} \end{aligned}$$

Here *we used the obvious relation:*

$$\min_{x_i \in \{0,1\}} d^T x = \min_{x_i \in \{0,1\}} \sum_i d_i x_i = \sum_i \min\{0, d_i\}$$

The corresponding L-dual is:

$$(D_1) \qquad \max_{y \ge 0} L_1(y)$$

We now compare $L_1(y)$ with the full L-dual function

$$\begin{aligned} L(y,u) = \min_x \quad & \{c^T x + y^T(Ax - b) \\ & \quad + \textstyle\sum_i u_i(x_i^2 - x_i)\} \\ = \min_x \quad & \{-b^T y + \sum_i \big[x_i(A^T y + c - u)_i \\ & \qquad\qquad + u_i x_i^2\big]\ \} \end{aligned}$$

<u>Ex.6.5</u> Let $q(\alpha, u)$, $\alpha, u \in \mathbb{R}$ be defined by $q(\alpha, u) := \min_{x \in \mathbb{R}} (\alpha - u)x + ux^2$.
Show

$$q(\alpha, u) = \begin{cases} -\frac{(\alpha - u)^2}{4u} & \text{if } u > 0 \\ 0 & \text{if } u = \alpha = 0 \\ -\infty & \text{otherwise} \end{cases}$$

and: $$\max_u q(\alpha, u) = \begin{cases} \alpha & \text{if } \alpha < 0 \\ 0 & \text{if } \alpha \geq 0 \end{cases} = \min\{0, \alpha\}$$

Using Ex.6.5 we find

Lem.6.1 $L_1(y) = \max_u L(y,u)$ **so that** $v(D_1) = v(D)$.

PROOF: $\max_u L(y,u) =$

$$
\begin{aligned}
&= -b^T y + \sum_i \max_{u_i} \min_{x_i} \{ x_i [(A^T y)_i + c_i - u_i] + u_i x_i^2 \} \\
\text{Ex. 6.5} \rightarrow \ &= -b^T y + \sum_i \min\{0,\ (A^T y)_i + c_i\} \\
&= L_1(y)
\end{aligned}
$$

By taking $\max_{y \geq 0}$ *we obtain* $v(D_1) = v(D)$. $\square$

Altogether without any assumption (using $(\star)$ *on p.11) we find.*

Cor. 6.2. For linear (IP) the relations hold:

$$v(LP) = v(D_1) = v(D) = v(P_{SDP}) \leq v(IP)$$

Maxcut problem: Let $G = (V, E)$ be an undirected graph with $n = |V|$ nodes and $m = |E|$ edges. Each edge (i, j) is assigned a nonnegative weight w_{ij} with $w_{ij} = w_{ji}$ and $w_{ii} = 0$ The *maxcut problem* asks for a set $S \subseteq V$ of nodes such that the weight of the associated cut is maximized:

$$\max_{S \subseteq V} w(S) , \quad \text{where} \quad w(S) = \sum_{i \in S, j \notin S} w_{ij} .$$

IP formulation: For $S \subseteq V$ we define a vector $y = (y_1, \ldots, y_n)$ via

$$y_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{if } i \notin S . \end{cases}$$

Then

$$(1 - y_i y_j) = \begin{cases} 2 & \text{if } i \in S, j \notin S \text{ or } j \in S, i \notin S \\ 0 & \text{otherwise} \end{cases}$$

The maxcut problem is thus equivalent with

$$\max_{y \in \mathbb{R}^n} \frac{1}{4} \sum_{i,j} w_{ij}(1 - y_i y_j) \quad \text{s.t. } y_i \in \{-1, 1\} \; .$$

or (*skipping the constant***) equivalent with:**

$$(IP) \qquad \min_{y \in \mathbb{R}^n} \frac{1}{4} y^T W y \quad \text{s.t. } y_i \in \{-1, 1\} \; .$$

SDP relaxation: Writing $y^T W y = W \bullet yy^T$, putting $Y = yy^T$, using $y_i = \pm 1 \Leftrightarrow y_i^2 = 1$ we find $Y_{ii} = 1$. Replace $Y = yy^T$ by $Y \succeq 0$ to obtain the relaxation of (IP):

$$(SDP) \qquad \min_{Y} \; W \bullet Y \quad \text{s.t.} \quad \begin{array}{l} Y_{ii} = 1, \; \forall i \\ Y \succeq 0 \end{array}$$

Rem. *By an Ex. in Chap 3, any* $n \times n$*-matrix* $Y \succeq 0$ *can be decomposed as* $Y = U^T U$ *with* U *a* $n \times n$*-matrix.*

Goemans and Williamson *proposed the following algorithmic approach to the maxcut problem.*

Randomized Algorithm for the Maxcut Problem:

- **Compute a solution Y of the problem (SDP).**
- **Compute the factorization $Y = U^T U$ with u_i the columns of U.**
- **Choose randomly a vector r, and set**
$$S = \{i \in V \mid u_i^T r \geq 0\} \ .$$

Let v_{opt} be the max-value of a maximum cut in G. It is shown by Goemans and Williamson (1005) that the solution S obtained by the randomized algorithm has an expected value:

$$E(v(S)) \ \geq \ \alpha \ v_{opt} \qquad \text{with} \quad \alpha > 0.878 \ .$$

6.6 Completely positive relaxation of integer programs

We reconsider the 0-1 program

$$(IP) \qquad \min_x \; x^T Q x + 2c^T x \qquad \text{s.t.}$$
$$\begin{aligned} a_j^T x &= b_j \quad j = 1, \dots, m \\ x &\geq 0 \\ x_i^2 - x_i &= 0 \;\; \forall i \end{aligned}$$

Basic idea: *As in the SDP relaxation we introduce* $X = xx^T$ *and we* **add the redundant conditions**

$$(a_j^T x)^2 = a_j^T x x^T a_j = b_j^2 \quad \text{or} \quad \boxed{a_j^T X a_j = b_j^2}\,.$$

(IP) can then equivalently be written as:

$$
\begin{array}{llrll}
(P') & \min\limits_{x,X} \; Q \bullet X + 2c^T x & & \text{s.t.} & \\
 & a_j^T X a_j = b_j^2 \,,\; a_j^T x & = b_j & j = 1, \ldots, m & \\
 & x & \geq 0 & & \\
 & X_{ii} - x_i & = 0 \;\; \forall i & & \\
 & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} & = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^T & &
\end{array}
$$

We again obtain a semi-definite relaxation:

(P_{SDP}) **program as in** (P') **but**

$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^T$ **replaced by** $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$

Ex. 6.6 **Show that the set** $\mathcal{S}_+^{n+1}$ *of psd matrices* **can be written as:**

$$\mathcal{S}_+^{n+1} = \{A = \sum_{k=1}^{N} a_k a_k^T \mid a_k \in \mathbb{R}^{n+1},\ N \text{ finite}\}$$

completely positive matrices (CP): *Consider now the set* $\mathcal{C}^{n+1}$ *of CP-matrices:*

$$\mathcal{C}^{n+1} := \{A = \sum_{k=1}^{N} a_k a_k^T \mid a_k \in \mathbb{R}^{n+1},\ N \text{ finite}, \boxed{a_k \geq 0}\}$$

Obviously: **For** $x \geq 0$

$$\binom{1}{x} \cdot \binom{1}{x}^T \in \mathcal{C}^{n+1} \subset \mathcal{S}_+^{n+1}$$

So the original program (*IP*) (or P') can be replaced by the (*stronger*) CP-relaxation

$$(P_{CP}) \quad \text{program as in } (P') \ (\textit{skipping } x \geq 0) \text{ and}$$
$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot (1 \ x^T) \quad \text{replaced by} \quad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{C}^{n+1}$$

Rem. **Both, (P_{SDP}), (P_{CP}) are convex programs and obviously:**

$$v(P_{SDP}) \leq v(P_{CP}) \leq v(IP)$$

In: ***"Burer,*** *On the copositive representation of binary and continuous nonconvex quadratic programs, Math. Program. 120 (2009), no. 2, Ser. A, 479–495.*" **the following is shown:**

Under a condition on (IP) which can be satisfied by introducing an extra slack variable we have

$$v(P_{CP}) = v(IP)$$

Some final remarks on relaxations

- LP- and Lagrange-relaxation are classical methods for (approximately) solving IP.
- SDP-relaxation are studied extensively since ≈20 years and often lead to sharper approximations. SDP is "polynomially approximable."
- The recent completely positive relaxations (CP) represent continuous convex programs. CP describes some NP-hard IP's exactly.
  - Therefore, CP must be NP-hard
  - Not every convex program is easy.
  - The convex structure of CO hopefully helps to solve IP's "more efficiently".