No-gap Second-order Optimality Conditions for State Constrained Optimal Control Problems

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Outline

1. Presentation of the problem, motivations
2. Definitions and assumptions
3. Main result
4. Application to the shooting algorithm

The Optimal Control Problem

$$(\mathcal{P}) \qquad \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \int_0^T \ell(u(t),y(t))\mathrm{d}t + \phi(y(T)) \text{ subject to:}$$

$$\dot{y}(t) = f(u(t),y(t)) \text{ a.e. on } [0,T]\,;\ y(0) = y_0$$
$$g(y(t)) \le 0 \quad \text{on } [0,T].$$

- Control and state spaces: $\mathcal{U} := L^\infty(0,T;\mathbb{R})$, $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$.
- Assumptions:

(A0) The mappings $\ell : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, $\phi : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are C^∞; f is Lipschitz continuous.

(A1) The initial condition satisfies $g(y_0) < 0$.

Why study Second-Order Optimality Conditions ?

- Second-order Sufficient Conditions: analysis of convergence of numerical algorithms, stability and sensitivity analysis.
- Strong second-order sufficient conditions known, in e.g. [Malanowski-Maurer et al. 1997,1998,2001,2004] ...
- To weaken the sufficient condition, find a Second-order Sufficient Condition as close as possible to the Second-order Necessary Condition (no gap).
- No-gap Second-order conditions known for *mixed* control-state constraints [Milyutin-Osmolovskii 1998], [Zeidan 1994] ...

Abstract formulation of Optimal Control Problem

- State mapping $\mathcal{U} \to \mathcal{Y}$, $u \mapsto y_u$, where y_u is the solution of:

$$\dot{y}_u(t) = f(u(t), y_u(t)) \quad \text{for a.a. } t \in [0, T]; \quad y_u(0) = y_0$$

- Cost and constraint mappings $J : \mathcal{U} \to \mathbb{R}$, $G : \mathcal{U} \to C[0, T]$:

$$J(u) = \int_0^T \ell(u(t), y_u(t))\mathrm{d}t + \phi(y_u(T)) \;\; ; \;\; G(u) = g(y_u).$$

- Abstract formulation of $(\mathcal{P})$ is:

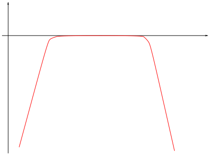
$$\min_{u \in \mathcal{U}} J(u); \qquad G(u) \in K,$$

where K is the cone of nonpositive continuous functions $C_-[0, T]$.

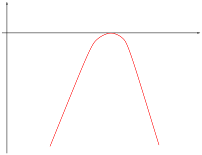
Definitions (1/3) Structure of a trajectory

- State constraint: $g(y_u(t)) \le 0, \forall t \in [0, T]$.
- Contact set:

$$I(g(y_u)) := \{t \in [0, T] \; ; \; g(y_u(t)) = 0\}.$$



boundary arc $[\tau_{en}, \tau_{ex}]$
→ entry and exit points



isolated contact point $\{\tau_{to}\}$
→ touch points

- Junction points:

$$\mathcal{T} := \partial I(g(y_u)).$$

Definitions (2/3) Order of the state constraint

- Order of the state constraint q: smallest number of time-derivation of the function

$$t \mapsto g(y_u(t)),$$

so that an explicit dependence in the control variable u appears.

$$g^{(j)}(u,y) := g_y^{(j-1)}(y) f(u,y) \quad 1 \leq j \leq q, \quad (u,y) \in \mathbb{R} \times \mathbb{R}^n$$

$$g_u^{(j)} \equiv 0, \quad 0 \leq j \leq q-1 \qquad \text{and} \qquad g_u^{(q)} \not\equiv 0.$$

- Example of a state constraint of order q:

$$y^{(q)}(t) = u(t) \qquad ; \qquad y(t) \leq 0.$$

Definitions (3/3)

$$(\mathcal{P}) \qquad \min J(u)\ ;\ \ G(u) \in K$$

- **Lagrangian** $L : \mathcal{U} \times \mathcal{M}[0,T] \to \mathbb{R}$

$$\begin{aligned} L(u,\eta) &:= J(u) + \langle \eta, G(u) \rangle \\ &= \int_0^T \ell(u(t), y_u(t))\mathrm{d}t + \phi(y_u(T)) + \int_0^T g(y_u(t))\mathrm{d}\eta(t) \end{aligned}$$

- **Hamiltonian** $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}$,

$$H(u,y,p) := \ell(u,y) + pf(u,y).$$

- **Costate** $p_{u,\eta}$: the solution in $BV([0,T];\mathbb{R}^{n*})$ of:

$$-\mathrm{d}p_{u,\eta} = H_y(u, y_u, p_{u,\eta})\mathrm{d}t + g_y(y_u)\mathrm{d}\eta\ ;\ \ p_{u,\eta}(T) = \phi_y(y_u(T)).$$

Assumptions (1/2)

(A2) Strong convexity of the Hamiltonian w.r.t. the control variable: $\exists\alpha > 0$ such that

$$\alpha \leq H_{uu}(w, y_u(t), p_{u,\eta}(t^-)) \quad \text{for all } w \in \mathbb{R} \text{ and } t \in [0, T].$$

(A3) Constraint Regularity: $\exists\gamma, \varepsilon > 0$ such that

$$\gamma \leq |g_u^{(q)}(u(t), y_u(t))| \quad \text{for a.a. } t, \ \mathrm{dist}\{t \ ; \ I(g(y_u))\} \leq \varepsilon.$$

(A4) Finite set of junctions points $\mathcal{T}$, and $g(y_u(T)) < 0$.

Junctions conditions results

- $u \in \mathcal{U}$ is a stationary point of $(\mathcal{P})$, if there exists a Lagrange multiplier $\eta \in \mathcal{M}_+[0,T]$ such that

$$\begin{cases} D_u L(u,\eta) = H_u(u(\cdot), y_u(\cdot), p_{u,\eta}(\cdot)) = 0, \text{ a.e. on } [0,T] \\ \eta \in N_K(G(u)). \end{cases}$$

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Proposition (Jacobson, Lele and Speyer, 1971)

Let $(u,\eta) \in \mathcal{U} \times \mathcal{M}_+[0,T]$ a stationary point and its (unique) Lagrange multiplier, satisfying (A2)-(A4). Then:

- *u and η are C^∞ on $[0,T] \setminus \mathcal{T}$ $\Rightarrow$ $\mathrm{d}\eta = \eta_0 \mathrm{d}t + \sum_{\tau \in \mathcal{T}} \nu_\tau \delta_\tau$*
- *$u, \ldots, u^{(q-2)}$ are continuous at junctions times;*
- *If q is odd, $u^{(q-1)}$ and η are continuous at entry/exit times;*
- *If $q = 1$, η is continous at touch points.*

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Proposition (Jacobson, Lele and Speyer, 1971)

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- $u, \ldots, u^{(q-2)}$ *are continuous at junctions times;*
- *If* q *is odd,* $u^{(q-1)}$ *and* η *are continuous at entry/exit times;*
- *If* $q = 1$, η *is continous at touch points.*

- Consequence: the time-derivatives of $t \mapsto g(y_u(t))$ are continuous at entry/exit points until order $\hat{q}$, with $\hat{q} := 2q - 2$ if q is even, and $\hat{q} = 2q - 1$ if q is odd.
- A touch point τ is said to be essential, if $\tau \in \mathrm{supp}(\eta)$ (equivalently, if $\nu_\tau \neq 0$ or if η is discontinuous at τ).

Assumptions (2/2)

(A5)(i) Non-Tangentiality condition at entry/exit points:

$$(-1)^{\hat{q}+1}\frac{\mathrm{d}^{\hat{q}+1}}{\mathrm{d}t^{\hat{q}+1}}g(y_u(t))|_{t=\tau_{en}^-} < 0 \quad ; \quad \frac{\mathrm{d}^{\hat{q}+1}}{\mathrm{d}t^{\hat{q}+1}}g(y_u(t))|_{t=\tau_{ex}^+} < 0$$

(A5)(ii) Reducibility Condition at essential touch points ($q \geq 2$):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(y_u(t))|_{t=\tau_{to}^{ess}} = g^{(2)}(u(\tau_{to}^{ess}), y_u(\tau_{to}^{ess})) < 0$$

(A6) Strict Complementarity on boundary arcs:

$$\overline{\mathrm{int}\, I(g(y_u))} = \cup[\tau_{en}, \tau_{ex}] \subset \mathrm{supp}(\eta)$$

Main Result

- For $u \in \mathcal{U}$ and $v \in L^2(0,T)$, the linearized state $z_{u,v}$ is the solution in $H^1(0,T;\mathbb{R}^n)$ of

$$\dot{z}_{u,v} = f_u(u,y_u)v + f_y(u,y_u)z_{u,v} \text{ on } [0,T]\ ;\ z_{u,v}(0) = 0.$$

Note that $(DG(u)v)(t) = g_y(y_u(t))z_{u,v}(t)$.

- For u a stationary solution with multiplier η, the critical cone is

$$\begin{aligned} C_2(u) &:= \{v \in L^2\ ;\ DG(u)v \in T_K(G(u))\ ;\ DJ(u)v \leq 0\} \\ &= \{v \in L^2\ ;\ DG(u)v \in T_K(G(u))\ ;\ \mathrm{supp}(\eta) \subset I^2_{u,v}\} \end{aligned}$$

with the second-order contact set:

$$I^2_{u,v} := \{t \in I(g(y_u))\ ;\ g_y(y_u(t))z_{u,v}(t) = 0\}.$$

Main Result

Theorem (No-gap Second-order Necessary Condition)

Let $u \in \mathcal{U}$ *a local optimal solution of* $(\mathcal{P})$*, with (unique) Lagrange multiplier* η*, satisfying (A1)-(A6). Denote by* $\mathcal{T}_{to}^{ess}$ *the set of essential touch points of the trajectory* (u, y_u) *and* $\nu_\tau = [\eta(\tau)]$*. Then, for all* $v \in C_2(u)$*:*

$$D^2_{uu}L(u,\eta)(v,v) - \sum_{\tau \in \mathcal{T}_{to}^{ess}} \nu_\tau \frac{(g_y^{(1)}(y_u(\tau))z_{u,v}(\tau))^2}{g^{(2)}(u(\tau), y_u(\tau))} \geq 0.$$

Main Result

Theorem (No-gap Second-order Necessary Condition)

Let $u \in \mathcal{U}$ a local optimal solution of $(\mathcal{P})$, with (unique) Lagrange multiplier η, satisfying (A1)-(A6). Denote by $\mathcal{T}_{to}^{ess}$ the set of essential touch points of the trajectory (u, y_u) and $\nu_\tau = [\eta(\tau)]$. Then, for all $v \in C_2(u)$:

$$D_{uu}^2 L(u,\eta)(v,v) - \sum_{\tau \in \mathcal{T}_{to}^{ess}} \nu_\tau \frac{(g_y^{(1)}(y_u(\tau))z_{u,v}(\tau))^2}{g^{(2)}(u(\tau), y_u(\tau))} \geq 0.$$

- Additional term (in blue), called the curvature term [Kawasaki, 1988].
- Only essential touch points have a contribution to the curvature term (the contribution of boundary arcs is null).

Main Result

Theorem (No-gap Second-order Sufficient Condition)

Let $(u, \eta) \in \mathcal{U} \times \mathcal{M}_+[0, T]$ *a stationary point and its multiplier, satisfying (A1)-(A6). The following assertions are equivalent:*
(i) *For all* $v \in C_2(u) \setminus \{0\}$,

$$D^2_{uu} L(u, \eta)(v, v) - \sum_{\tau \in \mathcal{T}^{ess}_{to}} \nu_\tau \frac{(g_y^{(1)}(y_u(\tau)) z_{u,v}(\tau))^2}{g^{(2)}(u(\tau), y_u(\tau))} > 0.$$

(ii) *u is a local optimal solution of* $(\mathcal{P})$ *satisfying the quadratic growth condition: there exists* $\beta, r > 0$ *such that*

$$J(\tilde{u}) \geq J(u) + \beta \|\tilde{u} - u\|_2^2 \quad \text{for all} \quad G(\tilde{u}) \in K, \ \|\tilde{u} - u\|_\infty < r.$$

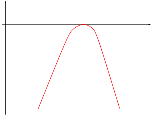
Reduction Approach (cf. semi-infinite programming)

- Let $x_0 \in W^{2,\infty}(0,T)$ having a local maximum at $t_0 \in (0,T)$, the latter being reducible:

$$\ddot{x}_0 \text{ is continuous at } t_0 \quad \text{and} \quad \ddot{x}_0(t_0) < 0.$$

- Then there exists $\varepsilon, \delta > 0$ such that for all $x \in W^{2,\infty}(0,T)$, $\|x - x_0\|_{2,\infty} \leq \delta$, x attains its unique maximum on $[t_0 - \varepsilon, t_0 + \varepsilon]$ at time t_x, and

$$x(t) \leq 0 \text{ on } [0,T] \Leftrightarrow \begin{cases} x(t) \leq 0 \text{ on } [0,T] \setminus (t_0 - \varepsilon, t_0 + \varepsilon) \\ x(t_x) \leq 0. \end{cases}$$



- The additional term comes from the second-order derivative of the mapping $x \mapsto x(t_x)$.

Application to the shooting algorithm

- Unconstrained case:

$$\begin{cases} \dot{y} = f(u,y) & ; \quad y(0) = y_0 \\ -\dot{p} = H_y(u,y,p) & ; \quad p(T) = \phi_y(y(T)) \\ 0 = H_u(u,y,p). & \end{cases}$$

- By (A2),

$$0 = H_u(u(t), y(t), p(t)) \;\Leftrightarrow\; u(t) = \Upsilon(y(t), p(t)).$$

- Shooting mapping $\mathcal{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$,

$$p_0 \;\mapsto\; p(T) - \phi_y(y(T)),$$

with (y,p) solution of:

$$\begin{cases} \dot{y} = f(\Upsilon(y,p), y) & ; \quad y(0) = y_0 \\ -\dot{p} = H_y(\Upsilon(y,p), y, p) & ; \quad p(0) = p_0. \end{cases}$$

Application to the shooting algorithm

- Assume that $q \geq 2$ and there is one isolated contact point τ. Then the shooting mapping is defined by $\mathcal{F} : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$,

$$\begin{pmatrix} p_0 \\ \nu \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} p(T) - \phi_y(y(T)) \\ g(y(\tau)) \\ g^{(1)}(y(\tau)) \end{pmatrix},$$

where (y, p) is solution of:

$$\left\{ \begin{array}{rcll} \dot{y} & = & f(\Upsilon(y,p),y) & \text{on } [0,T]; \quad y(0) = y_0 \\ -\dot{p} & = & H_y(\Upsilon(y,p),y,p) & \text{on } [0,\tau) \cup (\tau,T]; \quad p(0) = p_0 \\ [p(\tau)] & = & -\nu g_y(y(\tau)). & \end{array} \right.$$

- Additional conditions: $g(y(t)) \leq 0$ and $\nu \geq 0$.

Application to the shooting algorithm

- Shooting algorithm well-posed $\Leftrightarrow$ the Jacobian of the shooting mapping $D\mathcal{F}(p_0, \nu, \tau)$ is invertible.
- Solution of $D\mathcal{F}(p_0, \nu, \tau)(\pi_0, \gamma, \sigma) = 0$?

$$(PQ) \qquad \min_{v,z \in L^2 \times H^1} \frac{1}{2} \Big\{ \int_0^T D^2_{(u,y)(u,y)} H(u,y,p)((v,z),(v,z))\mathrm{d}t$$

$$+ \phi_{yy}(y(T))(z(T),z(T)) + \nu g_{yy}(y(\tau))(z(\tau),z(\tau))$$

$$- \nu \frac{(g_y^{(1)}(y(\tau))z(\tau))^2}{g^{(2)}(u(\tau),y(\tau))} \Big\}$$

$$\text{subject to} \begin{cases} \dot{z} = f_y(u,y)z + f_u(u,y)v \ ; \quad z(0) = 0 \\ g_y(y(\tau))z(\tau) = 0. \end{cases}$$

Application to the shooting algorithm

- The solution of $D\mathcal{F}(p_0, \nu, \tau)(\pi_0, \gamma, \sigma) = 0$ is as follows:
  - π_0 initial costate associated with a stationary solution (v, z) of (PQ)
  - γ multiplier associated with the punctual constraint $g_y(y(\tau))z(\tau) = 0$, and

$$\sigma = -\frac{g_y^{(1)}(y(\tau))z(\tau)}{g^{(2)}(u(\tau), y(\tau))}.$$

- The no-gap second-order sufficient condition implies that $(v, z) = 0$ is the only stationary solution of (PQ), and hence, $(\pi_0, \gamma, \sigma) = 0$

  $\Rightarrow$ the shooting algorithm is well-posed.
- Similar results when boundary arcs are present and $q \leq 2$ can be derived.

Proof

$$\mathcal{F} : \begin{pmatrix} p_0 \\ \nu \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} p(T) - \phi_y(y(T)) \\ g(y(\tau)) \\ g^{(1)}(y(\tau)) \end{pmatrix},$$

$$\left\{ \begin{array}{rcllr} \dot{y} & = & f(\Upsilon(y,p),y) & \text{on } [0,T]; & y(0) = y_0 \\ -\dot{p} & = & H_y(\Upsilon(y,p),y,p) & \text{on } [0,\tau) \cup (\tau,T]; & p(0) = p_0 \\ [p(\tau)] & = & -\nu g_y(y(\tau)). & & \end{array} \right.$$

- Differentiate, and obtain:

$$\begin{array}{rcl} 0 & = & g_y^{(1)}(y(\tau))z(\tau) + \sigma g^{(2)}(u(\tau),y(\tau)) \quad \Rightarrow \quad \sigma \\ [\pi(\tau)] & = & -\nu g_{yy}(y(\tau))z(\tau) - \gamma g_y(y(\tau)) - \nu\sigma g_y^{(1)}(y(\tau)) \\ & = & -\nu g_{yy}(y(\tau))z(\tau) - \gamma g_y(y(\tau)) \\ & & + \nu \dfrac{g_y^{(1)}(y(\tau))z(\tau)}{g^{(2)}(u(\tau),y(\tau))} g_y^{(1)}(y(\tau)). \quad \blacksquare \end{array}$$

Summary

- We give necessary and sufficient second-order optimality conditions for optimal control problems with a state constraint of arbitrary order q.
- We compute the curvature term: only essential touch points have a contribution to the curvature term; the contribution of boundary arcs is zero.
- Application of this no-gap second-order optimality conditions: well-posedness of the shooting algorithm.

References

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