Abstract Convexity and Augmented Lagrangians

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Workshop on Advances in Continuous Optimization

Reykjavik, Iceland

Friday June 30 and Saturday July 1, 2006

Joint work with

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Victoria, Australia

Outline

1 Abstract convexity: basic tools

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3 Duality Scheme

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4. Strong Duality

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5 Exact Penalty

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6. Existence of Optimal Path

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4 Strong Duality

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6 Existence of Optimal Path

7 Criterion for Exact Penalty Representation

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Then f is abstract convex or H-convex

Classical Case

X Banach space, $H = X^*$ and f lsc

f convex $\Longleftrightarrow$ f is X^*-convex

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$$g^*(l) := \sup_{z \in Z} \{\rho(z, w) - g(z)\}$$

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$$\begin{array}{rccc} \rho : & Z \times \Omega & \to & \mathbb{R} \\ & (z, r, (y, u)) & \mapsto & \nu_y(z) - r\sigma_u(z) \end{array}$$

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For which choices of ρ can we get exact Lagrangian multiplier?

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$$p(a, b_1) - p(a, b_2) \geq q(b_1 - b_2),\ \forall\, a \in \mathbb{R}$$

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$$\inf_{z \notin V} \sigma_u(z) > 0$$

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Studied by Zhou & Yang, 2004, 2005, 2006.

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C_1• p verifies a growth condition w.r.t. ν_y, σ_u if $\exists r = r(y, u)$ s.t.

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Proposition: If f is WLC then β is weakly lsc.

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$(a) \Longleftrightarrow\ (r, \widehat{y}, \widehat{u}) \in \partial \beta(0),\ \forall\, r > r'$

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(b) z_r converges weakly to 0, and every weak acc. point of x_r solves (P)

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$$\boxed{\beta(z) \geq \beta(0) + p(\nu_{\widehat{y}}(z), -r'\sigma_{\widehat{u}}(z))}$$

Application to SIP

$$\min h(x) \textbf{ s.t. } g(x,\xi) \leq 0 \textbf{ a.e } \xi$$

where $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ lsc and

(A_0) h coercive

(A_1) $\forall\,\xi,\ g(\cdot,\xi) : \mathbb{R}^n \to \mathbb{R},$ with $g(\cdot,\xi)$ lsc

Duality parameterization

Let $\theta : L_p(\Xi) \to L_p(\Xi)$ s.t.

(A_2) $\theta(\cdot)(\xi) : L_p(\Xi) \to \mathbb{R}$ is weakly lsc, $\theta(0) = 0$

$$\forall\, z \in L_p,\ D(z) := \{x \in \mathbb{R}^n \ :\ g(x,\xi) + \theta(z)(\xi) \leq 0,\ \text{a.e. } \xi\}$$

$$f(x,z) := h(x) + \delta_{D(z)}(x)$$

Under our assumptions, f is (sw) lower semicontinuous and WLC