**Exam Queueing Theory — May 21, 2007**

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| This exam consists of four problems and one sheet of formulas<br>Please put your name, university and student number on each sheet of paper<br>Motivate your answers |

1. Cars arrive at a small gas station to refuel according to a Poisson process with rate 30 per hour, and have an exponential service time distribution with mean 4 minutes. Since there are four gas pumps available, four cars can refuel simultaneously, but unfortunately there is no room for cars to wait. Hence, if a car arrives when all pumps are busy, the driver leaves immediately. For each customer that is served, an average profit is made of 7 euros.

   (a) Determine the probability that an arriving car is not refueled.

   (b) What is the long-run expected profit per day (consisting of eight hours)?

   Some day, the manager has the opportunity to buy an adjacent parking lot, so there is room to wait for all cars that arrive when the four pumps are busy. Suppose all drivers decide to wait instead of leaving when this happens.

   (c) Determine the probability that a car has to wait.

   (d) What is the long-run expected profit per day in this case?

   (e) Indicate shortly if the answers for (a), (b), (c) and (d) increase, decrease, or remain the same when the refueling times have a different distribution with equal mean 4 minutes, but smaller variance.

2. We consider two identical queueing systems, S1 and S2, each with a single server and an infinite waiting room. Jobs arrive according to a Poisson process with rate 2 per hour, and service times are exponential with mean 1/4 hour. The arrival stream is split into two (stochastically equal) arrival streams, which form the arrival processes for S1 and S2. The questions below are only about S1, since the situation for S2 is completely symmetric.
   First we suppose that the splitting is done *independently*: each arriving job is routed to S1 with probability 1/2 independently of all else, and to S2 otherwise.

   (a) Determine the probability that an arriving customer needs to wait upon arrival in S1.

   (b) Determine the expected sojourn time $ES$ and the expected number of jobs $EL$ in stationarity in S1.

   Now suppose that the splitting is done such that the jobs are *alternatingly* routed to S1 and S2; as a result the interarrival times at S1 have an Erlang-2 distribution.

   (c) Determine the probability that an arriving customer needs to wait upon arrival in S1.

   (d) Determine the expected sojourn time $ES$ and the expected number of jobs $EL$ in stationarity in S1.

3. Two types of jobs arrive at a machine. Type 1 jobs arrive according to a Poisson process with rate 45 per hour and need an exponential service time with mean 1/2 minute. Type 2 jobs arrive according to a Poisson process with rate 15 per hour and need an exponential service time with mean 1 minute.

   (a) What is the probability that an arbitrary arriving job is of type 1? Use this to find an expression for the service time distribution and the corresponding Laplace-Stieltjes transform of an arbitrary job.

   (b) Show that the Laplace-Stieltjes transform of the sojourn time $S$ of an arbitrary job is given by

   $$\tilde{S}(s) = \frac{5}{32}\frac{3}{3+2s} + \frac{27}{32}\frac{1}{1+2s}$$

   (c) What is the probability that an arbitrary job stays in the system (including service) for more than 3 minutes?

   (d) Find the mean busy period of the machine, as well as the mean idle period.

   Suppose now that the machine is turned off as soon as a busy period ends. When a new customer enters during an idle period, it takes some fixed setup time $T = 1/2$ minute before the job can be taken into service. Other jobs that are served in the same busy period do not need this setup time, since the machine is then already on.

   (e) Find the mean stationary sojourn time $ES$.

   (f) Find the mean busy period of the machine (excluding the initial setup time).

4. Consider an $M/G/1$ queue with unknown service time distribution with mean $EB$. Consider also the corresponding $M/M/1$ queue where service times are exponentially distributed with mean $\mu^{-1} = EB$. In both case the arrival rate is $\lambda$, and we assume that $\rho = \lambda EB < 1$. The random variable $W$ denotes the waiting time of any customer in stationarity in the $M/G/1$ system, while $W_M$ is the waiting time in the corresponding $M/M/1$ system.
   Show that $EW > EW_M$ *if and only if* (dan en slechts dan als) $c_B^2 > 1$, where

   $$c_B^2 = \frac{\mathrm{var}(B)}{(EB)^2}$$

   is the squared coefficient of variation of the service time distribution.

Norm (grade = total/4) :

| 1 | | | | | 2 | | | | 3 | | | | | | 4 | total |
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| a | b | c | d | e | a | b | c | d | a | b | c | d | e | f | | |
| 2 | 2 | 2 | 2 | 1 | 2 | 2 | 3 | 3 | 2 | 2 | 2 | 3 | 3 | 1 | 4 | + 4 = 40 |

Final grade = (average homework grade + 4 * written exam grade) / 5