Introduction to Stochastic Processes
Lectures 1-2

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General information

- **Info:** Mastermath web-site

- **Instructors:** Nelly Litvak and Werner Scheinhardt (University of Twente)

- **Examination:** Written exam in Utrecht. Re-exam at the UTwente.

- **Text:** Sheldon M. Ross. Introduction to Probability Models. Academic Press, 10th ed., 2007, (7th, 8th, 9th editions can also be used)
Agenda

- **Morning**
  - Markov chain definition and examples (section 4.1)
  - Chapman-Kolmogorov equations (section 4.2)
  - Classification of states (section 4.3)

- **Afternoon**
  - Limiting probabilities (section 4.4)
  - Some applications and examples (section 4.5.1, to read yourself: 4.5.2)
  - Mean time spent in transient states (section 4.6)

- **Homework:** Ross, IPM, problems: 4.1, 4.2-4.3, 4.10, 4.14, 4.15, 4.20, 4.25 (hint: use 4.20), 4.29, 4.30, 4.57, 4.63, problems 10 and 13 from the list of extra problems (to be found on the web, representative for the exam)
Example (Forecasting the weather)

Suppose that the chance of rain tomorrow depends on previous conditions only through whether or not it is raining today.

- If it rains today, it will rain tomorrow with probability $\alpha$
- If it does not rain today, it will rain tomorrow with probability $\beta$

The process is in state 0 if it rains and in state 1 if it does not rain. Then we have a two-state Markov chain whose transition probabilities are given by

$$
\mathbf{P} = \begin{bmatrix}
\alpha & 1 - \alpha \\
\beta & 1 - \beta
\end{bmatrix}
$$
Transforming a process into a Markov chain

Suppose that the chance of rain tomorrow depends on previous conditions only through the last *two* days.

<table>
<thead>
<tr>
<th>conditions yesterday</th>
<th>conditions today</th>
<th>chance of rain tomorrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>rain</td>
<td>rain</td>
<td>0.7</td>
</tr>
<tr>
<td>no rain</td>
<td>rain</td>
<td>0.5</td>
</tr>
<tr>
<td>rain</td>
<td>no rain</td>
<td>0.4</td>
</tr>
<tr>
<td>no rain</td>
<td>no rain</td>
<td>0.2</td>
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Transforming a process into a Markov chain

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</tr>
<tr>
<td>no rain</td>
<td>no rain</td>
<td>0.2</td>
</tr>
</tbody>
</table>

States description:
- **State 0**: if it rained today and yesterday
- **State 1**: if it rained today but not yesterday
- **State 2**: if it rained yesterday but not today
- **State 3**: if it did not rain either yesterday or today.
Transforming a process into a Markov chain

Suppose that the chance of rain tomorrow depends on previous conditions only through the last two days.

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States description:

- State 0: if it rained today and yesterday
- State 1: if it rained today but not yesterday
- State 2: if it rained yesterday but not today
- State 3: if it did not rain either yesterday or today.

ASSINGMENT: Write the transition probability matrix $P$ for this Markov chain.
A Random Walk model

State space of a Markov chain is given by \( i = 0, \pm 1, \pm 2, \ldots \). For some number \( 0 < p < 1 \), holds:

\[
P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \ldots
\]

Such Markov chain is called a random walk.
A Gambling model

A gambler either wins 1 EUR with probability $p$ or loses 1 EUR w.p. $1 - p$. The gambler quits if he either goes broke or attains a fortune $N$ EUR. Then the gambler's fortune is a Markov chain with state space $\{0, 1, \ldots, N\}$ having transition probabilities:

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 1, \ldots, N - 1,$$

$$P_{00} = P_{NN} = 1$$

Such Markov chain is a random walk with barriers (states 0 and $N$). The states 0 and $N$ are absorbing: once a Markov chain enter an absorbing state, it always stays in this state.
**Bonus Malus in car insurance**

Annual premium depends on the last year premium and on the number of claims made last year. Typically, no claims result in a lower premium and many claims result in a higher premium.

<table>
<thead>
<tr>
<th>State</th>
<th>Annual Premium</th>
<th>0 claims</th>
<th>1 claim</th>
<th>2 claims</th>
<th>⩾ 3 claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>250</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>400</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>600</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The number of claims per year has a Poisson distribution with parameter \( \lambda \):

\[
a_k = P\{\# \text{ claims in a year} = k\} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0.
\]

The state space is \( \{1, 2, 3, 4\} \).
Four-days weather forecast

Consider Example 4.1. State 0: rain, State 1: no rain

\[ P = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix} \]

Take \( \alpha = 0.7, \beta = 0.4 \). Given it is raining today, what is the chance that it will rain after 4 days from today?

\[
P(4) = P^4 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \]

\[ P(4)_{00} = 0.5749 \]

ASSIGNMENT: Complete the solution: what is the desired probability?

ANSWER: \( P(4)_{00} = 0.5749 \)
Four-days weather forecast

Consider Example 4.1. State 0: rain, State 1: no rain

\[
\begin{pmatrix}
\alpha & 1 - \alpha \\
\beta & 1 - \beta
\end{pmatrix}
\]

Take \(\alpha = 0.7\), \(\beta = 0.4\). Given it is raining today, what is the chance that it will rain after 4 days from today?

Solution:

\[
P^{(4)} = P^4 =
\begin{pmatrix}
0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 \\
0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6
\end{pmatrix}
\cdot
\begin{pmatrix}
0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 \\
0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6
\end{pmatrix}
\cdot
\begin{pmatrix}
0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 \\
0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6
\end{pmatrix}
\cdot
\begin{pmatrix}
0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 & 0.7 & 0.3 \\
0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6 & 0.4 & 0.6
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
0.61 & 0.39 & 0.61 & 0.39 & 0.5749 & 0.4251 \\
0.52 & 0.48 & 0.52 & 0.48 & 0.5668 & 0.4332
\end{pmatrix}
\]
Four-days weather forecast

Consider Example 4.1. State 0: rain, State 1: no rain

\[ P = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix} \]

Take \( \alpha = 0.7, \beta = 0.4 \). Given it is raining today, what is the chance that it will rain after 4 days from today?

**Solution:**

\[ P^{(4)} = P^4 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \cdot \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = \begin{pmatrix} 0.5668 & 0.4332 \\ 0.5749 & 0.4251 \end{pmatrix} \]

**ASSIGNMENT:** Complete the solution: what is the desired probability?
Four-days weather forecast

Consider Example 4.1. State 0: rain, State 1: no rain

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P = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix}
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Take \( \alpha = 0.7, \beta = 0.4 \). Given it is raining today, what is the chance that it will rain after 4 days from today?

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\[
= \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \cdot \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}
\]

**ASSIGNMENT:** Complete the solution: what is the desired probability?

**ANSWER:** \( P_{00}^{(4)} = 0.5749 \)
Example 4.9 (Cont. Example 4.4.)

▶ State 0: it rained today and yesterday
▶ State 1: it rained today but not yesterday
▶ State 2: it rained yesterday but not today
▶ State 3: if did not rain either yesterday or today.

ASSIGNMENT: Given it rained on Monday and Tuesday what is the chance that it will rain on Thursday?

\[
P = \begin{pmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8 \\
\end{pmatrix},
\]

\[
P(2) = P^2 = \begin{pmatrix}
0.49 & 0.12 & 0.21 & 0.18 \\
0.35 & 0.20 & 0.15 & 0.30 \\
0.20 & 0.12 & 0.20 & 0.48 \\
0.10 & 0.16 & 0.10 & 0.64 \\
\end{pmatrix},
\]

Rain on Thursday \equiv \text{state 0 or state 1 after two steps. The desired probability is } P_{200} + P_{201} = 0.49 + 0.12 = 0.61.
Example 4.9 (Cont. Example 4.4.)

- State 0: it rained today and yesterday
- State 1: it rained today but not yesterday
- State 2: it rained yesterday but not today
- State 3: it did not rain either yesterday or today.

**ASSIGNMENT:** Given it rained on Monday and Tuesday what is the chance that it will rain on Thursday?

\[
P = \begin{array}{cccc}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8 \\
\end{array}, \quad P^{(2)} = P^2 = \begin{array}{cccc}
0.49 & 0.12 & 0.21 & 0.18 \\
0.35 & 0.20 & 0.15 & 0.30 \\
0.20 & 0.12 & 0.20 & 0.48 \\
0.10 & 0.16 & 0.10 & 0.64 \\
\end{array}
\]
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**ASSIGNMENT:** Given it rained on Monday and Tuesday what is the chance that it will rain on Thursday?

\[
P = \begin{pmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8 \\
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0.35 & 0.20 & 0.15 & 0.30 \\
0.20 & 0.12 & 0.20 & 0.48 \\
0.10 & 0.16 & 0.10 & 0.64 \\
\end{pmatrix}
\]

Rain on Thursday \equiv state 0 or state 1 after two steps. The desired probability is \( P^{2}_{00} + P^{2}_{01} = 0.49 + 0.12 = 0.61 \)
Unconditional probabilities

Consider Example 4.1. State 0: rain, State 1: no rain

\[
P = \begin{pmatrix}
0.7 & 0.3 \\
0.4 & 0.6
\end{pmatrix}
\]

\[
P^{(4)} = P^4 = \begin{pmatrix}
0.5749 & 0.4251 \\
0.5668 & 0.4332
\end{pmatrix}
\]

\(\{X_n, n \geq 0\}\) is a Markov chain that describes the weather conditions (rain or no rain). If \(\alpha_0 = P\{X_0 = 0\} = 0.4\) and \(\alpha_1 = P\{X_0 = 1\} = 0.6\) then the unconditional probability that it will rain four days after we begin to keep the weather records is

\[
P\{X_4 = 0\} = 0.4P_{00}^4 + 0.6P_{10}^4 = (0.4)(0.5749) + (0.6)(0.5668) = 0.5700
\]
Pr-ty to enter set $\mathcal{A}$ by time $n$

State 0: rain, State 1: cloudy, State 2: sunny

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}$$

Today is sunny. What is the probability that it will not rain 3 days in a stretch?

**Solution:** Here $\mathcal{A} = \{0\}$. 
Pr-ty to enter set $\mathcal{A}$ by time $n$

State 0: rain, State 1: cloudy, State 2: sunny

\[ P = \begin{pmatrix}
0.4 & 0.6 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{pmatrix} \]

Today is sunny. What is the probability that it will not rain 3 days in a stretch?

**Solution:** Here $\mathcal{A} = \{0\}$. Make state 0 absorbing.
Pr-ty to enter set \( \mathcal{A} \) by time \( n \)

State 0: rain, State 1: cloudy, State 2: sunny

\[
P = \begin{bmatrix}
0.4 & 0.6 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{bmatrix}
\]

Today is sunny. What is the probability that it will not rain 3 days in a stretch?

**Solution:** Here \( \mathcal{A} = \{0\} \). Make state 0 absorbing.

\[
Q = \begin{bmatrix}
1 & 0 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{bmatrix}, \quad Q^3 = \begin{bmatrix}
1 & 0 & 0 \\
0.4430 & 0.3770 & 0.1800 \\
0.3830 & 0.4200 & 0.1970 \\
\end{bmatrix}
\]

The desired probability is the probability that the Markov chain with transition matrix \( Q \) is in state 1 or 2 after 3 steps.

\[
P\{\text{no rain within 3 days}\} = Q^3_{21} + Q^3_{22} = 0.4200 + 0.1970 = 0.6130
\]
Pr-ty to enter set $A$ by time $n$

State 0: rain, State 1: cloudy, State 2: sunny

\[
P = \begin{pmatrix}
0.4 & 0.6 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{pmatrix}
\]

Today is sunny. What is the probability that it will not rain 3 days in a stretch?

**Solution:** Here $A = \{0\}$. Make state 0 absorbing.

\[
Q = \begin{pmatrix}
1 & 0 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{pmatrix}, \quad Q^3 = \begin{pmatrix}
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0.4430 & 0.3770 & 0.1800 \\
0.3830 & 0.4200 & 0.1970 \\
\end{pmatrix}
\]

The desired probability is the probability that the Markov chain with transition matrix $Q$ is in state 1 or 2 after 3 steps.

\[
P\{\text{no rain within 3 days}\} = Q_{21}^3 + Q_{22}^3 = 0.4200 + 0.1970 = 0.6130
\]

$Q^n_{i,j}$, $i, j \notin A$ – probability to get to $i$ from $j$ in $n$ steps without hitting st $A$. 

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Classes of communicating states

\( j \) is \textit{accessible} from \( i \) if there exists \( k > 0 \) such that \( P^k_{ij} > 0 \)

States \( i \) and \( j \) \textit{communicate} if they are accessible from each other

Communicating states form a class
If there is only one class, MC is said to be \textit{irreducible}

Example 4.11.
\[
\begin{bmatrix}
0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Example 4.12.
\[
\begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
1/4 & 1/4 & 0 & 0 & 1/2 \\
\end{bmatrix}
\]
Classes of communicating states

$j$ is accessible from $i$ if there exists $k > 0$ such that $P_{ij}^k > 0$
States $i$ and $j$ communicate if they are accessible from each other
Communicating states form a class
If there is only one class, MC is said to be irreducible

Example 4.11.

\[
P = \begin{bmatrix}
0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

Example 4.12.

\[
P = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
1/4 & 1/4 & 0 & 0 & 1/2
\end{bmatrix}
\]

ASSIGNMENT: Find communication classes in the examples above.
Classes of communicating states

$j$ is *accessible* from $i$ if there exists $k > 0$ such that $P_{ij}^k > 0$

States $i$ and $j$ *communicate* if they are accessible from each other

Communicating states form a class

If there is only one class, MC is said to be *irreducible*

Example 4.11.

$$P = \begin{bmatrix}
0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}$$

Example 4.12.

$$P = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
1/4 & 1/4 & 0 & 0 & 1/2
\end{bmatrix}$$

**ASSIGNMENT:** Find communication classes in the examples above.

4.11. Irreducible chain

4.12. Communicating classes $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$
Proposition 4.1.

State $i$ is

- recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$
- transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$

Proof. Assume the process starts at $i$. At time $n$, there can be one visit to $i$ with probability $P_{ii}^n$. Note that

$$E[\# \text{ visits to } i \text{ at time } n] = 1 \cdot P_{ii}^n + 0 \cdot (1 - P_{ii}^n) = P_{ii}^n.$$ 

Thus, $\sum_{n=1}^{\infty} P_{ii}^n$ is the average number of visits to $i$ if the process initiates in $i$. If state $i$ is recurrent, the process returns there again and again infinitely often. If the state is transient then, after each visit, with positive probability the process never returns, the average number of visits is finite.
Proposition 4.1.

State \( i \) is

recurrent if \( \sum_{n=1}^{\infty} P_{ii}^n = \infty \) (and state \( i \) is visited infinitely often)

transient if \( \sum_{n=1}^{\infty} P_{ii}^n < \infty \) (and state \( i \) is visited finitely many times)
Proposition 4.1.

State $i$ is

recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$ (and state $i$ is visited infinitely often)

transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$ (and state $i$ is visited finitely many times)

Implication: In a finite Markov chain not all states are transient.

Intuition. If they all are transient then after finitely many steps the chain can not return to any of the states. Contradiction.
Proposition 4.1.

State $i$ is

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Corollary 4.2. If state $i$ is recurrent and state $i$ communicates with state $j$ then state $j$ is also recurrent. Recurrence is a class property!
Proposition 4.1.

State $i$ is recurrent if
\[ \sum_{n=1}^{\infty} P_{ii}^n = \infty \] (and state $i$ is visited infinitely often)

transient if
\[ \sum_{n=1}^{\infty} P_{ii}^n < \infty \] (and state $i$ is visited finitely many times)

Implication: In a finite Markov chain not all states are transient.

Intuition. If they all are transient then after finitely many steps the chain can not return to any of the states. Contradiction.

Corollary 4.2. If state $i$ is recurrent and state $i$ communicates with state $j$ then state $j$ is also recurrent. Recurrence is a class property!

Remarks (i) If state $i$ is transient and state $i$ communicates with state $j$ then state $j$ is also transient. Transience is a class property!
Proposition 4.1.

State $i$ is

- recurrent if $\sum_{n=1}^{\infty} P^n_{ii} = \infty$ (and state $i$ is visited infinitely often)
- transient if $\sum_{n=1}^{\infty} P^n_{ii} < \infty$ (and state $i$ is visited finitely many times)

Implication: In a finite Markov chain not all states are transient.

Intuition. If they all are transient then after finitely many steps the chain can not return to any of the states. Contradiction.

Corollary 4.2. If state $i$ is recurrent and state $i$ communicates with state $j$ then state $j$ is also recurrent. Recurrence is a class property!

Remarks (i) If state $i$ is transient and state $i$ communicates with state $j$ then state $j$ is also transient. Transience is a class property!
(ii) All states of an irreducible finite Markov chain are recurrent.
One-dimensional random walk

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots \]

All states communicate \( \Rightarrow \) one recurrent or transient class
One-dimensional random walk

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots \]

All states communicate \( \Rightarrow \) one recurrent or transient class

For \( n = 1, 2, \ldots \)

\[ P_{00}^{2n-1} = 0, \quad P_{00}^{2n} = \binom{2n}{n} p^n (1 - p)^n = \frac{(2n)!}{n! n!} (p(1 - p))^n \]
One-dimensional random walk

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots \]

All states communicate \( \Rightarrow \) one recurrent or transient class

For \( n = 1, 2, \ldots \)

\[ P_{00}^{2^n - 1} = 0, \quad P_{00}^{2^n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!} (p(1-p))^n \]

Stirling formula: \( n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi} \Rightarrow P_{00}^{2^n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}} \)
One-dimensional random walk

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots \]

All states communicate \( \Rightarrow \) one recurrent or transient class

For \( n = 1, 2, \ldots \)

\[ P_{00}^{2n-1} = 0, \quad P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!} (p(1-p))^n \]

Stirling formula: \( n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi} \) \( \Rightarrow \)

\[ P_{00}^{2n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}} \]

\( 4p(1-p) < 1 \) if \( p \neq 1/2 \) and \( 4p(1-p) = 1 \) if \( p = 1/2 \)

Hence, \( \sum_{n=1}^{\infty} P_{00}^{2n} = \infty \) iff \( p = 1/2 \) (random walk is symmetric)
One-dimensional random walk

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots \]

All states communicate \( \Rightarrow \) one recurrent or transient class

For \( n = 1, 2, \ldots \)

\[ P_{00}^{2n-1} = 0, \quad P_{00}^{2n} = \binom{2n}{n} p^n (1 - p)^n = \frac{(2n)!}{n!n!} (p(1 - p))^n \]

Stirling formula: \( n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi} \Rightarrow P_{00}^{2n} \sim \frac{(4p(1 - p))^n}{\sqrt{\pi n}} \)

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Hence, \( \sum_{n=1}^{\infty} P_{00}^{2n} = \infty \) iff \( p = 1/2 \) (random walk is symmetric)

**Note:** Two-dim. symmetric random walk is still recurrent but symmetric random walk in higher dimensions (\( \geq 3 \)) is not!
Summary

$\{X_n, n \geq 0\}$ - Markov chain
Countable or finite state space $S = \{0, 1, 2, \ldots\}$

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P_{ij}$$
$$P_{ij} \geq 0, \sum_j P_{ij} = 1$$

$P = (P_{ij})$ – transition matrix
$P^n = (P^n_{ij})$ – $n$-step transition probabilities

$$T_i = \inf\{n > 0 : X_n = i\}, \; i \in S$$ - 1st hitting time of state $i$
$$m_{ii} = \mathbb{E}(T_i | X_0 = i)$$ – mean return time to state $i$
The weather example

\[ P(\text{rain tomorrow} \mid \text{rain today}) = 0.7 \]

\[ P(\text{rain tomorrow} \mid \text{no rain today}) = 0.4 \]

\[
\begin{pmatrix}
0.7 & 0.3 \\
0.4 & 0.6
\end{pmatrix}
\]

\[
\begin{pmatrix}
0.61 & 0.39 \\
0.52 & 0.48
\end{pmatrix}
\]

\[
\begin{pmatrix}
0.5749 & 0.4251 \\
0.5668 & 0.4332
\end{pmatrix}
\]

\[
\begin{pmatrix}
0.5715 & 0.4285 \\
0.5714 & 0.4286
\end{pmatrix}
\]
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So, after many days, today’s weather does not affect the forecast anymore. The chance of rain will ‘stabilize’
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\end{pmatrix}
\]

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\end{pmatrix}
\quad \begin{pmatrix}
0.5715 & 0.4285 \\
0.5714 & 0.4286
\end{pmatrix}
\]

So, after many days, today’s weather does not affect the forecast anymore. The chance of rain will ‘stabilize’

\[ \pi_0 - \text{chance of rain}, \pi_1 - \text{chance of no rain}, \text{after infinit. long time:} \]
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\[ \pi_0 - \text{chance of rain, } \pi_1 - \text{chance of no rain, after infinit. long time:} \]

\[ \pi_0 = 0.7\pi_0 + 0.4\pi_1 \]
The weather example

\[ P(\text{rain tomorrow} \mid \text{rain today}) = 0.7 \]
\[ P(\text{rain tomorrow} \mid \text{no rain today}) = 0.4 \]

\[
P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \quad P^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix} \quad P^8 = \begin{pmatrix} 0.5715 & 0.4285 \\ 0.5714 & 0.4286 \end{pmatrix}
\]

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\[ \pi_0 + \pi_1 = 1 \]
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\[ \pi_1 = 0.3\pi_0 + 0.6\pi_1 \]
\[ \pi_0 + \pi_1 = 1 \]

Solution: \[ \pi_0 = 8/14 \approx 0.5714, \pi_1 = 6/14 \approx 0.4286 \]
**Convergence theorem**

**Def.** Positive recurrent aperiodic states are called *ergodic*
Convergence theorem

**Def.** Positive recurrent aperiodic states are called *ergodic*

**Theorem.** For an irreducible ergodic Markov chain, \( \lim_{n \to \infty} P_{ij}^n \) exists and is independent of \( i \). Furthermore, letting

\[
\pi_j = \lim_{n \to \infty} P_{ij}^n, \quad j \geq 0
\]

then \( \pi_j \) is the unique non-negative solution of

\[
\pi_j = \sum_i \pi_i P_{ij}, \quad j \geq 0, \quad \sum_i \pi_i = 1.
\]

Moreover, \( \pi_j = 1/m_{jj} \) where \( m_{jj} \) is the expected time between two successive visits to \( j \).

\( \pi_j \)'s represent a *stationary* distribution of a Markov chain.
Convergence theorem

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\( \pi_j \)'s represent a *stationary* distribution of a Markov chain
Example 4.19 (A model of class mobility)

Occupation classes: 0: lower; 1: middle; 2: upper. Assume that occupation of children depends only on the occupation of parents, and the transition matrix is:

\[
P = \begin{pmatrix}
0.45 & 0.48 & 0.07 \\
0.05 & 0.70 & 0.25 \\
0.01 & 0.5 & 0.49
\end{pmatrix}
\]

In the long-run, what percentage of population will have lower-, middle-, or upper-class jobs?
Example 4.19 (A model of class mobility)

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0.05 & 0.70 & 0.25 \\
0.01 & 0.5 & 0.49 \\
\end{pmatrix}
\]

In the long-run, what percentage of population will have lower-, middle-, or upper-class jobs?

**ASSIGNMENT:** Write down the linear equations for \( \pi_0, \pi_1, \pi_2 \).
Solution

The limiting probabilities \( \pi_i \) satisfy

\[
\begin{align*}
\pi_0 &= 0.45\pi_0 + 0.05\pi_1 + 0.01\pi_2 \\
\pi_1 &= 0.48\pi_0 + 0.70\pi_1 + 0.25\pi_2 \\
\pi_2 &= 0.07\pi_0 + 0.25\pi_1 + 0.49\pi_2 \\
\pi_0 + \pi_1 + \pi_2 &= 1
\end{align*}
\]

Hence, \( \pi_0 = 0.07, \pi_1 = 0.62, \pi_2 = 0.31 \)
Notes on convergence

Stationary distribution $\pi$: $\sum_i \pi_i P_{ij} = \pi_j$, $\sum_j \pi_j = 1$

- A finite aperiodic irreducible Markov chain is ergodic.
Notes on convergence

Stationary distribution $\pi$: $\sum_i \pi_i P_{ij} = \pi_j$, $\sum_j \pi_j = 1$

- A finite aperiodic irreducible Markov chain is ergodic.
- MC chain irreducible and ergodic (all states are positive recurrent and aperiodic) $\Rightarrow \lim_{n \to \infty} P^n_{ij} \to \pi_j = 1/m_{jj} > 0$ for all $i, j \in S$
Notes on convergence

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- $j$ is transient or null-recurrent $\Rightarrow \lim_{n \to \infty} P^n_{ij} = 0$, $i \in S$. All states trans. or null-rec. $\Rightarrow$ stationary distribution does not exist.

[$\text{Nelly Litvak, ISP}$] [22/32]
Notes on convergence

Stationary distribution $\pi$: $\sum_i \pi_i P_{ij} = \pi_j$, $\sum_j \pi_j = 1$

- A finite aperiodic irreducible Markov chain is ergodic.
- MC chain irreducible and ergodic (all states are *positive recurrent* and *aperiodic*) $\Rightarrow \lim_{n \to \infty} P_{ij}^n \to \pi_j = 1/m_{jj} > 0$ for all $i, j \in S$
- $j$ is *transient* or *null-recurrent* $\Rightarrow \lim_{n \to \infty} P_{ij}^n = 0$, $i \in S$. All states trans. or null-rec. $\Rightarrow$ stationary distribution does not exist.
- MC is irreducible, positive recurrent but *periodic* with period $d \Rightarrow \pi_j = 1/m_{jj}$, $j \in S$, is a stationary distribution, but $\lim_{n \to \infty} P_{ij}^n$ does not exists. Interpret $\pi_j$ as the fraction of time spent in $j$ or consider $\lim_{n \to \infty} P_{jj}^{nd} = d/m_{jj}$
Notes on convergence

Stationary distribution $\pi$: $\sum_i \pi_i P_{ij} = \pi_j$, $\sum_j \pi_j = 1$

- A finite aperiodic irreducible Markov chain is ergodic.
- MC chain irreducible and ergodic (all states are positive recurrent and aperiodic) $\Rightarrow \lim_{n \to \infty} P_{ij}^n \to \pi_j = 1/m_{jj} > 0$ for all $i, j \in S$
- $j$ is transient or null-recurrent $\Rightarrow \lim_{n \to \infty} P_{ij}^n = 0$, $i \in S$. All states trans. or null-rec. $\Rightarrow$ stationary distribution does not exist.
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- MC is reducible with several positive recurrent classes $\Rightarrow$ stationary distribution is not unique
Example 4.21 (partially)

A production process changes its states according to a positive recurrent Markov chain with transition probabilities $P_{ij}, i, j = 1, \ldots, n$. The process is up in a set $A$ and down in a set $A^c$. What is the average time between two break-downs?
Example 4.21 (partially)

A production process changes its states according to a positive recurrent Markov chain with transition probabilities $P_{ij}, i, j = 1, \ldots, n$. The process is *up* in a set $A$ and *down* in a set $A^c$. What is the average time between two break-downs?

**Solution.**

- Rate visit $i = \pi_i$
- Rate enter $j$ from $i = \pi_i P_{ij}$
- Rate enter $j$ from $A = \sum_{i \in A} \pi_i P_{ij}$
- Rate breakdown occurs $= \sum_{j \in A^c} \sum_{i \in A} \pi_i P_{ij}$
- Average time between break downs $= \left[ \sum_{j \in A^c} \sum_{i \in A} \pi_i P_{ij} \right]^{-1}$

(see numerical example in the book)
Google PageRank

The PageRank citation ranking: bringing order to the web.

\[ \pi_i = \sum_{j \to i} c d_j \pi_j + 1 - c n \]

Page is important if many important pages link to it!
PageRank $\pi_i$ of page $i$ is the long run fraction of time that a random surfer spends on page $i$. 

PageRank citation ranking: bringing order to the web.

**Google PageRank**

The PageRank citation ranking: bringing order to the web.

*PageRank* $\pi_i$ of page $i$ is the long run fraction of time that a random surfer spends on page $i$.

**‘Easily bored surfer’ model.** With probability $c$ (=0.85), a surfer follows a randomly chosen outgoing link. Otherwise, he/she jumps to a random page.

$$\pi_i = \sum_{j \rightarrow i} \frac{c}{d_j} \pi_j + \frac{1-c}{n}$$

Page is important if many important pages link to it!
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‘Easily bored surfer’ model. With probability $c (=0.85)$, a surfer follows a randomly chosen outgoing link. Otherwise, he/she jumps to a random page.

$$\pi_i = \sum_{j \to i} \frac{c}{d_j} \pi_j + \frac{1-c}{n}$$

Page is important if many important pages link to it!

**ASSIGNMENT:** Argue that the PageRank is a uniquely defined positive vector.
Strong Law of Large Numbers

It can be shown that $\pi_j$ is the long-run fraction of time spent in $j$ (the Law of Large Numbers)

$$\pi_j = \lim_{N \to \infty} \frac{\text{[# visits to } j \text{ on } [0, N]]}{N}$$
Strong Law of Large Numbers

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**Note:** Compare with $\lim_{n \to \infty} P(X_n = j|X_0 = i) = \pi_j$. Note that this is not these are two different limits!
Strong Law of Large Numbers

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**Note:** Compare with $\lim_{n \to \infty} P(X_n = j | X_0 = i) = \pi_j$. Note that this is not these are two different limits!
Proposition 4.3

Let \( \{X_n, n \geq 1\} \) be an irreducible Markov chain with stationary probabilities \( \pi_j, j \geq 0 \) and let \( r \) be a bounded function on the state space. Then, w.p. 1,

\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} r(X_n)}{N} = \sum_{j=0}^{\infty} r(j) \pi_j
\]
Proposition 4.3

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\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} r(X_n)}{N} = \sum_{j=0}^{\infty} r(j) \pi_j
\]

**Proof.** Let \( a_j(N) \) be the number of visits to state \( j \) during the time \( 1, \ldots, N \), then

\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} r(X_n)}{N} = \sum_{j=0}^{\infty} r(j) \lim_{N \to \infty} \frac{a_j(N)}{N}.
\]

Then apply the Law of Large Numbers.
Average premium in Bonus Malus car insurance

4 states for the premium. Assume that the number of claims by a customer has a Poisson distribution with mean 1/2. Then the transition matrix is:

\[
P = \begin{pmatrix}
0.6065 & 0.3033 & 0.0758 & 0.0144 \\
0.6065 & 0.0000 & 0.3033 & 0.0902 \\
0.0000 & 0.6065 & 0.0000 & 0.3935 \\
0.0000 & 0.0000 & 0.6065 & 0.3935 \\
\end{pmatrix}
\]

Hence, \( \pi_1 = 0.3692, \pi_2 = 0.2395, \pi_3 = 0.2103, \pi_4 = 0.1809 \)
Average premium in Bonus Malus car insurance

4 states for the premium. Assume that the number of claims by a customer has a Poisson distribution with mean $1/2$. Then the transition matrix is:

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Hence, $\pi_1 = 0.3692, \pi_2 = 0.2395, \pi_3 = 0.2103, \pi_4 = 0.1809$

The function $r(i)$ is a premium in state $i$. Thus, $r(1) = 200, r(2) = 250, r(3) = 400, r(4) = 600$. 


Average premium in Bonus Malus car insurance

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The function \( r(i) \) is a premium in state \( i \). Thus, \( r(1) = 200 \), \( r(2) = 250 \), \( r(3) = 400 \), \( r(4) = 600 \).

The long-run average premium per year is

\[
200\pi_1 + 250\pi_2 + 400\pi_3 + 600\pi_4 = 326.375
\]
Some history

A.A. Markov (1856-1922)

St. Petersburg school:
P.L. Chebyshev and his students
A.A. Markov, A.M. Lyapunov, G.F. Voronoi

Weak law of large numbers (J. Bernoulli, 1713):
\[ X_1, X_2, \ldots - \text{i.i.d.} \]

\[
\text{for any } \varepsilon > 0, \lim_{n \to \infty} P\{\left|X_1 + \cdots + X_n - \mathbb{E}(X)\right| \geq \varepsilon\} = 0
\]

P.A. Nekrasov (1898, 1902): Law of large numbers holds only for independent random variables. In 1906, A.A. Markov presented a sequence of dependent random variables for which the Bernoulli’s theorem holds.

In 1926, S.N. Bernstein called this type of sequences a Markov chain [Nelly Litvak, ISP] 28/32
Some history

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\[
\text{for any } \varepsilon > 0, \quad \lim_{n \to \infty} P \left\{ \left| \frac{X_1 + \cdots + X_n}{n} - E(X) \right| \geq \varepsilon \right\} = 0
\]

P.A. Nekrasov (1898, 1902): Law of large numbers holds only for independent random variables. In 1906, A.A. Markov presented a sequence of dependent random variables for which the Bernoulli’s theorem holds. In 1926, S.N. Bernstein called this type of sequences a Markov chain.
Linear algebra point of view

- Consider a finite irreducible MC with $n$ states
  Let vector $\pi$ be defined as $(\pi_1, \pi_2, \ldots, \pi_n)$
- Then stationary probabilities satisfy $\pi P = \pi$, so $\lambda = 1$ is an eigenvalue and $\pi$ is the corresponding eigenvector of $P$. The whole problem of finding $\pi$ can be formulated in framework of linear algebra!
- Applying Perron-Frobenius theory: If $P$ induces an irreducible aperiodic MC then $\lambda_1 = 1$ is a simple eigenvalue of $P$, and all other eigenvalues $\lambda_2, \ldots, \lambda_p$, where $p \leq n$, lie on a disk $|z| < 1$.
- The existence and uniqueness of $\pi$ can be proved as well as convergence. This is however not a simple theory!
The Gambler Ruin Problem (section 4.5.1)

A Gambler wins a game w.p. $p$ and loses w.p. $q = 1 - p$. Initial fortune is $i$. What is the chance that the Gambler reaches fortune $N$ before going broke?
The Gambler Ruin Problem (section 4.5.1)

A Gambler wins a game w.p. $p$ and loses w.p. $q = 1 - p$. Initial fortune is $i$. What is the chance that the Gambler reaches fortune $N$ before going broke?

$X_n$ - the player’s fortune at time $n$. Clearly, $\{X_n, n \geq 0\}$ – MC

$P_{00} = P_{NN} = 1$; $P_{i,i+1} = p = 1 - P_{i,i-1}$, $i = 1, \ldots, N - 1$
The Gambler Ruin Problem (section 4.5.1)

A Gambler wins a game w.p. \( p \) and loses w.p. \( q = 1 - p \). Initial fortune is \( i \). What is the chance that the Gambler reaches fortune \( N \) before going broke?

\( X_n \) - the player’s fortune at time \( n \). Clearly, \( \{X_n, n \geq 0\} \) – MC

\[
P_{00} = P_{NN} = 1; \quad P_{i,i+1} = p = 1 - P_{i,i-1}, \ i = 1, \ldots, N - 1
\]

\( P_i, \ i = 0, \ldots, N \) - prob-ty to reach \( N \) starting with \( i \)

\[
P_i = pP_{i+1} + qP_{i-1}, \ i = 1, \ldots, N - 1; \quad P_N = 1; \quad P_0 = 0
\]
The Gambler Ruin Problem (section 4.5.1)

A Gambler wins a game w.p. $p$ and loses w.p. $q = 1 - p$. Initial fortune is $i$. What is the chance that the Gambler reaches fortune $N$ before going broke?

$X_n$ - the player’s fortune at time $n$. Clearly, $\{X_n, n \geq 0\}$ – MC

\[ P_{00} = P_{NN} = 1; \quad P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 1, \ldots, N - 1 \]

\[ P_i, \quad i = 0, \ldots, N \text{ - prob-ty to reach } N \text{ starting with } i \]

\[ P_i = pP_{i+1} + qP_{i-1}, \quad i = 1, \ldots, N - 1; \quad P_N = 1; \quad P_0 = 0 \]

Equivalently, since $p + q = 1$,

\[ (p + q)P_i = pP_{i+1} + qP_{i-1}, \quad i = 1, \ldots, N - 1 \]
The Gambler Ruin Problem (continued)

Denote \( Z_i = P_i - P_{i-1} \). Then from the last equation,

\[
Z_{i+1} = \frac{q}{p} Z_i, \quad i = 1, \ldots, N - 1
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By induction, we get $Z_i = Z_1 (q/p)^{i-1}, \quad i = 1, \ldots, N$. Also,

$$P_i = Z_1 + \cdots + Z_i, \quad i = 1, \ldots, N$$
The Gambler Ruin Problem (continued)

Denote \( Z_i = P_i - P_{i-1} \). Then from the last equation,

\[
Z_{i+1} = \frac{q}{p} Z_i, \quad i = 1, \ldots, N - 1
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By induction, we get \( Z_i = Z_1 (q/p)^{i-1} \), \( i = 1, \ldots, N \). Also,

\[
P_i = Z_1 + \cdots + Z_i, \quad i = 1, \ldots, N
\]

**Case 1.** If \( p \neq q \), then \( P_N = Z_1 \frac{1-(q/p)^N}{1-q/p} = 1 \)

So, \( P_1 = Z_1 = \frac{1-q/p}{1-(q/p)^N} \) and \( P_i = \frac{1-(q/p)^i}{1-(q/p)^N}, \quad i = 1, \ldots, N \)
The Gambler Ruin Problem (continued)

Denote $Z_i = P_i - P_{i-1}$. Then from the last equation,

$$Z_{i+1} = \frac{q}{p} Z_i, \ i = 1, \ldots, N - 1$$

By induction, we get $Z_i = Z_1 (q/p)^{i-1}, \ i = 1, \ldots, N$. Also,

$$P_i = Z_1 + \cdots + Z_i, \ i = 1, \ldots, N$$

Case 1. If $p \neq q$, then $P_N = Z_1 \frac{1 - (q/p)^N}{1 - q/p} = 1$

So, $P_1 = Z_1 = \frac{1 - q/p}{1 - (q/p)^N}$ and $P_i = \frac{1 - (q/p)^i}{1 - (q/p)^N}, \ i = 1, \ldots, N$

Case 2. If $p = q$ then $Z_1 = Z_2 = \ldots = Z_N \Rightarrow P_N = NZ_1 = 1$

So, $Z_1 = 1/N$ and $P_i = i/N, \ i = 1, \ldots, N$
The Gambler Ruin Problem (transient states)

Let \( p = 0.4 \) and \( q = 0.6 \), \( N = 7 \), initial capital=3

\[
P_T = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0.4 & 0 & 0 & 0 \\
2 & 0.6 & 0 & 0.4 & 0 & 0 \\
3 & 0 & 0.6 & 0 & 0.4 & 0 \\
4 & 0 & 0 & 0.6 & 0 & 0.4 \\
5 & 0 & 0 & 0 & 0.6 & 0 \\
6 & 0 & 0 & 0 & 0 & 0.6
\end{bmatrix}
\]

\[ S = (s_{ij}) = [I - P_T]^{-1} \]

\[ \mathbb{E}[\# \text{times the gambler has 5 units}] = s_{3,5} = 0.9228 \]
\[ \mathbb{E}[\# \text{times the gambler has 2 units}] = s_{3,2} = 2.3677 \]
\[ P[\text{gambler ever has a fortune of 1}] = f_{3,1} = s_{3,1}/s_{1,1} = 0.8797 \]

**Check:** last prob-ty is equivalent to
\[ P(2 \text{ steps down before 4 steps up}) = 1 - \frac{1-(0.6/0.4)^2}{1-(0.6/0.4)^6} = 0.8797 \]