Answers of the exercises of ISP (September 2006), Week 1

1. \[ \mathbb{E}[X^2] = \int_{v=0}^\infty f(v) \int_{u=0}^v 2udvdv = \int_{u=0}^\infty 2u\mathbb{P}(X > u)du. \]

2. a: \( e^{-\lambda_1 v} \).
   b: \( \frac{\lambda_1}{\lambda_1 + \lambda_2} \).
   c: \( 1 - e^{-(\lambda_1 + \lambda_2)x} \).

3. a: \( 1/2 \).
   b: \( \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{2\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} \).

4. \[ \mathbb{P}(Y > x) = pe^{-\lambda_1 x} + (1 - p)e^{-\lambda_2 x}. \]
Density: \( p\lambda_1 e^{-\lambda_1 x} + (1 - p)\lambda_2 e^{-\lambda_2 x}, \ x > 0. \)

5. a: \( \lambda^2 te^{-\lambda t} \).

6. a: \( (1/5, 2/5, 2/5) \).
   b: \( m_i = 1/\pi_i, \ i = 1, 2, 3. \)

7. \( k = 1: 0. \ k = 2: (1/4)(2/5) = 1/10. \ k = 3: (1/4)(3/5) = (3/20). \ k = 4: 3/4. \)

8. a: \( 1 - e^{-12} - 12e^{-12} \).
   b: \( e^{-12/4} = e^{-3} \).
   c: \( 32e^{-32} \) (notice that clicking students form a Poisson process with intensity \((2/5) \times 10\))

9. Equations: \( \pi_p = 0.7\pi_p + \pi_b; \ \pi_g = 0.2\pi_p + 0.6\pi_g; \ \pi_r = 0.1\pi_p + 0.2\pi_g + 0.5\pi_r; \ \pi_b = 0.2\pi_g + 0.5\pi_r; \ \sum \pi_i = 1. \)
Solution: \( (\frac{10}{27}, \frac{5}{27}, \frac{4}{27}, \frac{3}{27}) \).

10. a: \( \{1, 5\}; \ \{2\}; \ \{3, 4, 6\} \).
    b: When starting in 1 or 5: \( x_A = (3/5, 0, 0, 0, 2/5, 0) \). When starting in 3, 4 or 6: \( x_B = (0, 0, 4/14, 5/14, 0, 5/14) \). When starting in 2: \( (1/3)x_A + (2/3)x_B. \)
    c: \( 1/3 \)

11. a: \( e^{-\lambda_R t}. \)
    b: \( \frac{\lambda_R}{\lambda_R + \lambda_W}. \)
    c: \( \sum_{j=0}^{3} e^{-\lambda_W t}(\frac{\lambda_W}{\lambda_R + \lambda_W})^j(\frac{\lambda_R}{\lambda_R + \lambda_W})^{3-j} \)
    d: \( 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}, \ \text{with} \ \lambda = \lambda_R + \lambda_W. \)
    e: \[ \mathbb{P}(j \text{ read}|n) = \binom{n}{j} (\frac{\lambda_R}{\lambda_R + \lambda_W})^j(\frac{\lambda_W}{\lambda_R + \lambda_W})^{n-j}. \]

12. a: \( a_{03} = 1 + \frac{2}{p}. \) Use equations like \( a_{03} = 1 + a_{13}. \)
    b: Use local balance equations like \( \pi_0 = (1/2)\pi_1 \) and \( \pi_1 = \pi_2 \) to obtain: \( (\frac{1}{2N}, \frac{1}{N}, \ldots, \frac{1}{N}, \frac{1}{2N}). \)
13. a. All states can reach each other; $P_{00} > 0$ implies aperiodicity; negative drift beyond $N$ implies positive recurrence.

b. Use local balance equations: $\pi_i = \pi_{i+1}$ for $i = 0, \ldots, N - 1$ and $(1/2)\pi_N = (1 - p)\pi_{N+1}$.

and $\pi_{N+j+1} = \frac{p}{1-p}\pi_{N+j}$ for $j = 1, 2, \ldots$, plus normalization.

c. $\alpha_{i3} = 12$. 