7.2

a. $S_n$ is Poisson with mean $n\mu$.

b. 

\[
P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n + 1) = P(S_n \leq t) - P(S_{n+1} \leq t) = \sum_{k=0}^{\lfloor t \rfloor} e^{-n\mu} \frac{(n\mu)^k}{k!} - \sum_{k=0}^{\lfloor t \rfloor} e^{-(n+1)\mu} \frac{(n+1)\mu)^k}{k!}.
\]

7.4

a. No. Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process with rate $\lambda$. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to $3/4$, then we can be certain that the next one is less than or equal to $1/4$.

b. No. Use the same processes as in a for a counterexample. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$. The probability will be different for the next interarrival.

c. No, because of a or b.

7.5. The random variable $N$ is equal to $N(1) + 1$ where $\{N(t), t \geq 0\}$ is the renewal process whose interarrival distribution is uniform on $(0, 1)$. By the result of Example 7.3,

\[E[N(t)] = m(1) + 1 = e.\]

7.10. Yes, $p/\mu$.

7.11.

\[
\frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t)}{t}.
\]

Since $X_1 < \infty$, Proposition 7.1 implies that, as $t \to \infty$,

\[
\frac{\text{number of renewals in } (X_1, t)}{t} \to \frac{1}{\mu}.
\]
7.12. Let $X$ be the time between successive $d$-events. Conditioning on the time until the next event following a $d$-event gives

$$E[X] = \int_0^d x \lambda e^{-\lambda x} dx + \int_d^{\infty} (x + E[X]) \lambda e^{-\lambda x} dx = \frac{1}{\lambda} + E[X] e^{-\lambda d}.$$ 

Therefore,

$$E[X] = \frac{1}{\lambda(1 - e^{-\lambda d})}.$$ 

a. $\frac{1}{E[X]} = \lambda(1 - e^{-\lambda d})$.

b. $1 - e^{-\lambda d}$.

7.15.

a. $X_i$ is the amount of time he has to travel after his $i$th choice (we will assume that he keeps making choices even after becoming free). $N$ is the number of choices he makes until becoming free.

b. 

$$E[T] = E\left[ \sum_{i=1}^{N} X_i \right] = E[N] E[X].$$

$N$ is a geometric random variable with $p = 1/3$, so $E[N] = 3$, $E[X] = \frac{1}{3} (2 + 4 + 6) = 4$. Hence, $E[T] = 12$.

c. 

$$E\left[ \sum_{i=1}^{N} X_i | N = n \right] = (n - 1) \frac{1}{2} (4 + 6) + 2 = 5n - 3,$$

since, given $N = n$, $X_1, \ldots, X_{n-1}$ are equally likely to be either 4 or 6, $X_n = 2$. Further, $E[\sum_{i=1}^{n} X_i] = 4n$.

d. From c,

$$E\left[ \sum_{i=1}^{N} X_i \right] = E[5N - 3] = 15 - 3 = 12.$$ 

7.19. Since, from Example 7.3, $m(t) = e^t - 1$, $0 < t \leq 1$, we obtain upon using the identity $t + E[Y(t)] = \mu(m(t) + 1)$ that $E[Y(1)] = e/2 - 1$.

7.21. This is an alternative renewal process, which is on when the server is busy and off when the server is vacant. The long-run proportion of time that the server is busy is

$$\frac{E[on\ time\ in\ a\ cycle]}{E[cycle\ length]} = \frac{\mu_G}{\mu_G + 1/\lambda}.$$
where $\mu_G$ is the mean of the distribution $G$ and the average cycle length follows from the memory-less property of the Poisson process (see also Examples 7.7, 7.11).

7.22. See the solution in the book.
7.26. The long-run average cost is

$$\frac{c + 2c + \cdots + (N - 1)c}{N/\lambda + K} = \frac{c(N - 1)/2 + KNc + \lambda K^2c}{N/\lambda + K}.$$ 

7.32. Say that the system is on at $t$ if $X_{N(t)+1}$, the interarrival time at $t$, is less than $c$ (and off otherwise). Hence, the proportion of time that $X_{N(t)+1}$ is less than $c$ is

$$\frac{E[\text{on time in a renewal cycle}]}{E[\text{cycle time}]} = \frac{\int_0^c tf(t)dt}{E[X]}.$$ 

7.37.

a. This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$E[\text{off}] = \sum_{i=1}^3 E[\text{off}|i \text{ fails}]P(i \text{ fails})$$

$$= \frac{\lambda_1}{5 \lambda_1 + \lambda_2 + \lambda_3} + 2 \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3\lambda_3}{2 \lambda_1 + \lambda_2 + \lambda_3}.$$ 

As the on time in a cycle is exponential with rate equal to $\lambda_1 + \lambda_2 + \lambda_3$, we obtain that $p$, the proportion of time that the system is working is

$$p = \frac{1/\lambda_1 + \lambda_2 + \lambda_3}{E[C]},$$

where

$$E[C] = E[\text{cycle time}] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + E[\text{off}].$$

b. Think of the system as a renewal reward process by supposing that we earn 1 per unit of time that machine 1 is being repaired. Then, $r_1$, the proportion of time that machine 1 is being repaired is

$$r_1 = \frac{\lambda_1}{5 \lambda_1 + \lambda_2 + \lambda_3}.$$
c. By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with \( s_2 \) being the proportion of time that 2 is in a state of suspended animation,

\[
s_2 = \frac{\frac{1}{5} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3}{2} \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}.
\]

**7.41.** This is a renewal process where a renewal happens when a machine is replaced by a new one. A machine is not older than one year iff the age of the renewal process is not greater than one. By Example 7.23, we have:

(a) 
\[
\int_0^1 (1 - x/2) \, dx = \frac{3}{4}
\]

(b) 
\[
\int_0^1 e^{-x} \, dx = 1 - e^{-1} \approx 0.6321
\]

**7.44.** Let \( T \) be the time it takes the shuttle to return. Now, given \( T \), \( X \) is Poisson with mean \( \lambda T \). Thus,

\[
E[X|T] = \lambda T, \quad \text{Var}(X|T) = \lambda T.
\]

Consequently,

a. \( E[X] = E[E[X|T]] = \lambda E[T] \).

b. \( \text{Var}(X) = E[\text{Var}(X|T)] + \text{Var}(E[X|T]) \) (see Proposition 3.1) = \( \lambda E[T] + \lambda^2 \text{Var}(T) \).

c. Assume that a reward of 1 is earned each time the shuttle returns empty. Then, from the renewal reward theory, \( r \), the rate at which the shuttle returns empty, is

\[
r = \frac{P(\text{empty})}{E[T]} = \frac{\int_0^\infty P(\text{empty}|T = t)f(t) \, dt}{E[T]} = \frac{\int_0^\infty e^{-\lambda t} f(t) \, dt}{E[T]} = \frac{E[e^{-\lambda T}]}{E[T]}.
\]

d. Assume that a reward of 1 is earned each time that a customer writes an angry letter. Then, with \( N_a \) equal to the number of angry letters written in a cycle, it follows that...
\( r_a \), the rate at which angry letters are written, is

\[
\begin{align*}
\frac{r_a}{\lambda} &= \frac{E[N_a]}{E[T]} = \int_0^\infty E[N_a | T = t] f(t) dt / E[T] \\
&= \int_c^\infty \lambda(t - c) f(t) dt / E[T] \\
&= \lambda E[\max\{0, T - c\}] / E[T].
\end{align*}
\]

Since passengers arrive at rate \( \lambda \), this implies that the proportion of passengers that write angry letters is \( r_a/\lambda \).

e. Because passengers arrive at a constant rate, the proportion of them that have to wait more than \( c \) will equal the proportion of time that the age of the renewal process (whose event times are the return times of the shuttle) is greater than \( c \). It is thus equal to \( 1 - F_e(c) \).

7.45.

(a) Let \( \{X_n, n \geq 0\} \) be a Markov chain in a discrete time \( n = 0, 1, 2, \ldots \), with transition matrix \( P \). Let \( \pi_1, \pi_2, \pi_3 \) denote the stationary probabilities that the Markov chain is in states 1, 2 or 3, respectively. Solving

\[
\pi_1 = (1/2)\pi_2 + \pi_3, \quad \pi_2 = \pi_1, \quad \pi_3 = (1/2)\pi_2, \quad \pi_1 + \pi_2 + \pi_3 = 1,
\]

we get \( \pi_1 = 2/5, \pi_2 = 2/5, \pi_3 = 1/5 \). Now, the long-run proportion of transitions that brings the system to state 1 is equal to the long-run fraction of visits of the Markov chain \( \{X_n, n \geq 0\} \) to state 1, that is, \( 2/5 \).

(b) Assume that a renewal happens when the system makes a transition to state 1. The long-run fraction of time of state 1 is

\[
\frac{E[\text{time in state 1 in a cycle}]}{E[\text{cycle length}]} = \frac{\mu_1}{\mu_1 + \sum_{i=2,3} E[\# \text{ visits to } i \text{ in a cycle}] \mu_i}.
\]

Note that

\[
\text{long-run fraction of visits to 2} = \pi_2 = \frac{E[\# \text{ visits to 2 in a cycle}]}{E[\# \text{ visits to 2 in a cycle}]} = \frac{E[\# \text{ visits to 2 in a cycle}]}{1/\pi_1}.
\]

Thus,

\[
E[\# \text{ visits to 2 in a cycle}] = \pi_2/\pi_1 \text{ and similarly } E[\# \text{ visits to 3 in a cycle}] = \pi_3/\pi_1.
\]
The answer is:

\[ \text{long-run fraction of time the system is in state 1} = \frac{\mu_1}{\mu_1 + \mu_2 \pi_2 / \pi_1 + \mu_3 \pi_3 / \pi_1}. \]

Further,

\[ \text{long-run fraction of time the system is in state 2} = \frac{\mu_2 \pi_2 / \pi_1}{\mu_1 + \mu_2 \pi_2 / \pi_1 + \mu_3 \pi_3 / \pi_1}, \]

\[ \text{long-run fraction of time the system is in state 3} = \frac{\mu_3 \pi_3 / \pi_1}{\mu_1 + \mu_2 \pi_2 / \pi_1 + \mu_3 \pi_3 / \pi_1}. \]

7.51. What we observe here is the inspection paradox. Let \( t \) be the time of an inspection in a hotel room. Consider arrivals of visitors to a hotel room as a renewal process where interarrival times \( X_1, X_2, \ldots \) are the lengths of stay with a common distribution function \( F \) and expectation \( \mu \). Then the interarrival time containing the instant \( t \) is typically larger than an arbitrary interarrival time. In other words, we are more likely to find a visitor who stays for a longer time. If we inspect many hotel rooms, then, by the law of large numbers, we will obtain a larger average length of stay than \( \mu \). At the airport, however, we simply have a sample from \( F \). Again, by the law of large numbers, the average length of stay in this sample is approximately \( \mu \).