parameters $i, p$. The result follows since starting
with an initial population of $1$ is equivalent to
having $1$ independent Yule processes, each starting
with a single individual.

12. (a) If the state is the number of individuals at time $t$,
we get a birth and death process with

\[ \lambda_n = n \lambda + \theta \quad n < N \]
\[ \lambda_n = n \lambda \quad n \geq N \]
\[ \mu_n = \mu. \]

(b) Let $P_i$ be the long-run probability that the system
is in state $i$. Since this is also the proportion of time
the system is in state $i$, we are looking for $\sum_{i=0}^{\infty} P_i$.
We have $\lambda_i P_k = \rho_{k+1} P_{k+1}$.
This yields

\[ P_1 = \frac{\theta}{\mu} P_0 \]
\[ P_2 = \frac{1+\theta}{2\mu} P_1 = \frac{\theta(1+\theta)}{2\mu^2} P_0 \]
\[ P_3 = \frac{2+\theta}{2\mu} P_2 = \frac{\theta(1+\theta)(2+\theta)}{6\mu^3} P_0 \]

For $k \geq 4$, we get

\[ P_k = \frac{(k-1)\mu}{2\mu} P_{k-1}. \]
which implies

\[ P_k = \frac{(k-1)(k-2)\ldots(3)}{(k)(k-1)\ldots(4)} \left( \frac{3}{k} \right)^{k-3} P_3 = \frac{2}{k} \left( \frac{3}{k} \right)^{k-3} P_3. \]

Therefore

\[ \sum_{k=3}^{\infty} P_k = 3 \left( \frac{3}{5} \right)^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k} \left( \frac{3}{k} \right)^k. \]

But

\[ \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{3}{k} \right)^k = \log \left[ \frac{1}{1 - \frac{3}{5^2}} \right] = \log \left( \frac{5^2}{5^2 - 1} \right) \text{ if } \frac{3}{5} < 1. \]

So

\[ \sum_{k=3}^{\infty} P_k = 3 \left( \frac{3}{5} \right)^3 P_3 \left[ \log \left( \frac{5^2}{5^2 - 1} \right) - \frac{1}{5} - \frac{1}{5} \left( \frac{3}{5} \right)^2 \right]. \]

Now \( \sum \tilde{P}_1 = 1 \) implies

\[ P_0 = \left[ 1 + \frac{\theta}{\beta} + \frac{\theta(1+\theta)}{2 \beta^2} + \frac{1}{21^3} \theta(1+\theta)(21+\theta) \right]^{-1}. \]

And finally,

\[ \sum_{k=3}^{\infty} P_k = \left[ \frac{1}{\beta^3} \left( \log \left( \frac{5^2}{5^2 - 1} \right) - \frac{1}{5} - \frac{1}{5} \left( \frac{3}{5} \right)^2 \right) \right]^{-1}. \]
With the number of customers in the shop as the state,
we get a birth and death process with
\[ \lambda_0 = \lambda_1 = 3 \quad \mu_1 = \mu_2 = 4. \]

Therefore
\[ P_1 = \frac{3}{4} P_0 \quad P_2 = \frac{3}{4} P_1 = \left[ \frac{3}{4} \right]^2 P_0. \]

And since \( \sum P_i = 1 \), we get
\[ P_0 = \left[ 1 + \frac{3}{4} + \left( \frac{3}{4} \right)^2 \right]^{-1} = \frac{16}{37}. \]

(a) The average number of customers in the shop is
\[ P_1 + 2P_2 = \left[ \frac{3}{4} + 2 \left( \frac{3}{4} \right)^2 \right] P_0 \]
\[ = \frac{30}{37} \left[ 1 + \frac{3}{4} + \left( \frac{3}{4} \right)^2 \right] = \frac{30}{37}. \]

(b) The proportion of customers that enter the shop is
\[ \frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16} = \frac{7}{16} = \frac{22}{37}. \]

(c) Now \( \mu = 8 \), and so

\[ P_0 = \left[ 1 + \frac{3}{8} + \left( \frac{3}{8} \right)^2 \right]^{-1} = \frac{54}{97} \]

So the proportion of customers who now enter the shop is

\[ 1 - P_2 = 1 - \left( \frac{9}{8} \right)^2 \frac{54}{97} = 1 - \frac{9}{97} = \frac{88}{97}. \]

The rate of added customers is therefore

\[ \lambda \left( \frac{88}{97} \right) - \lambda \left( \frac{22}{97} \right) = \lambda \left( \frac{88 - 22}{97} \right) = 0.45. \]

The business he does would improve by 0.45 customers per hour.

14. Letting the number of cars in the station be the state, we have a birth and death process with

\[ \lambda_0 = \lambda_1 = \lambda_2 = 20, \lambda_1 = 0, \lambda_2 = \mu = \mu_2 = 12. \]

Hence,

\[ P_0 = \frac{5}{3} P_0, P_1 = \frac{5}{3} P_0, P_2 = \frac{3}{5} P_0, P_3 = \frac{3}{5} P_0, \]

and as \( \sum P_i = 1 \), we have
\[ P_0 = \left[ 1 + \frac{5}{3} + \left( \frac{5}{3} \right)^2 + \left( \frac{5}{3} \right)^3 \right]^{-1} = \frac{27}{272} \]

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore \( 1 - P_0 = 245/272 \).

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore \[ P_B = \left( \frac{3}{2} \right)^3 P_0 = 125/272. \]

15. With the number of customers in the system as the state, we get a birth and death process with

\[ \lambda_0 = \lambda_1 = \lambda_2 = 3, \quad \lambda_3 = 0, \quad \mu_1 = 2, \quad \mu_2 = \mu_3 = 4. \]

Therefore, the balance equations reduce to

\[ P_1 = \frac{3}{2} P_0, \quad P_2 = \frac{9}{8} P_0, \quad P_3 = \frac{9}{4} P_2 = \frac{27}{32} P_0. \]

And therefore,

\[ P_0 = \left[ 1 + \frac{2}{3} + \frac{9}{8} + \frac{27}{32} \right]^{-1} = \frac{32}{143}. \]

(a) The fraction of potential customers that enter the system is
\[ \frac{1}{1-P_0} = 1 - P_0 = 1 - \frac{27}{82} = \frac{55}{82} = \frac{116}{164}. \]

(b) With a server working twice as fast we would get

\[ P_1 = \frac{9}{4} P_0, \quad P_2 = \frac{3}{4} P_1 = \left(\frac{9}{4}\right)^2 P_0, \quad P_3 = \left(\frac{9}{4}\right)^3 P_0, \]

and

\[ P_0 = \left[1 + \frac{9}{4} + \left(\frac{9}{4}\right)^2 + \left(\frac{9}{4}\right)^3\right]^{-1} = \frac{64}{175}. \]

So that now

\[ 1 - P_0 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = 1 - \frac{148}{175}. \]

16. Let the state be

- 0: an acceptable molecule is attached
- 1: no molecule attached
- 2: an unacceptable molecule is attached.

Then this is a birth and death process with balance equations:

\[ P_{12} = \frac{\beta}{\lambda} P_0, \]

\[ P_2 = \frac{\lambda(1-\varepsilon)}{\mu} P_1 = \frac{(1-\varepsilon)}{\mu} \frac{\beta}{\lambda} P_0. \]

Since \( P_0 P_1 = 1 \), we get
\[ P_0 = \left[ 1 + \mu_2 + \frac{1-\rho}{\mu_1} \right]^{-1} \frac{\lambda \rho \mu_1}{\lambda \rho \mu_1 + \mu_1 \mu_2 \lambda (1-\rho) \rho_0} \]

\( P_0 \) is the percentage of time the site is occupied by an acceptable molecule.

The percentage of time the site is occupied by an unacceptable molecule is

\[ P_2 = \frac{1-\rho}{\mu_1} P_0 = \frac{\lambda (1-\rho) \rho_0}{\lambda \rho \mu_1 + \mu_1 \mu_2 \lambda (1-\rho) \rho_0} \]

17. Say the state is 0 if the machine is up, say it is 1 when it is down due to a type 1 failure, i.e., the balance equations for the limiting probabilities are as follows.

\[ \lambda P_0 = \rho_1 P_1 + \rho_2 P_2 \]
\[ \rho_1 P_1 = \lambda P_0 \]
\[ \mu_1 P_1 = \lambda (1-\rho) P_0 \]
\[ \rho_0 = P_1 * P_2 = 1. \]

These equations are easily solved to give the results

\[ P_0 = \frac{1 - \rho \rho_1 \lambda}{(1 - \rho) \mu_1 + \lambda \mu_2 \lambda (1 - \rho) \rho_0} \]
\[ P_1 = \frac{\lambda \rho \mu_1}{\lambda \rho \mu_1 + \mu_1 \mu_2 \lambda (1-\rho) \rho_0} \]
\[ P_2 = \frac{\lambda (1-\rho) \rho_0}{\lambda \rho \mu_1 + \mu_1 \mu_2 \lambda (1-\rho) \rho_0} \]
There are $k+1$ states; state 0 means the machine is working, state $i$ means that it is in repair phase $i$, $i=1,...,k$. The balance equations for the limiting probabilities are

\[ \lambda P_0 = \mu_k P_k, \]
\[ \lambda P_1 = \mu_0 P_0, \]
\[ \lambda P_i = \mu_{i-1} P_{i-1}, \quad i=2,...,k. \]
\[ P_0 + ... + P_k = 1. \]

To solve, note that

\[ \lambda P_1 = \mu_1 P_1 = \mu_1 P_0 + \mu_2 P_2 = ... = \lambda P_0. \]

Hence,

\[ P_1 = \frac{\lambda}{\lambda + \mu_1} P_0, \]

and, upon summing,

\[ 1 = P_0 \left[ 1 - \sum_{i=1}^{k} \frac{\lambda}{\lambda + \mu_i} \right]. \]

Therefore,

\[ P_0 = \left[ 1 - \sum_{i=1}^{k} \frac{\lambda}{\lambda + \mu_i} \right]^{-1}, \quad P_1 = \frac{\lambda}{\lambda + \mu_1} P_0, \quad i=1,...,k. \]

The answer to part (a) is $P_1$ and to part (b) is $P_0$. 

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19. There are 4 states. Let state 0 mean that no machines are down, state 1 that machine one is down and two is up, state 2 that machine one is up and two is down, and 3 that both machines are down. The balance equations are as follows.

\[ \begin{align*}
(1_i - \lambda_j) p_0 &= \mu_i p_i - \mu_j p_1 \\
(\lambda_i - \mu_j) p_1 &= \mu_i p_0 - \mu_j p_3 \\
(\lambda_i - \lambda_j) p_2 &= \mu_i p_2 - \mu_2 p_0 \\
\mu_i p_3 &= \mu_i p_1 + \mu_j p_2 \\
p_0 + p_1 + p_2 + p_3 &= 1
\end{align*} \]

These equations are easily solved and the proportion of time machine 2 is down is \( p_2 \).

20. Letting the state be the number of down machines, this is a birth and death process with parameters

\[ \begin{align*}
\lambda_i &= \lambda, \quad i = 0, 1 \\
\mu_i &= \mu, \quad i = 1, 2
\end{align*} \]

By the results of Example 36, we have that

\[ E[\text{time to go from } 0 \text{ to } 2] = \frac{2}{\lambda} + \frac{\mu}{\lambda^2} \]

Using the formula at the end of Section 3, we have that

\[ \text{Var}(\text{time to go from } 0 \text{ to } 2) = \text{Var}(T) + \text{Var}(T) \]

\[ = \frac{1}{\lambda^2} + \frac{1}{2(\lambda + \mu)} - \frac{\mu}{\lambda^2} + \frac{\mu}{\mu - \lambda - \mu} (2/\lambda + \mu/2)^2 \]
Using Equation (5.3) for the limiting probabilities of a birth and death process, we have that

$$P_0 + P_1 = \frac{1 - \lambda/\mu}{1 - \lambda/\mu - (\lambda/\mu)^2}.$$

21. Now we have a birth and death process with parameters

$$\lambda_i = i, \quad i = 1, 2$$

$$\mu_i = i\alpha, \quad i = 1, 2.$$

Therefore,

$$P_0 + P_1 = \frac{1 - \lambda/\mu}{1 - \lambda/\mu - (\lambda/\mu)^2},$$

and so the probability that at least one machine is up is higher in this case.

22. The number in the system is a birth and death process with parameters

$$\lambda_n = 1/\mu (n-1), \quad n \geq 0$$

$$\mu_n = \mu, \quad n \geq 1.$$

From Equation (5.3),

$$1/P_0 = 1 - \sum_{n=1}^{\infty} (\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_0 = P_0 (\lambda/\mu)^{\infty}/n! = e^{-\lambda/\mu} (\lambda/\mu)^\infty/n! , \quad n \geq 0.$$
23. Let the state denote the number of machines that are
down. This yields a birth and death process with
\[ \lambda_0 = \frac{2}{3}, \quad \lambda_1 = \frac{2}{5}, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_3 = 0, 1 2 3 \]
\[ \mu_1 = \frac{1}{5}, \quad \mu_2 = \frac{2}{5}, \quad \mu_3 = \frac{3}{5} \]
The balance equations reduce to
\[ P_1 = \frac{3}{10} P_0 \quad \Rightarrow \quad P_0 = \frac{12}{5} P_0 \]
\[ P_2 = \frac{2}{7} P_1 \quad \Rightarrow \quad P_1 = \frac{48}{25} P_0 \]
\[ P_3 = \frac{1}{7} P_2 \quad \Rightarrow \quad P_2 = \frac{168}{250} P_0 \]
Hence, using the 1, yields
\[ P_0 = \left[ 1 + \frac{12}{5} + \frac{48}{25} + \frac{168}{250} \right]^{-1} = \frac{250}{1522} \]
(a) Average number not in use
\[ = P_1 + 2P_2 + 3P_3 = \frac{2135}{1522} = \frac{1058}{761} \]
(b) Proportion of time both repairmen are busy
\[ = P_2 + P_3 = \frac{672}{1522} = \frac{336}{761} \]