Branching Processes

Population model: each individual has \( k \) children (offspring) with probability \( p_k \), independent of all else.

- generation 0:  
  -

- generation 1:  
  -  

- generation 2:  
  -  

- generation 3:  
  -  

Applications: population models (family names), nuclear reactions, club membership, virus on internet, etc.

Let \( X_n \) = population size at generation \( n \), then \( \{X_n\} \) is a Markov chain: branching process.

Let, for \( i = 1, \ldots, X_n \),

\[
Z^{(n)}_i = \# \text{ children of } i^{th} \text{ individual in generation } n
\]

and suppose \( Z^{(n)}_i \) are i.i.d. r.v.'s with \( P(Z = k) = p_k \)

Then \( X_0 = 1 \) and for \( n = 0, 1, \ldots \):

\[
X_{n+1} = Z^{(n)}_1 + Z^{(n)}_2 + \cdots + Z^{(n)}_{X_n}
\]
Hence
\[ \Rightarrow EX_{n+1} = \sum_{k=0}^{\infty} E(X_{n+1}|X_n = k) \cdot P(X_n = k) \]
\[ = \sum_{k=0}^{\infty} k \cdot EZ \cdot P(X_n = k) \]
\[ = EZ \cdot EX_n \]

Since $EX_0 = 1$, we find with induction $EX_n = (EZ)^n$
Hence $\lim_{n \to \infty} EX_n = 0, 1, \text{or} \infty$

Similar derivation possible for $Var(X_n)$, based on the conditional variance formula:
\[ Var(X_n) = E[Var(X_n|X_{n-1})] + Var(E[X_n|X_{n-1}]) \]
(see book)
Other question: (when) does extinction occur?

Let $M = \min\{n \geq 1|X_n = 0\}$, extinction time and

$$u_n = P(M \leq n) = P(X_n = 0 \ X_0 = 1) = P_{1,0}^{(n)}$$

Then

$$u_n = P(M \leq n)$$

$$= \sum_{k=0}^{\infty} P(M \leq n|X_1 = k) \cdot P(X_1 = k)$$

$$= \sum_{k=0}^{\infty} (u_{n-1})^k \cdot p_k$$

Recursion starts with $u_0 = 0$

$$u_0 = 0 \rightarrow u_1 = p_0 \rightarrow u_2 = \sum_{k=0}^{\infty} p_0^k \cdot p_k \rightarrow \cdots$$

Note that $u_n = g(u_{n-1})$, where $g(z) = \sum z^k p_k$ is the probability generating function of the number of children $Z$. 
Intermezzo: Probability generating function

For nonnegative, integer-valued (discrete) r.v. $X$ with distribution $P(X = k) = p_k, \quad k = 0, 1, \ldots$ we define the probability generating function of $X$:

$$g(z) = E(z^X) = \sum_{k=0}^{\infty} p_k z^k \quad (0 \leq z \leq 1)$$

(similar to moment generating function $\phi(t) = E(e^{tX})$)

Properties:

- Differentiating $g$ and taking $z = 0$ yields $\{p_k\}$:
  
  $$g(z) = \sum_{k=0}^{\infty} p_k z^k \Rightarrow p_0 = g(0)$$
  
  $$g'(z) = \sum_{k=1}^{\infty} k p_k z^{k-1} \Rightarrow p_1 = g'(0)$$
  
  $$g''(z) = \sum_{k=2}^{\infty} k(k-1) p_k z^{k-2} \Rightarrow p_2 = \frac{1}{2} g''(0)$$
  
  $$g^{(n)}(z) = \sum_{k=0}^{\infty} \frac{k!}{(k-n)!} p_k z^{k-n} \Rightarrow p_n = \frac{1}{n!} g^{(n)}(0)$$

So relation $\{p_k\} \leftrightarrow g(z)$ is one to one!

Note also that $g(1) = 1$ and that $g(z)$ is positive, nondecreasing, convex.
• Differentiating $g$ and taking $z = 1$ yields moments:

\[
\begin{align*}
g'(1) & = EX \\
g''(1) & = EX(X - 1) = EX^2 - EX \\
g'''(1) & = EX(X - 1)(X - 2) = EX^3 - 3EX^2 + 2EX
\end{align*}
\]

etc.

• If $X_1, \ldots, X_n$ are independent r.v.'s with generating functions $g_1(z), \ldots, g_n(z)$, then $S = X_1 + \cdots + X_n$ has generating function:

\[
g_S(z) = Ez^S = Ez^{X_1 + \cdots + X_n} = Ez^{X_1} \cdot Ez^{X_2} \cdots Ez^{X_n} = g_1(z) \cdot g_2(z) \cdots g_n(z)
\]
(When) does extinction occur? (continued)

Scheme for \( u_n = g(u_{n-1}) \) starting at \( u_0 = 0 \):

\[ u_\infty \text{ is the smallest solution to } u = g(u) \]

Recall \( u_n = P(N \leq n) \implies u_\infty = P(N < \infty) \)

So \( u_\infty \) is probability of eventual extinction:

\[ u_\infty = P(N < \infty) = \pi_0 = \lim_{n \to \infty} P^{(n)}_{1,0} \]

Two cases:

- \( 0 < u_\infty < 1 \) when \( g'(1) > 1 \iff EZ > 1 \)
- \( u_\infty = 1 \) when \( EZ \leq 1 \)