General information

- **Meetings:** 10.15-12.00 and 13.00-14.45 (September 5, 6, 12, 13)
- **Instructors:** Nelly Litvak and Werner Scheinhardt
  (University of Twente)
- **Examination:** Written exam: September 26 2011, Buys Ballot Lab. room 201, 13.00-16.00
- **Text:** Sheldon M. Ross. Introduction to Probability Models. Academic Press, 10th ed., 2010, or 9th ed. (7th and 8th edition can also be used)
Agenda

- **Morning**
  - Markov chain definition and examples (section 4.1)
  - Chapman-Kolmogorov equations (section 4.2)
  - Classification of states (section 4.3)

- **Afternoon**
  - Limiting probabilities (section 4.4)
  - Some applications and examples (section 4.5.1, to read yourself: 4.5.2)
  - Mean time spent in transient states (section 4.6)

- **Homework:** Ross, IPM, problems: 4.1, 4.2-4.3, 4.10, 4.14, 4.15, 4.20, 4.25 (hint: use 4.20), 4.29, 4.30, 4.57, 4.63, problems 10 and 13 from the list of extra problems (to be found on the web, representative for the exam)
Example 4.1 (Forecasting the weather)

Suppose that the chance of rain tomorrow depends on previous conditions only through whether or not it is raining today.

- If it rains today, it will rain tomorrow with probability $\alpha$
- If it does not rain today, it will rain tomorrow with probability $\beta$

The process is in state 0 if it rains and in state 1 if it does not rain. Then we have a two-state Markov chain whose transition probabilities are given by

$$
P = \begin{pmatrix}
\alpha & 1 - \alpha \\
\beta & 1 - \beta
\end{pmatrix}
$$
Example 4.4 (Transforming a process into a Markov chain)

Suppose that the chance of rain tomorrow depends on previous conditions only through the last two days.

<table>
<thead>
<tr>
<th>conditions yesterday</th>
<th>conditions today</th>
<th>chance of rain tomorrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>rain</td>
<td>rain</td>
<td>0.7</td>
</tr>
<tr>
<td>no rain</td>
<td>rain</td>
<td>0.5</td>
</tr>
<tr>
<td>rain</td>
<td>no rain</td>
<td>0.4</td>
</tr>
<tr>
<td>no rain</td>
<td>no rain</td>
<td>0.2</td>
</tr>
</tbody>
</table>

States description:

- State 0: if it rained today and yesterday
- State 1: if it rained today but not yesterday
- State 2: if it rained yesterday but not today
- State 3: if it did not rain either yesterday or today.
Example 4.5 (A Random Walk model)

State space of a Markov chain is given by $i = 0, \pm 1, \pm 2, \ldots$. For some number $0 < p < 1$, holds:

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \ldots$$

Such Markov chain is called a random walk.
Example 4.6 (A Gambling model)

A gambler either wins 1 EUR with probability $p$ or loses 1 EUR w.p. $1 - p$. The gambler quits if he either goes broke or attains a fortune $N$ EUR. Then the gambler’s fortune is a Markov chain with state space $\{0, 1, \ldots, N\}$ having transition probabilities:

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 1, \ldots, N - 1,$$

$$P_{00} = P_{NN} = 1$$

Such Markov chain is a random walk with barriers (states 0 and $N$). The states 0 and $N$ are absorbing: once entered they are never left.
Example 4.7 (Bonus Malus in car insurance)

Annual premium depends on the last year premium and on the number of claims made last year. Typically, no claims result in a lower premium and many claims result in a higher premium.

<table>
<thead>
<tr>
<th>State</th>
<th>Annual Premium</th>
<th>0 claims</th>
<th>1 claim</th>
<th>2 claims</th>
<th>≥ 3 claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>250</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>400</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>600</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The number of claims per year has a Poisson distribution with parameter $\lambda$:

$$a_k = P\{\text{ [# claims in a year] = k} \} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0.$$ 

The state space is $\{1, 2, 3, 4\}$. 
Example 4.8 (Four-days weather forecast)

Consider Example 4.1. State 0: rain, State 1: no rain

\[
P = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix}
\]

Take \(\alpha = 0.7, \beta = 0.4\). Given it its raining today, what is the chance that it will rain after 4 days from today?

Solution:

\[
P^{(2)} = P^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}
\]

\[
P^{(4)} = P^4 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \cdot \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}
\]

The desired probability is \(P_{00}^4 = 0.5749\)
Example 4.9 (Cont. Example 4.4.)

- State 0: it rained today and yesterday
- State 1: it rained today but not yesterday
- State 2: it rained yesterday but not today
- State 3: if did not rain either yesterday or today.

Given it rained on Monday and Tuesday what is the chance that it will rain on Thursday?

\[
P = \begin{pmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8 \\
\end{pmatrix}, \quad P^{(2)} = P^2 = \begin{pmatrix}
0.49 & 0.12 & 0.21 & 0.18 \\
0.35 & 0.20 & 0.15 & 0.30 \\
0.20 & 0.12 & 0.20 & 0.48 \\
0.10 & 0.16 & 0.10 & 0.64 \\
\end{pmatrix}
\]

Rain on Thursday \equiv \text{state 0 or state 1 after two steps}. The desired probability is \( P_{00}^2 + P_{01}^2 = 0.49 + 0.12 = 0.61 \)
Example 4.8 (Four-days weather forecast)

Consider Example 4.1. State 0: rain, State 1: no rain

\[
P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}
\]

\[
P^{(4)} = P^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}
\]

\(\{X_n, n \geq 0\}\) is a Markov chain that describes the weather conditions (rain or no rain). If \(\alpha_0 = P\{X_0 = 0\} = 0.4\) and \(\alpha_1 = P\{X_0 = 1\} = 0.6\) then the unconditional probability that it will rain four days after we begin to keep the weather records is

\[
P\{X_4 = 0\} = 0.4P^4_{00} + 0.6P^4_{10} = (0.4)(0.5749) + (0.6)(0.5668) = 0.5700
\]
Another weather example

State 0: rain, State 1: cloudy, State 2: sunny

\[
P = \begin{bmatrix}
0.4 & 0.6 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{bmatrix}
\]

Today is sunny. What is the probability that it will not rain 3 days in a stretch?

Solution: Here \( \mathcal{A} = \{0\} \). Make state 0 absorbing.

\[
Q = \begin{bmatrix}
1 & 0 & 0 \\
0.2 & 0.5 & 0.3 \\
0.1 & 0.7 & 0.2 \\
\end{bmatrix}, \quad Q^3 = \begin{bmatrix}
1 & 0 & 0 \\
0.4430 & 0.3770 & 0.1800 \\
0.3830 & 0.4200 & 0.1970 \\
\end{bmatrix}
\]

The desired probability is the probability that the Markov chain with transition matrix \( Q \) is in state 1 or 2 after 3 steps.

\[
P\{\text{no rain within 3 days}\} = Q^3_{21} + Q^3_{22} = 0.4200 + 0.1970 = 0.6130
\]
Classes of communicating states

$j$ is *accessible* from $i$ if there exists $k > 0$ such that $P^k_{ij} > 0$

States $i$ and $j$ *communicate* if they are accessible from each other

Communicating states form a class

If there is only one class, MC is said to be *irreducible*

**Example 4.13/4.16**

\[
P = \begin{bmatrix}
0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

Irreducible chain

**Example 4.14/4.17**

\[
P = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
1/4 & 1/4 & 0 & 0 & 1/2
\end{bmatrix}
\]

Communicating classes \(\{0, 1\}, \{2, 3\}\) and \(\{4\}\)
Proposition 4.1. State $i$ is

recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$

Proof. Assume the process starts at $i$. At time $n$, there can be one visit to $i$ with probability $P_{ii}^n$. Note that

$$E[\# \text{ visits to } i \text{ at time } n] = 1 \cdot P_{ii}^n + 0 \cdot (1 - P_{ii}^n) = P_{ii}^n.$$

Thus, $\sum_{n=1}^{\infty} P_{ii}^n$ is the average number of visits to $i$ if the process initiates in $i$. If state $i$ is recurrent, the process returns there again and again infinitely often. If the state is transient then, after each visit, with positive probability the process never returns, the average number of visits is finite.
Proposition 4.1. State $i$ is

recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$ (and state $i$ is visited infinitely often)

transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$ (and state $i$ is visited finitely many times)

Implication: In a finite Markov chain not all states are transient.

Intuition. If they all are transient then after finitely many steps the chain can not return to any of the states. Contradiction.

Corollary 4.2. If state $i$ is recurrent and state $i$ communicates with state $j$ then state $j$ is also recurrent. Recurrence is a class property!

Remarks (i) If state $i$ is transient and state $i$ communicates with state $j$ then state $j$ is also transient. Transience is a class property!

(ii) All states of an irreducible finite Markov chain are recurrent.
One-dimensional random walk

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots \]

All states communicate \( \Rightarrow \) one recurrent or transient class

For \( n = 1, 2, \ldots \)

\[ P_{00}^{2n-1} = 0, \quad P_{00}^{2n} = \binom{2n}{n} p^n (1 - p)^n = \frac{(2n)!}{n!n!} (p(1 - p))^n \]

Stirling formula: \( n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi} \Rightarrow P_{00}^{2n} \sim \frac{(4p(1 - p))^n}{\sqrt{\pi n}} \)

\( 4p(1 - p) < 1 \) if \( p \neq 1/2 \) and \( 4p(1 - p) = 1 \) if \( p = 1/2 \)

Hence, \( \sum_{n=1}^{\infty} P_{00}^{2n} = \infty \) iff \( p = 1/2 \) (random walk is symmetric)

**Note:** Two-dim. symmetric random walk is still recurrent but symmetric random walk in higher dimensions (\( \geq 3 \)) is not!