MasterMath: Systems and Control Theory

Chapter 4: State Space Models

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Latent variables

\[ R\left(\frac{d}{dt}\right)(w) = M\left(\frac{d}{dt}\right)(\ell) \]

\( \ell \) - latent variables

\( w \) - manifest variables

Full behavior:  \( \mathcal{B}_f := \{ (w,\ell) \mid R\left(\frac{d}{dt}\right)(w) = M\left(\frac{d}{dt}\right)(\ell) \} \)

Manifest behavior:  \( \mathcal{B} := \{ w \mid \exists \ell : (w,\ell) \in \mathcal{B}_f \} \)

Example: Electrical circuit
Property of state

Intuition: $\ell$ in $\mathcal{B}_f := \{ (w, \ell) \mid R(\frac{d}{dt})(w) = M(\frac{d}{dt})(\ell) \}$ is called state, if $\ell(t_0)$ plays the role of memory up to time $t_0$ of the system.

Concatination operator $\wedge_t$:

$$(w_1 \wedge_t w_2)(t) := \begin{cases} w_1(\tau), & \tau < t \\ w_2(\tau), & \tau \geq t \end{cases}$$

Definition (State)

$\ell$ is called state of $\mathcal{B}_f :\iff \forall (w_1, \ell_1), (w_2, \ell_2) \in \mathcal{B}_f \forall t_0 \in \mathbb{R}$:

$$\ell_1(t_0) = \ell_2(t_0) \implies (w_1 \wedge_{t_0} w_2, \ell_1 \wedge_{t_0} \ell_2) \in \mathcal{B}_f$$
Input, non-anticipation, determinism implies state

**Theorem**

Consider a behavior $\mathcal{B}$ with trajectories $(u,x)$ and assume that

1. $u$ is free in $L^1_{\text{loc}}$
2. $x$ does not anticipate $u$ strictly, i.e. $\forall (u_1, x_1) \in \mathcal{B} \, \forall u_2 \in L^1_{\text{loc}} \, \forall t_0 \in \mathbb{R}$
   
   $$u_1 \equiv u_2 \text{ on } (t_0, -\infty) \implies \exists x_2 : (u_2, x_2) \in \mathcal{B} \text{ and } x_2 = x_1 \text{ on } (t_0, -\infty)$$

3. $\mathcal{B}$ has property of determinism, i.e. $\forall (u_1, x_1), (u_2, x_2) \in \mathcal{B} \, \forall t_0 \in \mathbb{R}$
   
   $$x_1(t_0) = x_2(t_0) \text{ and } u_1 \equiv u_2 \text{ on } [t_0, \infty) \implies x_1 \equiv x_2 \text{ on } [t_0, \infty)$$

Then $x$ is a state of $\mathcal{B}$. 
i/s/o Systems

For $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$ let

$$
\mathcal{B}_{i/s/o} := \left\{ (u,x,y) \mid \begin{array}{l}
\dot{x} = Ax + Bu \\
y = Cx + Du
\end{array} \right\}
$$

“input” $u$
“state” $x$
“output” $y$

latent variables $\ell = x$
manifest variables $w = (u,y)$

$$
\mathcal{B}_{i/o} := \{ (u,y) \mid \exists x : (u,x,y) \in \mathcal{B}_{i/s/o} \}
$$

Theorem

$x$ is a state of $\mathcal{B}_{i/s/o}$
$u$ is an input for $\mathcal{B}_{i/s/o}$ and $\mathcal{B}_{i/o}$
input/output form for $\mathcal{B}_{i/o}$ exists
Explicit solution behavior

Theorem

\[ \mathcal{B}_{i/s/o} = \left\{ (u,x,y) \mid \begin{array}{l}
\dot{x} = Ax + Bu \\
y = Cx + Du
\end{array} \right\} = \left\{ (u,x,y) \mid \exists c \in \mathbb{R}^n : x(t) = e^{At}c + \int_0^t e^{A(t-s)}Bu(s)\,ds \right\} \]

Recall matrix exponential: \( e^M = \sum_{i=0}^{\infty} \frac{M^i}{i!} \)

Crucial property: \( \frac{d}{dt}e^{At} = Ae^{At} \)
Some remarks concerning the calculation of $e^{At}$

1. $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} \\ & \ddots \\ & & e^{\lambda_n t} \end{bmatrix}$

2. $A = \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix} \Rightarrow e^{At} = e^{\lambda t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$

3. $A = TDT^{-1} \Rightarrow e^{At} = Te^{Dt}T^{-1}$

4. $A = N$ with $N^n = 0 \Rightarrow e^{Nt} = \sum_{i=0}^{n-1} N^i t^i = P_N(t)$

**Corollary (from Jordan Canonical Form)**

$\forall A \in \mathbb{R}^{n \times n} \exists$ (complex) diagonalizable $D \in \mathbb{R}^{n \times n} \exists$ nilpotent $N \in \mathbb{R}^{n \times n}$:

$A = N + D$ and $DN = ND$.

In particular,

$e^{At} = e^{Nt+Dt} = P_N(t)e^{Dt}$
i/o-equivalence of i/s/o systems

Definition

\[ B_1^{i/o} \simeq_{i/o} B_2^{i/o} \iff B_1^{i/o} = B_2^{i/o} \]

Theorem

\[ \begin{align*} 
\left\{ \begin{array}{l} 
(u, x, y) \mid \dot{x} = A_1x + B_1u \\
y = C_1x + D_1u 
\end{array} \right\} & \simeq_{i/o} \left\{ \begin{array}{l} 
(u, x, y) \mid \dot{x} = A_2x + B_2u \\
y = C_2x + D_2u 
\end{array} \right\} 
\end{align*} \]

if \( \exists S \in \mathbb{R}^{n \times n} \) invertible s.t.

\[ A_2 = SA_1S^{-1}, \quad B_2 = SB_1, \quad C_2 = C_1S^{-1}, \quad D_2 = D_1 \]
Controllability
Controllability definition and characterization

**Definition**

$\mathcal{B}$ is controllable at $t \in \mathbb{R}$ :

$$\forall w_1, w_2 \in \mathcal{B} \exists T > t \exists w_{12} \in \mathcal{B}:
\quad w_1 \land_t w_{12} \land_T w_2 \in \mathcal{B}$$

Trivially controllable:

$$\mathcal{B} = \{ w \mid 0w = 0 \} \quad \text{(completely free)}$$
$$\mathcal{B} = \{ w \mid w = 0 \} \quad \text{(completely constraint)}$$

Trivially non-controllable: All nontrivial autonomous behaviors

**Theorem (Controllability characterization)**

$$\mathcal{B} = \{ w \mid R(\frac{d}{dt})(w) = 0 \} \quad \text{controllable} \iff \lambda \mapsto \text{rank}_\mathbb{C} R(\lambda) \text{ is constant}$$

In particular: Controllability independent from $t$ and $T - t$ arbitrarily small

**Corollary (SISO-behaviours)**

The SISO-behavior $\mathcal{B} = \{ (u, y) \mid p(\frac{d}{dt})(y) = q(\frac{d}{dt})(u) \}$ is controllable

$$\iff p(\xi) \text{ and } q(\xi) \text{ have no common roots}$$
Decomposition into autonomous and controllable part

**Theorem**

Assume \( \mathcal{B} = \{ w \mid R\left( \frac{d}{dt} \right)(w) = 0 \} \) with *minimal* \( R(\xi) \in \mathbb{R}[\xi]^{g \times q} \), then \( \exists \mathcal{B}_{\text{aut}}, \mathcal{B}_{\text{contr}} \subseteq \mathcal{B} : \)

\[
\mathcal{B} = \mathcal{B}_{\text{aut}} \oplus \mathcal{B}_{\text{contr}}
\]

and

\[
\text{rank} \ R(\lambda) < g \iff \text{rank} \ R_{\text{aut}}(\lambda) < g
\]

**Remark:** For the proof we need

\[
\mathcal{B} \text{ controllable} \iff \mathcal{B} \cap C^\infty \text{ controllable}
\]
Controllability of i/s systems

\[ \mathcal{B}_{i/s} := \{ (u, x) \mid \dot{x} = Ax + Bu \} \]

Theorem

The following statements are equivalent

1. \( \mathcal{B}_{i/s} \) is controllable
2. \( \text{rank}[\lambda I - A, B] = n \ \forall \lambda \in \mathbb{C} \)
3. \( \forall z_1, z_2 \in \mathbb{R}^n \ \forall t_1 < t_2 \in \mathbb{R} \ \exists (u, x) \in \mathcal{B}_{i/s} : \ x(t_1) = z_1 \ \text{and} \ x(t_2) = z_2 \)
4. \( \text{rank}[B, AB, A^2B, \ldots, A^{n-1}B] = n \)
5. \( \mathcal{B}_{i/s/o} \) is controllable

\[ \mathcal{B}_{i/s/o} := \left\{ (u, x, y) \mid \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*} \right\} \]
Some properties of $[B, AB, \ldots, A^{n-1}B]$

**Corollary (from Cayley-Hamilton)**

The subspace $\text{im}[B, AB, \ldots, A^{n-1}B]$ is the smallest $A$-invariant subspace containing $\text{im}B$.

**Theorem (Reachable and controllable space)**

$\mathcal{R}(A, B) = \text{im}[B, AB, \ldots, A^{n-1}B]$ is the reachable space of $\mathcal{B}_{i/s}$, i.e.

$$z \in \mathcal{R}(A, B) \iff \forall T > 0 \exists (x, u) \in \mathcal{B}_{i/s} : x(0) = 0 \text{ and } x(T) = z$$

$\mathcal{R}(A, B)$ is the controllable space of $\mathcal{B}_{i/s}$, i.e.

$$z \in \mathcal{R}(A, B) \iff \forall T > 0 \exists (x, u) \in \mathcal{B}_{i/s} : x(0) = z \text{ and } x(T) = 0$$
Kalman controllability decomposition

**Theorem**

Choose invertible matrix $S = [V, W]$ such that

$$\text{im } V = \text{im}[B, AB, \ldots, A^{n-1}]$$

then

$$S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad S^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

and $(A_{11}, B_1)$ is controllable.
## Stabilizability (asymptotic controllability)

### Definition

\[ \mathcal{B} \text{ stabilizable } \iff \forall w_1 \in \mathcal{B} \ \exists w_2: \]

\[ w_1 \wedge_0 w_2 \in \mathcal{B} \quad \text{and} \quad w_2(t) \to 0 \text{ as } t \to \infty \]

### Theorem

\[ \mathcal{B} = \left\{ w \mid R \left( \frac{d}{dt} \right)(w) = 0 \right\} \text{ is stabilizable} \]

\[ \iff \lambda \mapsto \text{rank}_C R(\lambda) \text{ is constant on } \mathbb{C}_{\Re \geq 0} := \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq 0 \} \]

### Theorem

\[ \mathcal{B}_{i/s} \ (\text{and also } \mathcal{B}_{i/s/o}) \text{ is stabilizable} \]

\[ \iff A_{22} \text{ in Kalman decomposition is Hurwitz} \]
Summary controllability

- Controllability for general behaviors = concatenability of trajectories
  - Characterization: $\lambda \mapsto \text{rank}_C R(\lambda)$ is constant

- Controllability for i/s/o systems = reachability of any final state from any initial state
  - Characterization: $\text{rk}[B, AB, \ldots, A^{n-1} B] = n$

- Stabilizability = all trajectories can be driven asymptotically to zero
  - Characterization $\lambda \mapsto \text{rank}_C R(\lambda)$ is constant on $\mathbb{C}_{\Re \geq 0}$
Observability
Observability

**Definition (Observability)**

\( \ell \) is observable from \( w \) in \( \mathcal{B}_f \) (or, short, \( \mathcal{B}_f \) is observable)

\[ \iff \forall (w_1, \ell_1), (w_2, \ell_2) \in \mathcal{B}_f: \]

\[ w_1 = w_2 \implies \ell_1 = \ell_2 \]

**Lemma (Linear case)**

*If \( \mathcal{B}_f \) is linear then*

observability  \[ \iff [w = 0 \implies \ell = 0] \]
Observability characterization

Theorem

\[ \mathcal{B} = \{ (w, \ell) \mid R(\frac{d}{dt})(w) = M(\frac{d}{dt})(\ell) \} \text{ is observable } \iff \]

\[ M(\lambda) \text{ has full column rank } \forall \lambda \in \mathbb{C} \]

Intuition: If \( M(\lambda) \) has full column rank \( \forall \lambda \) then \( \exists \) polynomial left inverse \( M(\xi)^\dagger \), i.e. \( M(\xi)^\dagger M(\xi) = I \), hence \( \forall (w, I) \in \mathcal{B}_d: \)

\[ \ell = M(\frac{d}{dt})^\dagger R(\frac{d}{dt})(w) \]

On the other hand, if \( M(\lambda) \) does not have full column rank, there exists a nontrivial “kernel”, i.e. \( \exists \ell \neq 0 \) with \( M(\frac{d}{dt})(\ell) = 0 \).

Attention

In contrast to controllability, observability is not a property of the behavior itself, but of a specific choice of “internal” and “external” variables within a behavior.
### Observability of i/s/o systems

#### Corollary

\[ \mathcal{B}_{i/s/o} = \left\{ (u, x, y) \mid \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \right\} \text{is observable (i.e. } x \text{ is observable from } (u, y) \iff \right. \]

\[ \text{rank } \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C} \]

\[ \mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = [C^T, A^T C^T, (A^T)^2 C^T, \ldots, (A^T)^{n-1} C^T]^T \]

#### Corollary (Duality)

\[ \mathcal{B}_{i/s/o} \text{ is observable } \iff \text{rank } \mathcal{O} = n \iff \text{rank } \mathcal{O}^T = n \iff \]

\[ \mathcal{B}_{i/s/o}^d := \left\{ (u_d, x_d, y_d) \mid \begin{array}{l} \dot{x}_d = A^T x_d + C^T u_d \\ y_d = B^T x_d + D^T u_d \end{array} \right\} \text{ is controllable} \]
Unobservable subspace and reconstructability

Theorem

\[ \ker \mathcal{O} \] is the unobservable subspace of \( \mathcal{B}_i/s/o \), i.e.

\[
\ker \mathcal{O} = \{ x_0 \in \mathbb{R}^n \mid \exists (0, x, 0) \in \mathcal{B}_i/s/o \text{ with } x(0) = x^0 \}
\]

Furthermore, \( \ker \mathcal{O} \) is the largest \( A \)-invariant subspace contained in \( \ker C \)

Theorem

\( \mathcal{B}_i/s/o \) observable \iff \( \forall \varepsilon > 0 \forall (u_1, x_1, y_1), (u_2, x_2, y_2) \in \mathcal{B}_i/s/o: \)

\[
(u_1, y_1) = (u_2, y_2) \text{ on } [0, \varepsilon] \implies x_1 = x_2 \text{ on } [0, \varepsilon]
\]
Definition (Detectability)

\( \ell \) is detectable from \( w \) in \( \mathcal{B}_f \) (or, short, \( \mathcal{B}_f \) is detectable)

\[ \iff \forall (w_1, \ell_1), (w_2, \ell_2) \in \mathcal{B}_f: \]

\[ w_1 = w_2 \implies \ell_1(t) - \ell_2(t) \to 0 \text{ as } t \to \infty \]

Lemma (Linear case)

If \( \mathcal{B}_f \) is linear then

\[ \text{detectability } \iff [w = 0 \implies \ell \to 0] \]
Detectability characterizations

**Theorem**

\[ \mathcal{B} = \{ (w, \ell) \mid R(\frac{d}{dt})(w) = M(\frac{d}{dt})(\ell) \} \text{ is detectable} \iff \]

\[ M(\lambda) \text{ has full column rank} \quad \forall \lambda \in \mathbb{C}_{\text{Re} \geq 0} \]

**Corollary**

\[ \mathcal{B}_{i/s/o} \text{ is detectable} \iff \text{ rank} \left[ \begin{array}{c} \lambda I - A \\ C \end{array} \right] = n \quad \forall \lambda \in \mathbb{C}_{\text{Re} \geq 0} \]

\[ \iff \text{A}_{11} \text{ in Kalman observability decomposition is Hurwitz} \]

For \( \mathcal{B}_{i/s/o} \) choose invertible \( S = [V, W] \) with \( \text{im} \, V = \ker \mathcal{D} \) then

\[ S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad CS = [0, C_2] \]

and \( (C_2, A_{22}) \) is observable.
Kalman decomposition

For any \((A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}\) \exists invertible \(S \in \mathbb{R}^{n \times n}\):

\[
S^{-1}AS = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
0 & A_{22} & 0 & A_{24} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix}, \quad S^{-1}B = \begin{bmatrix}
B_1 \\
B_2 \\
0 \\
0
\end{bmatrix}
\]

\[
CS = \begin{bmatrix}
0 & C_2 & 0 & C_4
\end{bmatrix}
\]

where

\[
\left(\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}, \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}\right) \text{ is controllable}
\]

and

\[
\left(\begin{bmatrix}
A_{22} & A_{24} \\
0 & A_{44}
\end{bmatrix}, \begin{bmatrix}
C_2 \\
C_4
\end{bmatrix}\right) \text{ is observable}
\]
Elimination of latent variables

**Theorem**

\[ \forall \mathcal{B}_f = \{ (w, \ell) \mid R \left( \frac{d}{dt} \right)(w) = M \left( \frac{d}{dt} \right)(\ell) \}, \quad R(\xi) \in \mathbb{R}[\xi]^{g \times q}, \quad M(\xi) \in \mathbb{R}[\xi]^{g \times d} \quad \exists R'(\xi) \in \mathbb{R}[\xi]^{g' \times q} : \]

\[ \{ w \mid \exists \ell : (w, \ell) \in \mathcal{B}_f \} \cap \mathcal{C}^\infty = \{ w \mid R' \left( \frac{d}{dt} \right)(w) = 0 \} \cap \mathcal{C}^\infty \]

In fact, choose unimodular \( U(\xi) \in \mathbb{R}[\xi]^{g \times g} \) such that

\[ U(\xi)M(\xi) = \begin{bmatrix} 0 \\ M''(\xi) \end{bmatrix} \quad \text{and} \quad M''(\xi) \text{ has full row rank}, \]

then \( R'(\xi) \) is given by

\[ U(\xi)R(\xi) = \begin{bmatrix} R'(\xi) \\ R''(\xi) \end{bmatrix}. \]

Key observation:

\[ M''(\xi) \text{ has full row rank} \quad \Rightarrow \quad \forall w \exists \ell : R'' \left( \frac{d}{dt} \right)(w) = M'' \left( \frac{d}{dt} \right)(\ell) \]
Consequence for interconnections (not in the book)

Definition

Let \( \mathcal{B}_{i/o} = \{ (u, y) \mid p\left(\frac{d}{dt}\right)(y) = q\left(\frac{d}{dt}\right)(u) \} \) a scalar input-output form (i.e. \( p(\xi) \neq 0 \in \mathbb{R}[\xi] \) and \( q(\xi) \in \mathbb{R}[\xi] \)) then

\[
h(\xi) := \frac{q(\xi)}{p(\xi)} \in \mathbb{R}(\xi)
\]

is called transfer function of \( \mathcal{B}_{i/o} \).

Theorem

Consider \( \mathcal{B}_{i/o}^1, \mathcal{B}_{i/o}^1 \) with transfer functions \( h_1(\xi), h_2(\xi) \).

Then the transfer function of the

1. series connection is \( h_2(\xi)h_1(\xi) \)
2. parallel connection is \( h_1(\xi) + h_2(\xi) \)
3. feedback connection is \( \frac{h_1(\xi)}{1 - h_1(\xi)h_2(\xi)} \)
Theorem

Consider observable $\mathcal{B}_{i/o}$ given by $(A, b, c, d)$, then

$$\mathcal{B}_{i/o} = \{ (u, y) \mid p\left(\frac{d}{dt}\right)(y) = q\left(\frac{d}{dt}\right)(u) \}$$

with

$$p(\xi) = \det(\xi I - A), \quad q(\xi) = \det(\xi I - A)(c(\xi I - A)^{-1} b + d)$$

In particular, the corresponding transfer function is

$$h(\xi) = c(\xi I - A)^{-1} b + d$$
From i/o to i/s/o - the observer canonical form

**Theorem**

Let $\mathcal{B}_{i/o} = \{ (u,y) \mid p(\frac{d}{dt})(y) = q(\frac{d}{dt})(u) \}$ with monic $p(\xi)$ and $n = \deg p(\xi) > \deg q(\xi)$. Then $\mathcal{B}_{i/s/o}$ given by

$$
\begin{bmatrix}
0 & -p_0 \\
1 & 0 & -p_1 \\
& \ddots & \ddots & \ddots \\
& & 1 & 0 & -p_{n-2} \\
& & & 1 & 0 & -p_{n-1} \\
& & & & 1 & 0 & \cdots & 0 & 1 \\
\end{bmatrix},
\begin{bmatrix}
q_0 \\
q_1 \\
\vdots \\
q_{n-2} \\
q_{n-1}
\end{bmatrix}
$$

has $\mathcal{B}_{i/o}$ as corresponding input-output behavior.
From i/o to i/s/o - the controller canonical form

Theorem

Let $\mathcal{B}_{i/o} = \{ (u, y) \mid p\left(\frac{d}{dt}\right)(y) = q\left(\frac{d}{dt}\right)(u) \}$ with monic $p(\xi)$ and $n = \deg p(\xi) > \deg q(\xi)$. Assume additionally that $\mathcal{B}_{i/o}$ is controllable. Then $\mathcal{B}_{i/s/o}$ given by

$$A = \begin{bmatrix} 0 & 1 & & & \cdot & \cdot & \cdot \\ 0 & 1 & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \ddots & \ddots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \ddots & 0 & 1 & & \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} & \cdots & \\ \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} q_0 \\ q_1 \\ \cdots \\ q_{n-2} \\ q_{n-1} \end{bmatrix}$$

has $\mathcal{B}_{i/o}$ as corresponding input-output behavior.
Observability, minimality and equivalence

Recall from Chapter 4:

\[ \exists \text{ invertible } S : (A_2, b_2, c_2, d_2) = (S^{-1}A_1S, S^{-1}b_1, c_1S, d_1) \]

\[ \implies \mathcal{B}_{i/s/o}^2 \simeq_{i/o} \mathcal{B}_{i/s/o}^1 \quad (\Leftrightarrow \mathcal{B}_{i/o}^2 = \mathcal{B}_{i/o}^1) \]

Theorem ("Uniqueness" of i/s/o representation)

Let \((A_1, b_1, c_1, d_1), (A_2, b_2, c_2, d_2)\) observable. Then

\[ \mathcal{B}_{i/s/o}^2 \simeq_{i/o} \mathcal{B}_{i/s/o}^1 \quad (\Leftrightarrow \mathcal{B}_{i/o}^2 = \mathcal{B}_{i/o}^1) \]

\[ \iff \exists \text{ invertible } S : (A_2, b_2, c_2, d_2) = (S^{-1}A_1S, S^{-1}b_1, c_1S, d_1) \]

Theorem (Minimality)

\[(A, b, c, d)\] is a **minimal** i/s/o representation of \(\mathcal{B}_{i/o}\)

\[ \iff (A, b, c, d) \text{ is observable.} \]
Controllability and image representation

Kernel representation: \( \mathcal{B}_{\text{ker}} = \{ w \mid R\left(\frac{d}{dt}\right)(w) = 0 \} \)

Image representation: \( \mathcal{B}_{\text{im}} = \{ w \mid \exists \ell : w = M\left(\frac{d}{dt}\right)(\ell) \} \)

Elimination of latent variables:

\[
\forall M(\xi) \in \mathbb{R}[\xi]^{q \times m} \exists R(\xi) \in \mathbb{R}[\xi]^{g \times q} : \mathcal{B}_{\text{im}} = \mathcal{B}_{\text{ker}}
\]

**Theorem**

For \( R(\xi) \in \mathbb{R}[\xi]^{q \times m} \) exists \( M(\xi) \in \mathbb{R}[\xi]^{g \times q} \) with \( \mathcal{B}_{\text{ker}} = \mathcal{B}_{\text{im}} \)

\( \iff \) \( \mathcal{B}_{\text{ker}} \) is controllable.