The Behavioral Description of Linear
Time-invariant Systems

Jan Willem Polderman

Department of Applied Mathematics
University of Twente

September 9, 2019
General setting:

The aim of the intensive course SYSTEMS AND CONTROL is to get familiar with the basic concepts and more advanced notions from the mathematical theory of systems and control. The text for the course is the book *Introduction to Mathematical Systems theory: A Behavioral Approach*, by J.W. Polderman and J.C. Willems (Springer, New York, 1998). The hard copy book is out of print but is available as an e-book from Springer.
The book will be treated in several sessions, see below for a schedule. The students are expected to participate as follows:

- During the periods between the lecture days students are expected to study parts of the book independently and to work on exercises from the book.
- During the course a TAKE HOME EXAM consisting of four parts will be handed out. All assignments have to be handed in two weeks after publication.
- During the intensive week in Twente, students will get a reading assignment. On this material they have to give a PRESENTATION on Monday November 18th.
Students earn their course grade by the combination of the take home exam, the written exam, and the quality of the report and presentation. During the periods between the lecture days students can get individual help from the lecturers J.W. Polderman and S. Trenn

**Contact:**

All information on the course can be found on the web sites

MasterMath

and

Homepage JWP

On the latter web site there are downloadable pdf files, containing information on PROGRAM & GUIDELINES regarding the course (including a list of useful exercises) as well as the TAKE HOME EXAM. The information on the web site will be regularly updated. The lecturers J.W. Polderman and S. Trenn can be contacted by email.
Lecturers

J.W. Polderman
E-mail JWP
tel: +31 53 489 3438
Web site JWP

S. Trenn
E-mail ST
tel: +31 50 363 6496
Web site ST
Literature:

Schedule:

▶ First meeting.
The first part is used to get acquainted with each other. An inventory is made of the background knowledge of the students. During the second part we will describe in general terms what the first part of the book is about.
   a  Inventory of background knowledge.
   b  Sketch of Chapters 1,2.

▶ Second Meeting.
We start by discussing the problems with the book and the exercises. After that we will outline in general terms the second part of the book.
   a  Inventory of problems, exercises.
   b  Sketch of chapters 3 and 4.
Third meeting.
We start by discussing the problems with the book and the exercises. After that we will outline in general terms the third part of the book.
   a  Inventory of problems, exercises.
   b  Sketch of chapters 5 and 6.

Fourth meeting
We start by discussing the problems with the book and the exercises. After that we will outline in general terms the third part of the book.
   a  Inventory of problems, exercises.
   b  Sketch of chapters 7 and 9.
Intensive week in Twente.
During this week five HALF DAY SESSIONS (monday afternoon, tuesday-friday mornings) will be spend on: Chapter 10, further reading material. The group is divided into pairs (or triples) of students. To each sub group a subject for the final presentations is assigned. Students study their reading material independently. The lecturers are available for questions. Informal try out of final presentations

Additional activities:

- To be confirmed. Wednesday afternoon: Visit to the lab of Robotics and Mechatronics or excursion to a local high tech company, e.g. Demcon.
- Wednesday of Thursday evening Joint dinner (own cost)
- Some other evening: visit to the exciting night life of Enschede, to be organized by the UT students.
Monday, December 2nd: written exam
Monday, January 6th: retake

Monday, November 18th: Presentations.
  ▶ Presentations by students.
  ▶ Conclusive discussion.
Definition
A mathematical model is a pair \((U, \mathcal{B})\) with \(U\) a set, called the universum—its elements are called outcomes—and \(\mathcal{B}\) a subset of \(U\), called the behavior.

Example
\(U = \{\text{ice, water, steam}\} \times [-273, \infty)\) and \(\mathcal{B} = ((\{\text{ice}\} \times [-273, 0]) \cup (\{\text{water}\} \times [0, 100]) \cup (\{\text{steam}\} \times [100, \infty])).\)
Definition
A dynamical system $\Sigma$ is defined as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}),$$

with $\mathbb{T}$ a subset of $\mathbb{R}$, called the time axis, $\mathbb{W}$ a set called the signal space, and $\mathcal{B}$ a subset of $\mathbb{W}^\mathbb{T}$ called the behavior ($\mathbb{W}^\mathbb{T}$ is standard mathematical notation for the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$).
\[ M \frac{d^2}{dt^2} y + d \frac{d}{dt} y + ky = F. \]

\[
\begin{bmatrix}
M \left( \frac{d}{dt} \right)^2 + d \frac{d}{dt} + k & -1 \\
R \left( \frac{d}{dt} \right) & y \\
F & w
\end{bmatrix} = 0
\]
Example
Let $R(\xi) \in \mathbb{R}^{2 \times 3}[\xi]$ be given by
\[
R(\xi) = \begin{bmatrix}
\xi^3 & -2 + \xi & 3 \\
-1 + \xi^2 & 1 + \xi + \xi^2 & \xi
\end{bmatrix}.
\]

The multivariable differential equation $R(\frac{d}{dt})w = 0$ is
\[
\frac{d^3}{dt^3}w_1 - 2w_2 + \frac{d}{dt}w_2 + 3w_3 = 0,
\]
\[
-w_1 + \frac{d^2}{dt^2}w_1 + w_2 + \frac{d}{dt}w_2 + \frac{d^2}{dt^2}w_2 + \frac{d}{dt}w_3 = 0.
\]
Generally

\[ R\left(\frac{d}{dt}\right)w = 0 \quad R(\xi) \in \mathbb{R}^{g \times q}[\xi]. \]

Number of equation: \( g \).
Number of variables: \( q \).
Definition
A dynamical system $\Sigma = (T, W, B)$ is said to be linear if $W$ is a vector space (over a field $F$: for the purposes of this book, think of it as $\mathbb{R}$ or $\mathbb{C}$), and $B$ is a linear subspace of $W^T$ (which is a vector space in the obvious way by pointwise addition and multiplication by a scalar).
Definition
A dynamical system $\Sigma = (T, W, B)$ is said to be \textit{linear} if $W$ is a vector space (over a field $F$: for the purposes of this book, think of it as $\mathbb{R}$ or $\mathbb{C}$), and $B$ is a linear subspace of $W^T$ (which is a vector space in the obvious way by pointwise addition and multiplication by a scalar).

Definition
A dynamical system $\Sigma = (T, W, B)$ with $T = \mathbb{Z}$ or $\mathbb{R}$ is said to be \textit{time-invariant} if $\sigma^t B = B$ for all $t \in T$ ($\sigma^t$ denotes the \textit{backward $t$-shift}: $(\sigma^t f)(t') := f(t' + t)$). If $T = \mathbb{Z}$, then this condition is equivalent to $\sigma B = B$. If $T = \mathbb{Z}_+$ or $\mathbb{R}_+$, then time-invariance requires $\sigma^t B \subseteq B$ for all $t \in T$. In this course we almost exclusively deal with $T = \mathbb{R}$ or $\mathbb{Z}$, and therefore we may as well think of time-invariance as $\sigma^t B = B$. The condition $\sigma^t B = B$ is called \textit{shift-invariance} of $B$. 
Basic definitions and the calculus of representations
Recall that a dynamical system is defined by a triple \( \Sigma = (T, W, B) \) with \( T \) the time axis, \( W \) the signal space and \( B \subset W^T \) the behavior. We specialize to the case where:

- \( T = \mathbb{R} \)
- \( W = \mathbb{R}^q \)

Furthermore we restrict the attention to dynamical systems defined by linear, constant-coefficient, differential equations:

\[
R \left( \frac{d}{dt} \right) w = 0.
\]
Recall that a dynamical system is defined by a triple $\Sigma = (T, W, \mathcal{B})$ with $T$ the time axis, $W$ the signal space and $\mathcal{B} \subset W^T$ the behavior. We specialize to the case where:

- $T = \mathbb{R}$
- $W = \mathbb{R}^q$

Furthermore we restrict the attention to dynamical systems defined by linear, constant-coefficient, differential equations:

$$R\left(\frac{d}{dt}\right)w = 0.$$ 

**Definition**
This equation defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B})$, with $\mathcal{B}$ defined as

$$\mathcal{B} := \{ w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid w \text{ satisfies } R\left(\frac{d}{dt}\right)w = 0 \text{ weakly.} \}$$
With $C = 1$ and $R_1 = R_2 = 1$ we obtain:

$$V + \frac{d}{dt} V = 2I + \frac{d}{dt} I.$$

Take $V(t) = 1(t)$ then: $I(t) =$
With $C = 1$ and $R_1 = R_2 = 1$ we obtain:

$$V + \frac{d}{dt} V = 2I + \frac{d}{dt} I.$$ 

Take $V(t) = 1(t)$ then: $I(t) = \left(\frac{1}{2} + \frac{1}{2}e^{-2t}\right)1(t) + ke^{-2t}$
Remark

Notice that \( V(t) \) nor \( l(t) \) are differentiable. What then is the interpretation of solution?
Remark

Notice that $V(t)$ nor $l(t)$ are differentiable. What then is the interpretation of solution?

1. *In the sense of distributions?*
Remark

*Notice that $V(t)$ nor $I(t)$ are differentiable. What then is the interpretation of solution?*

1. *In the sense of distributions?*
2. *In the sense of ‘delta function calculus’?*
Remark

Notice that V(t) nor l(t) are differentiable. What then is the interpretation of solution?

1. In the sense of distributions?
2. In the sense of ‘delta function calculus’?
3. Just ignore that there could be a problem?
Remark

Notice that $V(t)$ nor $I(t)$ are differentiable. What then is the interpretation of solution?

1. In the sense of distributions?
2. In the sense of ‘delta function calculus’?
3. Just ignore that there could be a problem?
4. Integral equations!
Weak Solutions (Integral Equations)

Consider once more

\[ V + \frac{d}{dt} V = 2I + \frac{d}{dt} I. \]

‘Integrate’ both sides:

\[ \int_0^t V(\tau)d\tau + V(t) = 2 \int_0^t I(\tau)d\tau + I(t) + d_0. \]
Weak Solutions (Integral Equations)

Consider once more

\[ V + \frac{d}{dt} V = 2I + \frac{d}{dt} I. \]

‘Integrate’ both sides:

\[ \int_0^t V(\tau)d\tau + V(t) = 2 \int_0^t I(\tau)d\tau + I(t) + d_0. \]

Substituting non-smooth functions in this equation is no longer a problem. In fact: Smooth solutions of the differential equation are also solutions of the integral equation and vice versa.
Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ and consider

$$R\left(\frac{d}{dt}\right)w = 0.$$ 

Define the integral operator acting on $\mathcal{L}^{1}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ by

$$(\int w)(t) := \int_{0}^{t} w(\tau)d\tau,$$

$$(\int^{k+1} w)(t) := \int_{0}^{t} (\int^{k} w)(\tau)d\tau,$$

Let $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L$, $R_L \neq 0 \in \mathbb{R}^{g \times q}[\xi]$ be given. Define $R^*(\xi)$ as

$$R^*(\xi) := \xi^L R\left(\frac{1}{\xi}\right) = R_L + R_{L-1}\xi + \cdots + R_1\xi^{L-1} + R_0\xi^L.$$ 

and consider
\[(R^*(\int)w)(t) = c_0 + c_1 t + \cdots + c_{L-1} t^{L-1}, \quad c_i \in \mathbb{R}^g.\]

We call \(w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)\) a \textit{weak} solution of the system of differential equations if there exist constant vectors \(c_i \in \mathbb{R}^g\) such that the integral equation is satisfied for almost all \(t \in \mathbb{R}\).
\[(R^* (\int) w)(t) = c_0 + c_1 t + \cdots + c_{L-1} t^{L-1}, \quad c_i \in \mathbb{R}^g.\]

We call \(w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)\) a \textit{weak} solution of the system of differential equations if there exist constant vectors \(c_i \in \mathbb{R}^g\) such that the integral equation is satisfied for almost all \(t \in \mathbb{R}\). For \(R(\xi) \in \mathbb{R}^{g \times q}[\xi]\) we define

\[\mathcal{B} = \{ w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0, \text{ weakly}\}\]
\[(R^*(\int w)(t) = c_0 + c_1 t + \cdots + c_{L-1} t^{L-1}, \quad c_i \in \mathbb{R}^g.\]

We call \(w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)\) a **weak** solution of the system of differential equations if there exist constant vectors \(c_i \in \mathbb{R}^g\) such that the integral equation is satisfied for almost all \(t \in \mathbb{R}\). For \(R(\xi) \in \mathbb{R}^{g \times q}[\xi]\) we define

\[\mathcal{B} = \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0, \quad \text{weakly}\}\]

**Theorem**

\(\mathcal{B}\) is closed.
\[(R^*(\int)w)(t) = c_0 + c_1 t + \cdots + c_{L-1} t^{L-1}, \quad c_i \in \mathbb{R}^g.\]

We call \(w \in \mathcal{L}^{1}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)\) a \textit{weak} solution of the system of differential equations if there exist constant vectors \(c_i \in \mathbb{R}^g\) such that the integral equation is satisfied for almost all \(t \in \mathbb{R}\). For \(R(\xi) \in \mathbb{R}^{g \times q}[\xi]\) we define

\[
\mathcal{B} = \{ w \in \mathcal{L}^{1}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0, \quad \text{weakly} \}
\]

**Theorem**

\(\mathcal{B}\) is closed.

\(\mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q)\) is dense in \(\mathcal{B}\).
\[
(R^* (\int) w)(t) = c_0 + c_1 t + \cdots + c_{L-1} t^{L-1}, \quad c_i \in \mathbb{R}^g.
\]

We call \( w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \) a \textit{weak} solution of the system of differential equations if there exist constant vectors \( c_i \in \mathbb{R}^g \) such that the integral equation is satisfied for almost all \( t \in \mathbb{R} \).

For \( R(\xi) \in \mathbb{R}^{g \times q}[\xi] \) we define

\[
\mathcal{B} = \{ w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R \frac{d}{dt} w = 0, \quad \text{weakly} \}
\]

**Theorem**

\( \mathcal{B} \) is closed.

\( \mathcal{B} \cap C^\infty(\mathbb{R}, \mathbb{R}^q) \) is dense in \( \mathcal{B} \).

\( \mathcal{B}_1 \cap C^\infty(\mathbb{R}, \mathbb{R}^q) = \mathcal{B}_2 \cap C^\infty(\mathbb{R}, \mathbb{R}^q) \) implies \( \mathcal{B}_1 = \mathcal{B}_2 \).
Theorem

$\mathcal{B}$ is linear.

$\mathcal{B}$ is time-invariant.
Definition (Equivalent differential equations)

Let $R_i(\xi) \in \mathbb{R}^{g_i \times q}[\xi], i = 1, 2$. The differential equations

$$R_1\left(\frac{d}{dt}\right)w = 0 \quad \text{and} \quad R_2\left(\frac{d}{dt}\right)w = 0$$

are said to be equivalent if they define the same dynamical system. In other words, equivalence means that $w$ is a solution of

$$R_1\left(\frac{d}{dt}\right)w = 0$$

if and only if it is also a solution of

$$R_2\left(\frac{d}{dt}\right)w = 0.$$
Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ and $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Define $R'(\xi) := U(\xi)R(\xi)$. Denote the behaviors corresponding to $R(\xi)$ and $R'(\xi)$ by $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then:

1. $\mathcal{B} \subset \mathcal{B}'$.

2. If in addition, $U^{-1}(\xi)$ exists and if $U^{-1}(\xi) \in \mathbb{R}^{g \times q}[\xi]$, then $\mathcal{B} = \mathcal{B}'$. 

Proof.

1. Choose $w \in \mathcal{B}$. Since $R'(d/dt)w = 0$, we also have that $U(d/dt)R'(d/dt)w = 0$. It follows that then also $w \in \mathcal{B}'$.

2. By 1, it suffices to prove that $\mathcal{B}' \subset \mathcal{B}$. Since $U^{-1}(\xi)$ is a well-defined polynomial matrix, this follows by just applying Part 1 to $U^{-1}(\xi)R(\xi)$ and $R(\xi)$. 

Theorem
Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ and $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Define $R'(\xi) := U(\xi)R(\xi)$. Denote the behaviors corresponding to $R(\xi)$ and $R'(\xi)$ by $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then:

1. $\mathcal{B} \subset \mathcal{B}'$.
2. If in addition, $U^{-1}(\xi)$ exists and if $U^{-1}(\xi) \in \mathbb{R}^{g \times g}[\xi]$, then $\mathcal{B} = \mathcal{B}'$.

Proof.
1. Choose $w \in \mathcal{B}$. Since $R(\frac{d}{dt})w = 0$, we also have that $U(\frac{d}{dt})R(\frac{d}{dt})w = 0$. It follows that then also $w \in \mathcal{B}'$.
2. By 1, it suffices to prove that $\mathcal{B}' \subset \mathcal{B}$. Since $U^{-1}(\xi)$ is a well-defined polynomial matrix, this follows by just applying Part 1 to $U^{-1}(\xi)R(\xi)$ and $R(\xi)$. \hfill $\square$
The converse is also true:

**Theorem**

Let \( R_i(\xi) \in \mathbb{R}^{g_i \times q}[\xi] \) define the behaviors \( \mathcal{B}_i, \ i = 1, 2 \). If \( \mathcal{B}_1 \subset \mathcal{B}_2 \) then there exists matrix \( F(\xi) \in \mathbb{R}^{g_1 \times g_2}[\xi] \) such that

\[
R_2(\xi) = F(\xi)R_1(\xi).
\]
The converse is also true:

**Theorem**
Let $R_i(\xi) \in \mathbb{R}^{g_i \times q}[\xi]$ define the behaviors $\mathcal{B}_i$, $i = 1, 2$. If $\mathcal{B}_1 \subset \mathcal{B}_2$ then there exists matrix $F(\xi) \in \mathbb{R}^{g_1 \times g_2}[\xi]$ such that

$$R_2(\xi) = F(\xi)R_1(\xi).$$

**Remark**
The statement relates two systems of differential equations that have the same solution set. It appears logical that to prove the statement that somehow we need to know something about solutions of systems of differential equations. By transforming the matrix in diagonal form, the core of the proof relies on scalar differential equations only.
Corollary

Let $R_i(\xi) \in \mathbb{R}^{g \times q}[\xi]$ define the same behavior, then there exists a matrix $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that

$$R_2(\xi) = U(\xi)R_1(\xi),$$

and $U(\xi)$ has a polynomial inverse.
Polynomial matrices with a polynomial inverse play a very important role. Hence the following definition.

**Definition (Unimodular matrix)**

Let $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Then $U(\xi)$ is said to be a *unimodular* polynomial matrix if there exists a polynomial matrix $V(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that $V(\xi)U(\xi) = I$. Equivalently, if $\det U(\xi)$ is equal to a nonzero constant.
Polynomial matrices with a polynomial inverse play a very important role. Hence the following definition.

**Definition (Unimodular matrix)**

Let \( U(\xi) \in \mathbb{R}^{g \times g}[\xi] \). Then \( U(\xi) \) is said to be a unimodular polynomial matrix if there exists a polynomial matrix \( V(\xi) \in \mathbb{R}^{g \times g}[\xi] \) such that \( V(\xi)U(\xi) = I \). Equivalently, if \( \det U(\xi) \) is equal to a nonzero constant.

**Example**

\[
U(\xi) = \begin{bmatrix} -1 + 2\xi - \xi^2 & -2 + \xi \\ 2 + \xi^3 & 1 - \xi^2 \end{bmatrix},
\]
Polynomial matrices with a polynomial inverse play a very important role. Hence the following definition.

**Definition (Unimodular matrix)**

Let \( U(\xi) \in \mathbb{R}^{g \times g}[\xi] \). Then \( U(\xi) \) is said to be a unimodular polynomial matrix if there exists a polynomial matrix \( V(\xi) \in \mathbb{R}^{g \times g}[\xi] \) such that \( V(\xi)U(\xi) = I \). Equivalently, if \( \det U(\xi) \) is equal to a nonzero constant.

**Example**

\[
U(\xi) = \begin{bmatrix}
-1 + 2\xi - \xi^2 & -2 + \xi \\
2 + \xi^3 & 1 - \xi^2
\end{bmatrix},
\]
Polynomial matrices with a polynomial inverse play a very important role. Hence the following definition.

**Definition (Unimodular matrix)**

Let $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Then $U(\xi)$ is said to be a *unimodular* polynomial matrix if there exists a polynomial matrix $V(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that $V(\xi)U(\xi) = I$. Equivalently, if $\det U(\xi)$ is equal to a nonzero constant.

**Example**

$$U(\xi)^{-1} = \frac{1}{3} \begin{bmatrix} 1 - \xi^2 & 2 - \xi \\ -2 - \xi^3 & -1 + 2\xi - \xi^2 \end{bmatrix}$$
There are three types of elementary unimodular matrices:
There are three types of elementary unimodular matrices:
1. Identity matrix with two rows interchanged.
There are three types of elementary unimodular matrices:

1. Identity matrix with two rows interchanged.

2. \[ \text{diag} \left( 1, \ldots, 1, \alpha, 1, \ldots, 1 \right), \]

   \[ \uparrow \]

   \[ \text{ith place} \]
There are three types of elementary unimodular matrices:

3.

\[
\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & \xi^d & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

\[\leftarrow \text{ith row.}\]

\[\uparrow \text{ jth column}\]
Theorem

\( U(\xi) \in \mathbb{R}^{g \times g}[\xi] \) is unimodular if and only if it is a finite product of elementary unimodular matrices.
Theorem
\( U(\xi) \in \mathbb{R}^{g \times g}[\xi] \) is unimodular if and only if it is a finite product of elementary unimodular matrices.

Remark
The relevance of this theorem is that whenever statements are made about unimodular matrices it is often convenient to prove the statement for elementary unimodular matrices and subsequently that the property to which the statement refers carries over to products of unimodular matrices.
Theorem

\( U(\xi) \in \mathbb{R}^{g \times g}[\xi] \) is unimodular if and only if it is a finite product of elementary unimodular matrices.

Remark

The relevance of this theorem is that whenever statements are made about unimodular matrices it is often convenient to prove the statement for elementary unimodular matrices and subsequently that the property to which the statement refers carries over to products of unimodular matrices.

Proof.

The proof is based on the observation that the inverse of a given unimodular matrix may be transformed into the identity matrix by means of successive pre-multiplication with elementary unimodular matrices.
Theorem

Let \( r_1(\xi), \ldots, r_k(\xi) \in \mathbb{R}[\xi] \) and assume that \( r_1(\xi), \ldots, r_k(\xi) \) have no common factor. Then there exists a unimodular matrix \( U(\xi) \in \mathbb{R}^{k \times k}[\xi] \) such that the last row of \( U(\xi) \) equals \([r_1(\xi), \ldots, r_k(\xi)]\).
Theorem
Let \( r_1(\xi), \ldots, r_k(\xi) \in \mathbb{R}[\xi] \) and assume that \( r_1(\xi), \ldots, r_k(\xi) \) have no common factor. Then there exists a unimodular matrix \( U(\xi) \in \mathbb{R}^{k \times k}[\xi] \) such that the last row of \( U(\xi) \) equals \([r_1(\xi), \ldots, r_k(\xi)]\).

Proof.
Based on excessive use of the Euclidean algorithm (division with remainder). \( \square \)
Corollary (Bezout)

Let $r_1(\xi), \ldots, r_k(\xi) \in \mathbb{R}[\xi]$. Assume that the greatest common divisor of $r_1(\xi), \ldots, r_k(\xi)$ is 1, i.e., the $k$ polynomials have no common factor, i.e., they are coprime. Then there exist polynomials $a_1(\xi), \ldots, a_k(\xi) \in \mathbb{R}[\xi]$ such that

$$r_1(\xi)a_1(\xi) + \cdots + r_k(\xi)a_k(\xi) = 1.$$
Corollary (Bezout)

Let \( r_1(\xi), \ldots, r_k(\xi) \in \mathbb{R}[\xi] \). Assume that the greatest common divisor of \( r_1(\xi), \ldots, r_k(\xi) \) is 1, i.e., the k polynomials have no common factor, i.e., they are coprime. Then there exist polynomials \( a_1(\xi), \ldots, a_k(\xi) \in \mathbb{R}[\xi] \) such that

\[
r_1(\xi)a_1(\xi) + \cdots + r_k(\xi)a_k(\xi) = 1.
\]

Proof.

\[
\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} U(\xi) = \begin{bmatrix} r_1(\xi) & \cdots & r_k(\xi) \end{bmatrix} \implies \\
\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_1(\xi) & \cdots & r_k(\xi) \end{bmatrix} U(\xi)^{-1}
\]
Left unimodular transformations can be used to bring the polynomial matrix in a more suitable form, like the upper triangular form.

**Theorem (Upper triangular form)**

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. There exists a unimodular matrix $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that $U(\xi)R(\xi) = T(\xi)$ and $T_{ij}(\xi) = 0$ for $i = 1, \ldots, g$, $j < i$. 
Left unimodular transformations can be used to bring the polynomial matrix in a more suitable form, like the *upper triangular form*.

**Theorem (Upper triangular form)**

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. There exists a unimodular matrix $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that $U(\xi)R(\xi) = T(\xi)$ and $T_{ij}(\xi) = 0$ for $i = 1, \ldots, g$, $j < i$.

**Proof.**

Division with remainder. \(\square\)
By applying both left and right unimodular transformations we can bring a polynomial matrix into an even more convenient form, called the *Smith form*. 
By applying both left and right unimodular transformations we can bring a polynomial matrix into an even more convenient form, called the *Smith form*.

**Theorem**

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ and $V(\xi) \in \mathbb{R}^{q \times q}[\xi]$. Denote the behaviors defined by $R(\frac{d}{dt})w = 0$ and $R(\frac{d}{dt})V(\frac{d}{dt})w = 0$ by $\mathcal{B}$ and $\mathcal{B}'$ respectively. If $V(\xi)$ is unimodular, then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic as vector spaces.
By applying both left and right unimodular transformations we can bring a polynomial matrix into an even more convenient form, called the Smith form.

**Theorem**

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ and $V(\xi) \in \mathbb{R}^{q \times q}[\xi]$. Denote the behaviors defined by $R\left(\frac{d}{dt}\right)w = 0$ and $R\left(\frac{d}{dt}\right)V\left(\frac{d}{dt}\right)w = 0$ by $\mathcal{B}$ and $\mathcal{B}'$ respectively. If $V(\xi)$ is unimodular, then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic as vector spaces.

**Theorem (Smith form, square case)**

Let $R(\xi) \in \mathbb{R}^{g \times g}[\xi]$. There exist unimodular matrices $U(\xi), V(\xi) \in \mathbb{R}^{g \times g}$ such that

1. $U(\xi)R(\xi)V(\xi) = \text{diag}(d_1(\xi), \ldots, d_g(\xi))$.

2. There exist (scalar) polynomials $q_i(\xi)$ such that $d_{i+1}(\xi) = q_i(\xi)d_i(\xi)$, $i = 1, \ldots, g - 1$. 
Example
Let \( R(\xi) \) be given by

\[
R(\xi) = \begin{bmatrix}
-1 + \xi & 1 + \xi \\
-2 + \xi + \xi^2 & 2 + 3\xi + \xi^2
\end{bmatrix} .
\]
Example
Let $R(\xi)$ be given by

$$R(\xi) = \begin{bmatrix} -1 + \xi & 1 + \xi \\ -2 + \xi + \xi^2 & 2 + 3\xi + \xi^2 \end{bmatrix}.$$ 

Let $\mathcal{B}$ denote the behavior represented by $R\left(\frac{d}{dt}\right)w = 0$. By pre-multiplication with a suitable unimodular matrix and deleting the resulting zero row, we find that $\mathcal{B}$ is thus also represented by

$$\begin{bmatrix} -1 + \xi & 1 + \xi \end{bmatrix}.$$
Example
Let $R(\xi)$ be given by

$$R(\xi) = \begin{bmatrix} -1 + \xi & 1 + \xi \\ -2 + \xi + \xi^2 & 2 + 3\xi + \xi^2 \end{bmatrix}.$$

Let $\mathcal{B}$ denote the behavior represented by $R\left(\frac{d}{dt}\right)w = 0$. By pre-multiplication with a suitable unimodular matrix and deleting the resulting zero row, we find that $\mathcal{B}$ is thus also represented by

$$\begin{bmatrix} -1 + \xi & 1 + \xi \end{bmatrix}.$$

Intuitively, it is clear that no further reduction is possible. In more complicated situations, we would like to have a precise criterion on the basis of which we can conclude that we are dealing with the minimal number of equations.
Definition (Independence)

The polynomial vectors $r_1(\xi), \ldots, r_k(\xi)$ are said to be independent over $\mathbb{R}[\xi]$ if

$$\sum_{j=1}^{k} a_j(\xi) r_j(\xi) = 0 \iff a_1(\xi) = \cdots = a_k(\xi) = 0.$$ 

Here $a_1(\xi), \ldots, a_k(\xi)$ are scalar polynomials.
Definition (Row rank and column rank)

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. The row rank (column rank) of $R(\xi)$ is defined as the maximal number of independent rows (columns). We say that $R(\xi)$ has full row rank (full column rank) if the row rank (column rank) equals the number of rows (columns) in $R(\xi)$. 
Definition (Row rank and column rank)
Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. The row rank (column rank) of $R(\xi)$ is defined as the maximal number of independent rows (columns). We say that $R(\xi)$ has full row rank (full column rank) if the row rank (column rank) equals the number of rows (columns) in $R(\xi)$.

Theorem
Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. Denote the row rank and the column rank of $R(\xi)$ by $k_r$ and $k_c$ respectively.

1. The integers $k_r, k_c$ are invariant with respect to pre- and postmultiplication by unimodular matrices.
2. $k_r = k_c$, and hence we can speak about the rank of $R(\xi)$.
3. There exists a $k_r \times k_r$ submatrix\(^1\) of $R(\xi)$ with nonzero determinant.

\(^1\)Submatrix as the result of selecting $k_r$ rows and $k_r$ columns
Proof.
Proof.
Property 1. is easy to see for elementary unimodular matrices.
Proof.
Property 1. is easy to see for elementary unimodular matrices. Property 2. follows by transforming into Smith form, for which the statement is obvious, and using Property 1.
Proof.
Property 1. is easy to see for elementary unimodular matrices. Property 2. follows by transforming into Smith form, for which the statement is obvious, and using Property 1. Property 3. follows by selecting $k_r$ independent rows. The resulting matrix has row rank $k_r$ and therefore also column rank $k_r$. Hence it contains $k_r$ independent columns. These yield a square $k_r \times k_r$ submatrix of $R(\xi)$. By transforming this square submatrix into Smith form the result follows. \qed
Theorem

Every behavior $\mathcal{B}$ defined by $R \left( \frac{d}{dt} \right) w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ admits an equivalent full row rank representation, that is, there exists a representation $\tilde{R} \left( \frac{d}{dt} \right) w = 0$ of $\mathcal{B}$ with $\tilde{R}(\xi) \in \mathbb{R}^{\tilde{g} \times q}$ and rank $\mathbb{R} = \tilde{g}$. 
Theorem

Every behavior $\mathcal{B}$ defined by $R\left(\frac{d}{dt}\right)w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ admits an equivalent full row rank representation, that is, there exists a representation $\tilde{R}\left(\frac{d}{dt}\right)w = 0$ of $\mathcal{B}$ with $\tilde{R}(\xi) \in \mathbb{R}^{\tilde{g} \times q}$ and rank $\tilde{R} = \tilde{g}$.

Proof.

Find suitable unimodular $U(\xi)$ so that $U(\xi)R(\xi)$ has a zero row. Delete that zero row and repeat until no further reduction is possible. The resulting matrix $\tilde{R}(\xi)$ has full row rank. $\square$
Theorem
Every behavior $\mathcal{B}$ defined by $R\left(\frac{d}{dt}\right) w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ admits an equivalent full row rank representation, that is, there exists a representation $\tilde{R}\left(\frac{d}{dt}\right) w = 0$ of $\mathcal{B}$ with $\tilde{R}(\xi) \in \mathbb{R}^{\tilde{g} \times q}$ and rank $\mathbb{R} = \tilde{g}$.

Remark
It appears obvious that it is desirable to transform a given matrix $R(\xi)$ into full row rank form so as to reduce redundancy and the number of equations.
A further reduction of the number of equations could perhaps be possible by starting from some other matrix that defines the same behavior. This leads to the notion of minimality.
Definition (Minimality)

Let the behavior $\mathcal{B}$ be defined by $R\left(\frac{d}{dt}\right)w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. The representation $R\left(\frac{d}{dt}\right)w = 0$ is called \textit{minimal} if every other representation has at least $g$ rows, that is, if $w \in \mathcal{B}$ if and only if $R'(\frac{d}{dt})w = 0$ for some $R'(\xi) \in \mathbb{R}^{g' \times q}[\xi]$ implies $g' \geq g$. 
Definition (Minimality)

Let the behavior $\mathcal{B}$ be defined by $R(\frac{d}{dt})w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. The representation $R(\frac{d}{dt})w = 0$ is called minimal if every other representation has at least $g$ rows, that is, if $w \in \mathcal{B}$ if and only if $R'(\frac{d}{dt})w = 0$ for some $R'(\xi) \in \mathbb{R}^{g' \times q}[\xi]$ implies $g' \geq g$.

Theorem

If $R(\xi)$ is a minimal representation of $\mathcal{B}$ if and only if $R(\xi)$ has full row rank.