The Behavioral Description of Linear
Time-invariant Systems

Jan Willem Polderman

Department of Applied Mathematics
University of Twente

September 24, 2017
Time Domain Description of Linear Systems
Main Results Chapter 3

Consider

$$\mathcal{B} = \{ w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0 \} \quad R(\xi) \text{ full row rank}$$
Main Results Chapter 3

Consider

$$\mathcal{B} = \{ w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0 \} \quad R(\xi) \text{ full row rank}$$

The behavior $\mathcal{B}$ is autonomous if and only if $\det(R(\xi)) \neq 0$ (the zero polynomial).
Consider

\[ \mathcal{B} = \left\{ w \in \mathcal{L}^{1}_{\text{loc}}(\mathbb{R}, \mathbb{R}^{q}) \mid R \frac{d}{dt} w = 0 \right\} \quad R(\xi) \text{ full row rank} \]

The behavior \( \mathcal{B} \) is autonomous if and only if \( \det(R(\xi)) \neq 0 \) (the zero polynomial).

The behavior \( \mathcal{B} \) can be written in input-output form otherwise.
Main Results Chapter 3

Consider

\[ \mathcal{B} = \{ w \in \mathcal{L}_1^1(\mathbb{R}, \mathbb{R}^q) \mid R \left( \frac{d}{dt} \right) w = 0 \} \quad R(\xi) \text{ full row rank} \]

The behavior \( \mathcal{B} \) is autonomous if and only if \( \det(R(\xi)) \neq 0 \) (the zero polynomial).
The behavior \( \mathcal{B} \) can be written in input-output form otherwise.

In both case there is an explicit expression for the trajectories in \( \mathcal{B} \).
Definition

A behavior $\mathcal{B}$ is called **autonomous** if for all $w_1, w_2 \in \mathcal{B}$

$$w_1(t) = w_2(t) \text{ for } t \leq 0 \Rightarrow w_1(t) = w_2(t) \text{ for almost all } t.$$ 

In words: the future of every trajectory is completely determined by its past.
Theorem

Let $P(\xi) \in \mathbb{R}[\xi]$ be a monic polynomial and let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$, be the distinct roots of $P(\xi)$ of multiplicity $n_i$:

$$P(\xi) = \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}.$$  

The corresponding behavior $\mathcal{B}$ defined by $P\left(\frac{d}{dt}\right)w = 0$ is autonomous.
Theorem

Let $P(\xi) \in \mathbb{R}[\xi]$ be a monic polynomial and let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$, be the distinct roots of $P(\xi)$ of multiplicity $n_i$:

$P(\xi) = \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}.$

The corresponding behavior $\mathcal{B}$ defined by $P(\frac{d}{dt})w = 0$ is autonomous and is a finite-dimensional subspace of $C^\infty(\mathbb{R}, \mathbb{C})$ of dimension $n = \deg P(\xi)$. 
Theorem

Let \( P(\xi) \in \mathbb{R}[\xi] \) be a monic polynomial and let \( \lambda_i \in \mathbb{C}, i = 1, \ldots, N \), be the distinct roots of \( P(\xi) \) of multiplicity \( n_i \):

\[
P(\xi) = \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}.
\]

The corresponding behavior \( \mathcal{B} \) defined by \( P \left( \frac{d}{dt} \right) w = 0 \) is autonomous and is a finite-dimensional subspace of \( C^\infty(\mathbb{R}, \mathbb{C}) \) of dimension \( n = \deg P(\xi) \). Moreover, \( w \in \mathcal{B} \) if and only if it is of the form

\[
w(t) = \sum_{k=1}^{N} r_k(t) e^{\lambda_k t}
\]

with \( r_k(\xi) \in \mathbb{C}[\xi] \) an arbitrary polynomial of degree less than \( n_k \).
The polynomial $P(\xi)$ is called the \textit{characteristic polynomial} of $\mathcal{B}$, and the roots of $P(\xi)$ are called the \textit{characteristic values}.
The polynomial $P(\xi)$ is called the *characteristic polynomial* of $\mathcal{B}$, and the roots of $P(\xi)$ are called the *characteristic values*.

Steps of the proof
The polynomial $P(\xi)$ is called the *characteristic polynomial* of $\mathcal{B}$, and the roots of $P(\xi)$ are called the *characteristic values*.

Steps of the proof

1. Verification: each trajectory in the statement satisfies the differential equation
The polynomial $P(\xi)$ is called the characteristic polynomial of $\mathcal{B}$, and the roots of $P(\xi)$ are called the characteristic values.

Steps of the proof

1. Verification: each trajectory in the statement satisfies the differential equation
2. Independence: the separate trajectories form a linearly independent set. This yields a lowerbound on dimension
The polynomial $P(\xi)$ is called the *characteristic polynomial* of $\mathcal{B}$, and the roots of $P(\xi)$ are called the *characteristic values*.

Steps of the proof

1. Verification: each trajectory in the statement satisfies the differential equation

2. Independence: the separate trajectories form a linearly independent set. This yields a lowerbound on dimension

3. Upperbound on dimension. This step is the only place where existence and uniqueness of solutions of ODE is invoked, be it an extremely simple ODE!
Theorem

Let $P(\xi) \in \mathbb{R}^{q \times q}[\xi]$ and let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$, be the distinct roots of $\det P(\xi)$ of multiplicity $n_i$: $\det P(\xi) = c \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$ for some nonzero constant $c$. 
Theorem

Let $P(\xi) \in \mathbb{R}^{q \times q}[\xi]$ and let $\lambda_i \in \mathbb{C}, i = 1, \ldots, N$, be the distinct roots of $\det P(\xi)$ of multiplicity $n_i$: $\det P(\xi) = c \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$ for some nonzero constant $c$. The corresponding behavior $\mathcal{B}$ is autonomous and is a finite-dimensional subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ of dimension $n = \deg \det P(\xi)$. 
Theorem

Let $P(\xi) \in \mathbb{R}^{q \times q}[\xi]$ and let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$, be the distinct roots of $\det P(\xi)$ of multiplicity $n_i$: $\det P(\xi) = c \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$ for some nonzero constant $c$. The corresponding behavior $B$ is autonomous and is a finite-dimensional subspace of $\mathcal{C}^{\infty}(\mathbb{R}, \mathbb{C}^q)$ of dimension $n = \deg \det P(\xi)$. Moreover, $w \in B$ if and only if it is of the form

$$w(t) = \sum_{i=1}^{N} \sum_{j=0}^{n_i-1} B_{ij} t^j e^{\lambda_i t},$$
Theorem
Let $P(\xi) \in \mathbb{R}^{q \times q}[\xi]$ and let $\lambda_i \in \mathbb{C}, i = 1, \ldots, N$, be the distinct roots of $\det P(\xi)$ of multiplicity $n_i$: $\det P(\xi) = c \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$ for some nonzero constant $c$. The corresponding behavior $\mathcal{B}$ is autonomous and is a finite-dimensional subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ of dimension $n = \deg \det P(\xi)$. Moreover, $w \in \mathcal{B}$ if and only if it is of the form

$$w(t) = \sum_{i=1}^{N} \sum_{j=0}^{n_i-1} B_{ij} t^j e^{\lambda_i t},$$

where the vectors $B_{ij} \in \mathbb{C}^q$ satisfy the relations

$$\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} P^{(j-\ell)}(\lambda_i) B_{ij} = 0, \quad i = 1, \ldots, N; \quad \ell = 0, \ldots, n_i-1.$$
The polynomial $\prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$ is called the characteristic polynomial of the autonomous behavior $\mathcal{B}$. The roots of $\det P(\xi), \lambda_1, \cdots, \lambda_N$, are called the characteristic values of the behavior. For the case that $P(\xi) = I\xi - A$ for some matrix $A$, the characteristic values are just the eigenvalues of $A$. 
The polynomial $\prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$ is called the characteristic polynomial of the autonomous behavior $\mathcal{B}$. The roots of $\det P(\xi), \lambda_1, \cdots, \lambda_N$, are called the characteristic values of the behavior. For the case that $P(\xi) = I\xi - A$ for some matrix $A$, the characteristic values are just the eigenvalues of $A$. The linear relations

$$\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} P^{(j-\ell)}(\lambda_i) B_{ij} = 0, \quad i = 1, \ldots, N; \quad \ell = 0, \ldots, n_i-1.$$ 

are readily derived from the identity:

$$P\left(\frac{d}{dt}\right)Ae^{\lambda t} = P(\lambda)Ae^{\lambda t}$$
The polynomial $\prod_{k=1}^{N}(\xi - \lambda_k)^{n_k}$ is called the \textit{characteristic polynomial} of the autonomous behavior $\mathcal{B}$. The roots of $\det P(\xi), \lambda_1, \cdots, \lambda_N$, are called the characteristic values of the behavior. For the case that $P(\xi) = I\xi - A$ for some matrix $A$, the characteristic values are just the eigenvalues of $A$. The linear relations

$$\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} P(j-\ell)(\lambda_i)B_{ij} = 0, \quad i = 1, \ldots, N; \quad \ell = 0, \ldots, n_i-1.$$  

are readily derived from the identity:

$$\frac{d}{d\lambda} P\left(\frac{d}{dt}\right)Ae^{\lambda t} = P(\lambda)Ae^{\lambda t}$$
The polynomial \( \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k} \) is called the \textit{characteristic polynomial} of the autonomous behavior \( \mathcal{B} \). The roots of \( \det P(\xi), \lambda_1, \cdots, \lambda_N \), are called the characteristic values of the behavior. For the case that \( P(\xi) = I\xi - A \) for some matrix \( A \), the characteristic values are just the eigenvalues of \( A \). The linear relations

\[
\sum_{j=\ell}^{n_i-1} \left( \begin{array}{c} j \\ \ell \end{array} \right) P^{(j-\ell)}(\lambda_i) B_{ij} = 0, \quad i = 1, \ldots, N; \quad \ell = 0, \ldots, n_i-1.
\]

are readily derived from the identity:

\[
\frac{d}{d\lambda} P(\frac{d}{dt}) A e^{\lambda t} = \frac{d}{d\lambda} (P(\lambda) A e^{\lambda t})
\]
The polynomial \( \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k} \) is called the **characteristic polynomial** of the autonomous behavior \( \mathcal{B} \). The roots of \( \det P(\xi), \lambda_1, \ldots, \lambda_N \), are called the characteristic values of the behavior. For the case that \( P(\xi) = I\xi - A \) for some matrix \( A \), the characteristic values are just the eigenvalues of \( A \). The linear relations

\[
\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} P^{(j-\ell)}(\lambda_i) B_{ij} = 0, \quad i = 1, \ldots, N; \quad \ell = 0, \ldots, n_i-1.
\]

are readily derived from the identity:

\[
\frac{d}{d\lambda} P(\frac{d}{dt}) A e^{\lambda t} = \frac{d}{d\lambda} (P(\lambda) A e^{\lambda t})
\]
The polynomial $\prod_{k=1}^{N}(\xi - \lambda_k)^{n_k}$ is called the characteristic polynomial of the autonomous behavior $\mathcal{B}$. The roots of $\det P(\xi), \lambda_1, \cdots, \lambda_N$, are called the characteristic values of the behavior. For the case that $P(\xi) = l\xi - A$ for some matrix $A$, the characteristic values are just the eigenvalues of $A$. The linear relations

$$\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} P^{(j-\ell)}(\lambda_i)B_{ij} = 0, \quad i = 1, \ldots, N; \ \ell = 0, \ldots, n_i-1.$$  

are readily derived from the identity:

$$\frac{d}{d\lambda} P\left(\frac{d}{dt}\right) A e^{\lambda t} = \frac{d}{d\lambda} (P(\lambda) A e^{\lambda t})$$

$$= P'(\lambda) A e^{\lambda t} + P(\lambda) A te^{\lambda t}$$
The polynomial $\prod_{k=1}^{N}(\xi - \lambda_k)^{n_k}$ is called the \textit{characteristic polynomial} of the autonomous behavior $\mathcal{B}$. The roots of $\det P(\xi)$, $\lambda_1, \cdots, \lambda_N$, are called the characteristic values of the behavior. For the case that $P(\xi) = l\xi - A$ for some matrix $A$, the characteristic values are just the eigenvalues of $A$. The linear relations

$$\sum_{j=\ell}^{n_i-1} \binom{j}{\ell} P^{(j-\ell)}(\lambda_i)B_{ij} = 0, \quad i = 1, \ldots, N; \quad \ell = 0, \ldots, n_i-1.$$ 

are readily derived from the identity:

$$\frac{d}{d\lambda} P\left(\frac{d}{dt}\right)Ae^{\lambda t} = \frac{d}{d\lambda} (P(\lambda)Ae^{\lambda t})$$

$$= P'(\lambda)Ae^{\lambda t} + P(\lambda)Ate^{\lambda t}$$

and higher order analogues.
Definition
Let $\mathcal{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$). This partition is called an input/output partition if:

1. $w_1$ is free; i.e., for all $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in \mathcal{B}$
2. $w_2$ does not contain any further free components; i.e., given $w_1$, none of the components of $w_2$ can be chosen freely. Stated differently, $w_1$ is maximally free.

If these hold, then $w_1$ is called an input variable and $w_2$ is called an output variable.
Definition

Let $\mathcal{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$). This partition is called an input/output partition if:

1. $w_1$ is free; i.e., for all $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in \mathcal{B}$
Definition
Let $\mathcal{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$). This partition is called an input/output partition if:

1. $w_1$ is free; i.e., for all $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in \mathcal{B}$
2. $w_2$ does not contain any further free components; i.e., given $w_1$, none of the components of $w_2$ can be chosen freely. Stated differently, $w_1$ is maximally free.
Definition

Let $\mathcal{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$). This partition is called an input/output partition if:

1. $w_1$ is free; i.e., for all $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in \mathcal{B}$

2. $w_2$ does not contain any further free components; i.e., given $w_1$, none of the components of $w_2$ can be chosen freely. Stated differently, $w_1$ is maximally free.

If these hold, then $w_1$ is called an input variable and $w_2$ is called an output variable.
Definition
Let $B$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$). This partition is called a proper input/output partition if:

1. $w_1$ is properly free; i.e., for all $w_1 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in B$.
2. $w_2$ does not contain any further free components; i.e., given $w_1$, none of the components of $w_2$ can be chosen freely. Stated differently, $w_1$ is maximally free.

If these hold, then $w_1$ is called a proper input variable and $w_2$ is called an output variable.
Definition

Let $\mathcal{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$). This partition is called a proper input/output partition if:

1. $w_1$ is properly free; i.e., for all $w_1 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in \mathcal{B}$. 


Definition
Let $\mathcal{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \text{col}(w_1, w_2)$ ($w_1 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$). This partition is called a proper input/output partition if:

1. $w_1$ is properly free; i.e., for all $w_1 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$, there exists a $w_2 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$ such that $\text{col}(w_1, w_2) \in \mathcal{B}$.
2. $w_2$ does not contain any further free components; i.e., given $w_1$, none of the components of $w_2$ can be chosen freely. Stated differently, $w_1$ is maximally free.
Definition
Let \( \mathcal{B} \) be a behavior with signal space \( \mathbb{R}^q \). Partition the signal space as \( \mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p \) and \( w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \) correspondingly as \( w = \text{col}(w_1, w_2) \) (\( w_1 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \) and \( w_2 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^p) \)). This partition is called a proper input/output partition if:

1. \( w_1 \) is properly free; i.e., for all \( w_1 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \), there exists a \( w_2 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^p) \) such that \( \text{col}(w_1, w_2) \in \mathcal{B} \).

2. \( w_2 \) does not contain any further free components; i.e., given \( w_1 \), none of the components of \( w_2 \) can be chosen freely. Stated differently, \( w_1 \) is maximally free.

If these hold, then \( w_1 \) is called a proper input variable and \( w_2 \) is called an output variable.
Theorem

Let $\mathcal{B}$ be the behavior defined by

$$
\mathcal{B} = \{ w = \text{col}(u, y) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^p) \mid P \left( \frac{d}{dt} \right) y = Q \left( \frac{d}{dt} \right) u \},
$$

and let $u \in C^\infty(\mathbb{R}, \mathbb{R}^m)$. Let $P^{-1}(\xi)Q(\xi)$ be proper and its partial fraction expansion be given by

$$
P^{-1}(\xi)Q(\xi) = A_0 + \sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{A_{ij}}{(\xi - \lambda_i)^j}.
$$
Theorem

Let $\mathcal{B}$ be the behavior defined by

$$\mathcal{B} = \{ w = \text{col}(u, y) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^p) \mid P \frac{d}{dt} y = Q \frac{d}{dt} u \},$$

and let $u \in C^\infty(\mathbb{R}, \mathbb{R}^m)$. Let $P^{-1}(\xi)Q(\xi)$ be proper and its partial fraction expansion be given by

$$P^{-1}(\xi)Q(\xi) = A_0 + \sum_{i=1}^N \sum_{j=1}^{n_i} \frac{A_{ij}}{(\xi - \lambda_i)^j}.$$

Define $y$ by

$$y(t) := A_0 u(t) + \sum_{i=1}^N \sum_{j=1}^{n_i} A_{ij} \int_0^t \frac{(t - \tau)^{j-1}}{(j-1)!} e^{\lambda_i(t-\tau)} u(\tau) d\tau, \quad t \in \mathbb{R}.$$
Theorem

Let $\mathcal{B}$ be the behavior defined by

$$\mathcal{B} = \{ w = \text{col}(u, y) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^p) \mid P \frac{d}{dt} y = Q \frac{d}{dt} u \},$$

and let $u \in C^\infty(\mathbb{R}, \mathbb{R}^m)$. Let $P^{-1}(\xi)Q(\xi)$ be proper and its partial fraction expansion be given by

$$P^{-1}(\xi)Q(\xi) = A_0 + \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} \frac{1}{(\xi - \lambda_i)^j}.$$

Define $y$ by

$$y(t) := A_0 u(t) + \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} \int_{0}^{t} \frac{(t - \tau)^{j-1}}{(j-1)!} e^{\lambda_i(t-\tau)} u(\tau) d\tau, \quad t \in \mathbb{R}.$$
Corollary

Let $\mathcal{B}$ be as in the previous theorem. Then $(u, y)$ defines a proper input/output partition.
Corollary

Let $\mathcal{B}$ be as in the previous theorem. Then $(u, y)$ defines a proper input/output partition.

This motivates:

Definition

A dynamical system represented by $P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$ with $P(\xi) \in \mathbb{R}^{p \times p}[\xi]$ and $Q(\xi) \in \mathbb{R}^{p \times m}[\xi]$ is said to be in input/output form if it satisfies:

- $\det P(\xi) \neq 0$.
- $P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions. (By Cramer’s rule, the entries of $P^{-1}(\xi)Q(\xi)$ are rational functions.)
Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ be of full row rank. If $g < q$, then there exists a choice of columns of $R(\xi) : c_{i_1}(\xi), \ldots, c_{i_g}(\xi) \in \mathbb{R}^{g \times q}[\xi]$ such that
Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ be of full row rank. If $g < q$, then there exists a choice of columns of $R(\xi)$: $c_{i_1}(\xi), \ldots, c_{i_g}(\xi) \in \mathbb{R}^{g \times q}[\xi]$ such that

1. $\det \begin{bmatrix} c_{i_1}(\xi) & \cdots & c_{i_g}(\xi) \end{bmatrix} \neq 0$. 
Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ be of full row rank. If $g < q$, then there exists a choice of columns of $R(\xi) : c_{i_1}(\xi), \ldots, c_{i_g}(\xi) \in \mathbb{R}^{g \times q}[\xi]$ such that

1. $\det \begin{bmatrix} c_{i_1}(\xi) & \cdots & c_{i_g}(\xi) \end{bmatrix} \neq 0$.

2. $\begin{bmatrix} c_{i_1}(\xi) & \cdots & c_{i_g}(\xi) \end{bmatrix}^{-1} \begin{bmatrix} c_{i_{g+1}}(\xi) & \cdots & c_{i_q}(\xi) \end{bmatrix}$ is a proper rational matrix.
Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ be of full row rank. If $g < q$, then there exists a choice of columns of $R(\xi) : c_{i_1}(\xi), \ldots, c_{i_g}(\xi) \in \mathbb{R}^{g \times q}[\xi]$ such that

1. $\det \left[ \begin{array}{ccc} c_{i_1}(\xi) & \cdots & c_{i_g}(\xi) \end{array} \right] \neq 0$.

2. $\left[ \begin{array}{ccc} c_{i_1}(\xi) & \cdots & c_{i_g}(\xi) \end{array} \right]^{-1} \left[ \begin{array}{ccc} c_{i_{g+1}}(\xi) & \cdots & c_{i_q}(\xi) \end{array} \right]$ is a proper rational matrix.

Proof.

Choose $R_1(\xi)$ as a $g \times g$ nonsingular submatrix of $R(\xi)$ such that the degree of its determinant is maximal among the $g \times g$ submatrices of $R(\xi)$. Since $R(\xi)$ has full row rank, we know that $\det R_1(\xi)$ is not the zero polynomial. Denote the matrix formed by the remaining columns of $R(\xi)$ by $R_2(\xi)$. It follows from Cramer’s rule that $R_1^{-1}(\xi)R_2(\xi)$ is proper. \qed
Corollary

Let $R\left( \frac{d}{dt} \right)w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$, be a full row rank representation of the behavior $\mathcal{B}$. $\mathcal{B}$ admits an i/o representation with input $u = \text{col}(w_{i_{g+1}}, \ldots, w_{i_q})$ and output $y = \text{col}(w_{i_1}, \ldots, w_{i_g})$. 

Main Results Chapter 3

Consider

\[ \mathcal{B} = \{ w \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0 \} \quad R(\xi) \text{ full row rank} \]
Consider

$$\mathcal{B} = \{ w \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0 \} \quad R(\xi) \text{ full row rank}$$

The behavior $\mathcal{B}$ is autonomous if and only if $\det(R(\xi)) \neq 0$ (the zero polynomial).
Main Results Chapter 3

Consider

\[ \mathcal{B} = \{ w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R \left( \frac{d}{dt} \right) w = 0 \} \quad R(\xi) \text{ full row rank} \]

The behavior \( \mathcal{B} \) is autonomous if and only if \( \det(R(\xi)) \neq 0 \) (the zero polynomial).

The behavior \( \mathcal{B} \) can be written in input-output form otherwise.
Main Results Chapter 3

Consider

\[ \mathcal{B} = \{ w \in L_{1}^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R \left( \frac{d}{dt} \right) w = 0 \} \quad R(\xi) \text{ full row rank} \]

The behavior \( \mathcal{B} \) is autonomous if and only if \( \det(R(\xi)) \neq 0 \) (the zero polynomial).
The behavior \( \mathcal{B} \) can be written in input-output form otherwise.

In both case there is an explicit expression for the trajectories in \( \mathcal{B} \).
Models with Latent Variables

\[ R\left( \frac{d}{dt} \right) w = M\left( \frac{d}{dt} \right) \ell, \]

where \( w : \mathbb{R} \to \mathbb{R}^q \) denotes the manifest variable and \( \ell : \mathbb{R} \to \mathbb{R}^d \) the latent variable, and \( R(\xi) \in \mathbb{R}^{g \times q}[\xi] \) and \( M(\xi) \in \mathbb{R}^{g \times d}[\xi] \) are polynomial matrices with the same number of rows, namely \( g \), and with \( q \) columns. The full behavior \( \mathcal{B}_f \) and the manifest behavior \( \mathcal{B}_m \) are defined as

\[ \mathcal{B}_f = \{(w, \ell) \in \mid (w, \ell) \text{ satisfies the equations}\}, \]

\[ \mathcal{B}_m = \{w \mid \exists \ell \text{ such that } (w, \ell) \in \mathcal{B}_f\}. \]
Example

Consider the mass–spring system:

\[ (k_1 + k_2)q + M\left(\frac{d}{dt}\right)^2 q = F. \]
State Space Models

If we observe a trajectory $w = (F, q)$ up to $t = 0$ (the past), what can we then say about $(F, q)$ after $t = 0$ (the future)?

The future of $q$ depends, on the one hand, on the future of $F$, and on the other hand on the position and velocity at $t = 0$.

If $w_i = (q_i, F_i)$ are possible trajectories, then the trajectory $w = (q, F)$ that equals $w_1$ up to $t = 0$ and $w_2$ after $t = 0$, i.e., a trajectory that concatenates the past of $w_1$ and the future of $w_2$ at $t = 0$, is also an element of the behavior, if $q_1(0) = q_2(0)$ and $(\frac{d}{dt} q_1)(0) = (\frac{d}{dt} q_2)(0)$.

This observation, which still needs mathematical justification, inspires us to introduce the latent variable $x := \text{col}(q, \frac{d}{dt} q)$. 
Thus $x$ forms what we call the *state* of this mechanical system. Notice that we can rewrite the system equation in terms of $x$ as

$$
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 \\ -k_1 - k_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} F,
$$

$$
q = \begin{bmatrix} 1 & 0 \end{bmatrix} x,
$$

$$
w = \begin{bmatrix} q \\ F \end{bmatrix}.
$$

If $(w_1, x_1)$ and $(w_2, x_2)$ satisfy the equations, then $(w, x)$, the concatenation of $(w_1, x_1)$ and $(w_2, x_2)$ at $t = 0$, also satisfies these equations provided $x_1(0) = x_2(0)$.

We call $x$ the *state*.

The equation is first order in the latent variable $x$ and order zero (static) in the manifest variables $q$ and $F$. 
Definition (Property of state)

Consider the latent variable system. Let \((w_1, \ell_1), (w_2, \ell_2) \in \mathcal{B}_f\) and \(t_0 \in \mathbb{R}\) and suppose that \(\ell_1, \ell_2\) are continuous. Define the concatenation of \((w_1, \ell_1)\) and \((w_2, \ell_2)\) at \(t_0\) by \((w, \ell)\), with

\[
  w(t) = \begin{cases} 
    w_1(t) & t < t_0, \\
    w_2(t) & t \geq t_0,
  \end{cases} \quad \text{and} \quad \ell(t) = \begin{cases} 
    \ell_1(t) & t < t_0, \\
    \ell_2(t) & t \geq t_0.
  \end{cases}
\]

Then \(\mathcal{B}_f\) is said to be a state space model, and the latent variable \(\ell\) is called the state if \(\ell_1(t_0) = \ell_2(t_0)\) implies \((w, \ell) \in \mathcal{B}_f\).
General form of State Space Models

\[ E \frac{dx}{dt} + Fx + Gw = 0 \]
General form of State Space Models

\[
E \frac{dx}{dt} + Fx + Gw = 0
\]

Special case input-state-output

\[
\frac{d}{dt} x = Ax + Bu, \\
y = Cx + Du.
\]
General form of State Space Models

\[ E \frac{dx}{dt} + Fx + Gw = 0 \]

Special case input-state-output

\[
\begin{align*}
\frac{d}{dt} x &= Ax + Bu, \\
y &= Cx + Du.
\end{align*}
\]

Here \( u \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \) is the input, \( x \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \) is the state, and \( y \in L_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^p) \) is the output.
General form of State Space Models

$$E \frac{dx}{dt} + Fx + Gw = 0$$

Special case input-state-output

$$\frac{d}{dt} x = Ax + Bu,$$
$$y = Cx + Du.$$ 

Here \( u \in L^1_{loc}(\mathbb{R}, \mathbb{R}^m) \) is the input, \( x \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n) \) is the state, and \( y \in L^1_{loc}(\mathbb{R}, \mathbb{R}^p) \) is the output. Consequently, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \). The matrix \( D \) is called the \textit{feedthrough} term.
In polynomial form, the equations can be written as
\[ R\left( \frac{d}{dt} \right) w = M\left( \frac{d}{dt} \right) x, \text{ with } w = \text{col}(u, y) \text{ and } R(\xi), M(\xi) \text{ given by} \]

\[
R(\xi) := \begin{bmatrix} B & 0 \\ -D & I \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} l_\xi - A \\ C \end{bmatrix}.
\]
Note that the “dynamics”, the part of the equations that contains derivatives, of this representation is completely contained in the vector \((u, x)\) and that moreover, only first-order derivatives occur.
Note that the “dynamics”, the part of the equations that contains derivatives, of this representation is completely contained in the vector \((u, x)\) and that moreover, only first-order derivatives occur.

The equation determining \(y\) from \(x\) and \(u\) is static: it does not contain derivatives.
Note that the “dynamics”, the part of the equations that contains derivatives, of this representation is completely contained in the vector \((u, x)\) and that moreover, only first-order derivatives occur.

The equation determining \(y\) from \(x\) and \(u\) is static: it does not contain derivatives.

The relation between \(u\) and \(x\) is of a proper i/o nature.
Note that the “dynamics”, the part of the equations that contains derivatives, of this representation is completely contained in the vector \((u, x)\) and that moreover, only first-order derivatives occur.

The equation determining \(y\) from \(x\) and \(u\) is static: it does not contain derivatives.

The relation between \(u\) and \(x\) is of a proper i/o nature.

To see this, write \(\frac{d}{dt}x = Ax + Bu\) as \((\frac{d}{dt} I - A)x = Bu\). Since \(\det(I\xi - A) \neq 0\) and since \((I\xi - A)^{-1}B\) is strictly proper (why?), \(u\) is a maximally free variable in \(L_{1}\text{loc}(\mathbb{R}, \mathbb{R}^m)\).
Note that the “dynamics”, the part of the equations that contains derivatives, of this representation is completely contained in the vector \((u, x)\) and that moreover, only first-order derivatives occur. The equation determining \(y\) from \(x\) and \(u\) is static: it does not contain derivatives.

The relation between \(u\) and \(x\) is of a proper i/o nature. To see this, write \(\frac{d}{dt} x = Ax + Bu\) as \((\frac{d}{dt} I - A)x = Bu\). Since \(\det(I\xi - A) \neq 0\) and since \((I\xi - A)^{-1} B\) is strictly proper (why?), \(u\) is a maximally free variable in \(L^1_{loc}(\mathbb{R}, \mathbb{R}^m)\).

The full behavior:

\[
\mathcal{B}_{i/s/o} := \{(u, x, y) \in L^1_{loc}(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p) \mid \text{eq. satisfied weakly}\}
\]

The \textit{manifest} behavior is given by

\[
\mathcal{B}_{i/o} := \{(u, y) \in L^1_{loc}(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^p) \mid \exists x \text{ s.t. } (u, x, y) \in \mathcal{B}_{i/s/o}\}.
\]
**Theorem**

Let \( u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \), and define \( x \) by

\[
x(t) := \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau.
\]

Then \((u, x) \in \mathcal{B}_{\text{i/s}}\).

**Corollary**

Every element \((u, x)\) of \(\mathcal{B}_{\text{i/s}}\) is of the form

\[
x(t) = e^{At} c + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau, \quad c \in \mathbb{R}^n.
\]
State Space Transformations

Definition

Two i/s/o representations are called *input/output equivalent* if they represent the same input/output behavior.

The i/s/o representations of a given i/o system are not unique. Indeed, consider

\[
\begin{align*}
\frac{d}{dt} x &= Ax + Bu, \\
y &= Cx + Du.
\end{align*}
\]

Let \( S \in \mathbb{R}^{n \times n} \) be a nonsingular matrix and let \((u, x, y)\) satisfy the state equations Define \( \tilde{x} := Sx. \)
Note that this corresponds to expressing the state coordinates with respect to a new basis. The differential equation governing $\tilde{x}$ is

$$\frac{d}{dt} \tilde{x} = SAS^{-1} \tilde{x} + SBu,$$

$$y = CS^{-1} \tilde{x} + Du.$$

The equations show that every $(u, y)$ that belongs to the first i/o behavior also belongs to the second i/o and vice versa. This means that both state space representations represent the same i/o behavior, and therefore they are input/output equivalent.
Theorem

Two state space representations parametrized by 
\((A_1, B_1, C_1, D_1)\) and \((A_2, B_2, C_2, D_2)\) respectively, are 
input/output equivalent if there exists a nonsingular matrix 
\(S \in \mathbb{R}^{n \times n}\) such that 
\[SA_1 S^{-1} = A_2, \quad SB_1 = B_2, \quad C_1 S^{-1} = C_2, \quad D_1 = D_2.\]
Correspondingly, we call two quadruples 
\((A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2)\) \(\in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}\) 
equivalent, or similar, if there exists a nonsingular matrix 
\(S \in \mathbb{R}^{n \times n}\) such that 
\[SA_1 S^{-1} = A_2, \quad SB_1 = B_2, \quad C_1 S^{-1} = C_2, \quad D_1 = D_2.\]
The matrix \(S\) is called the corresponding state similarity transformation matrix.

Remark

Note that this theorem shows that similarity implies the same i/o behavior. However, if two representations are i/o equivalent, then the corresponding quadruples of system matrices need not be similar.
Important Issues

1. Property of State
2. General State Space Representation
3. State Space Representation Problem
4. State Elimination Problem