Controllability and Observability

Elimination of latent variables & state space representations

Jacob van der Woude, TU Delft
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Chapter 5: Controllability & observability

- Model or system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B})$ with time axis $\mathbb{R}$, signal space $\mathbb{R}^q$ and set of trajectories (the behavior)

$$\mathcal{B} = \{ w : \mathbb{R} \to \mathbb{R}^q | R(\frac{d}{dt})w = 0 \},$$

where $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$, ie., $R(\xi)$ is a $g \times q$ polynomial matrix in $\xi$.

- $\Sigma$ is **controllable** if for any two trajectories $w_1, w_2 \in \mathcal{B}$ there exists a time $t_1 \geq 0$ and a trajectory $w \in \mathcal{B}$, such that

$$w(t) = \begin{cases} w_1(t) & t \leq 0 \\ w_2(t - t_1) & t \geq t_1 \end{cases}$$

- Controllable behavior: there is a trajectory from any past behavior to any future behavior. Opposite to autonomous behavior. There the past behavior completely determines the future behavior.
Examples

- $R(\xi)$ is zero matrix. Then $B = \{ w | w : \mathbb{R} \to \mathbb{R}^q \}$. Hence, $B$ contains all possible trajectories, and therefore is controllable trivially.

- $R(\xi)$ is identity matrix. Then $B = \{ w | w(t) = 0, \forall t \}$. Hence, $B$ only contains zero trajectory. However, the requirements for controllability are satisfied, check this.

- $R(\xi) = \xi - 1$. Then $B = \{ w : \mathbb{R} \to \mathbb{R}^q | w(t) = ce^t \}$. Take $w_1, w_2 \in B$, then $w_i = c_i e^t$ for some $c_i \in \mathbb{R}$, $i = 1, 2$. If $c_1 \neq c_2$, then for no $t_1 \geq 0$ there is a $c \in \mathbb{R}$ such that $c_1 e^t = ce^t$ for $t \leq 0$, and $c_2 e^{t - t_1} = ce^t$ for $t \geq t_1$, check this. Hence, not controllable.

- If $\det R(\xi)$ is nonzero constant. Then $R(\xi)$ is unimodular and $B = \{ w | w(t) = 0, \forall t \}$. Hence, controllable behavior.

- If $\det R(\xi)$ is nonzero polynomial. Then $B$ is solution set of $R(\frac{d}{dt})w = 0$. Each solution fixed by initial conditions. Hence, behavior not controllable.
Theorems and lemmas

**Lemma** If $R(\xi)$ is $q \times q$ and $\det R(\xi) \neq 0$: $\Sigma$ controllable $\iff R(\xi)$ unimodular $\iff \det R(\xi) = c \neq 0$ with $c$ constant $\iff \text{rank } R(\lambda) = q$ for all $\lambda \in \mathbb{C}$.

**Theorem** General $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$: $\Sigma$ controllable $\iff \text{rank } R(\lambda)$ is the same for all $\lambda \in \mathbb{C}$.

The proof can be given using the Smith form of $R(\xi)$. From the proof it also follows that the time $t_1 \geq 0$ can be taken arbitrarily small, but positive.

**Special case.** Consider $p(\xi), q(\xi) \in \mathbb{R}[\xi]$ with $\deg p(\xi) \geq \deg q(\xi)$ and

$$R(\xi) = \begin{pmatrix} p(\xi) & -q(\xi) \end{pmatrix}, \quad w(t) = \begin{pmatrix} y(t) \\ u(t) \end{pmatrix},$$

then $\Sigma$ is controllable $\iff p(\xi)$ and $q(\xi)$ are coprime, i.e., have no common zero.
Theorems and lemmas

Take $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ with rank $R(\xi) = g$ (meaning $R(\xi)$ has full column rank), i.e., rank $R(\lambda) = g$ for almost all $\lambda \in \mathbb{C}$.

- rank $R(\lambda) = g$ for all $\lambda \in \mathbb{C} \iff \Sigma$ controllable
  $\left( \mathcal{B} = \mathcal{B}_{\text{contr}} = \{ w : \mathbb{R} \to \mathbb{R}^q | R \left( \frac{d}{dt} \right) w = 0 \} \right)$.

- More general, if $\mathcal{B} = \{ w : \mathbb{R} \to \mathbb{R}^q | R \left( \frac{d}{dt} \right) w = 0 \}$, then $\mathcal{B} = \mathcal{B}_{\text{contr}} \oplus \mathcal{B}_{\text{aut}}$, where
  - $\mathcal{B}_{\text{contr}}$ is the controllable part of $\mathcal{B}$.
  - $\mathcal{B}_{\text{aut}}$ is the autonomous part of $\mathcal{B}$, where the characteristic values in the solutions correspond to those numbers $\lambda \in \mathbb{C}$ for which the rank of $R(\lambda)$ drops.

- Example $R(\xi) = [p(\xi), q(\xi)] = [1 - \xi - \xi^2 + \xi^3, 2 - \xi - \xi^2]$. Note $p(\xi) = (1 - \xi)(1 - \xi^2)$ and $q(\xi) = (1 - \xi)(2 + \xi)$. Hence, $\Sigma$ not controllable. Controllable part described by $\tilde{R}(\xi) = [\tilde{p}(\xi), \tilde{q}(\xi)] = [1 - \xi^2, 2 + \xi]$. The autonomous part is more involved, has characteristic value at $\lambda = 1$. 
Consider
\[ \Sigma : \quad \frac{d}{dt}x = Ax + Bu, \]
with \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}. \)

\( \Sigma \) is state controllable if for any \( x_0, x_1 \in \mathbb{R}^n \), there is a time \( \tau > 0 \) and control \( u(t) \) for \( t \in [0, \tau) \) such that by the control \( u \) the state \( x \) goes from \( x(0) = x_0 \) to \( x(\tau) = x_1 \).

We have the well-known Kalman rank condition (1963):
\( \Sigma \) is state controllable \( \iff \) rank \( R = n \), where \( R \) is defined by \( R = [B, AB, A^2B, \ldots, A^{n-1}B] \), the so-called controllability matrix.

A well-known alternative condition by Hautus (1969):
\( \Sigma \) is state controllable \( \iff \) rank \( [A - \lambda I_n, B] = n \) for all \( \lambda \in \mathbb{C} \).

\( \Sigma \) can be seen as \( R\left(\frac{d}{dt}\right)w = 0 \), with \( w(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \) and \( R(\xi) = [A - \xi I_n, B] \).

Hence, controllability equivalent to state controllability.
Invariance properties

- The subspace \( \text{im} R \), i.e., the so-called controllable subspace given by \( \text{im} [B, AB, A^2B, \ldots, A^{n-1}B] \), has the following properties
  - it is \( A \)-invariant, \( (v \in \text{im} R \implies Av \in \text{im} R) \),
  - contains \( \text{im} B \),
  - is the smallest \( A \)-invariant subspace that contains \( \text{im} B \).

- Based on \( \text{im} R \) there is a nonsingular matrix \( S \) such that

\[
S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad S^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},
\]

and \((A_{11}, B_1)\) is controllable.

- Indeed, take a basis for \( \text{im} R \) and extend this basis to a basis for \( \mathbb{R}^n \). Collect the obtained vectors in the matrix \( S \).
Stabilizability

Let $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathcal{B})$ with $\mathcal{B} = \{ w : \mathbb{R} \rightarrow \mathbb{R}^q | R(\frac{d}{dt})w = 0 \}$. $\Sigma$ is called **stabilizable** if for any trajectory $w \in \mathcal{B}$ there exists a trajectory $w' \in \mathcal{B}$, such that

$$w'(t) = w(t) \text{ for } t \leq 0 \text{ and } \lim_{t \rightarrow \infty} w'(t) = 0.$$ 

Stabilizable behavior: there is a trajectory from any past behavior to the (future) zero behavior.

**Theorem** General $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$: $\Sigma$ stabilizable $\iff$ rank $R(\lambda)$ is the same for all $\lambda \in \mathbb{C}_+ (= \{ s \in \mathbb{C} | \Re s \geq 0 \})$.

State space version. When $R(\xi) = [A - \xi I_n, B]$, then state stabilizability, i.e., for any $x(0) = x_0$ there is a control $u(t), t \geq 0$, such that $\lim_{t \rightarrow \infty} x(t) = 0$, is equivalent to rank $R(\lambda) = n$ for all $\lambda \in \mathbb{C}_+$. 

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Observability

Let \((R, R^q, B)\) be a time-invariant dynamical system.

Assume that trajectories in \(B\) are partitioned as \((w_1, w_2)\) with \(w_i \in \mathbb{R}^{q_i}, i = 1, 2, \) and \(q_1 + q_2 = q.\)

We say that \(w_2\) is **observable** from \(w_1\) if for all trajectories of the form \((w_1, w_2), (w_1, w'_2) \in B\), it follows that \(w_2 = w'_2.\) Hence, \(w_1\) completely determines \(w_2.\)

**Theorem** Let \(R_1(\xi) \in \mathbb{R}^{g \times q_1}[\xi]\) and \(R_2(\xi) \in \mathbb{R}^{g \times q_2}[\xi].\)

Let \(B\) be the behavior defined by

\[
R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2.
\]

Then the variable \(w_2\) is observable from \(w_1\) if and only if

\[
\text{rank } R_2(\lambda) = q_2
\]

for all \(\lambda \in \mathbb{C}.\)
Observability of latent variable models

- Interesting case: observability of latent variable descriptions.
- Consider

\[ R\left( \frac{d}{dt} \right) w = M\left( \frac{d}{dt} \right) \ell, \]

with \( w \) the observed (manifest) variable, and \( \ell \) the variable to-be-observed (latent variable).

- Then the variable \( \ell \) is observable from \( w \) if and only if \( \text{rank } M(\lambda) \) has the same value for all \( \lambda \in \mathbb{C} \).

- Application: Consider \( p(\xi), q(\xi) \in \mathbb{R}[\xi] \) with \( \text{deg } p(\xi) \geq 1 \) then in

\[ p\left( \frac{d}{dt} \right) y = q\left( \frac{d}{dt} \right) u, \]

\( y \) is never observable from \( u \).
Observability of state space models

- Consider the $i/s/o$ system

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Denote the associated behavior by $\mathcal{B}_{i/s/o}$.

- The state $x$ is observable from $(u, y)$ if and only if for all $\lambda \in \mathbb{C}$

$$\text{rank} \left( \begin{pmatrix} \lambda I_n - A \\ C \end{pmatrix} \right) = n.$$ 

- Define the observability matrix

$$W = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$
Theorem  The following statements are equivalent.

- The state is observable from \((u, y)\).
- For all \(\lambda \in \mathbb{C}\)
  \[
  \text{rank } \begin{pmatrix} \lambda I_n - A \\ C \end{pmatrix} = n.
  \]
- \(\text{rank } W = n\).
- The input/output trajectory determines the state uniquely: if \(t_1 > 0\) and \((u_1, y_1), (u_2, y_2)\) satisfy the system equations such that for \(t \in [0, t_1)\) : \((u_1(t), y_1(t)) = (u_2(t), y_2(t))\), then also \(x_1(t) = x_2(t)\) for all \(t \in [0, t_1]\).
Invariance properties

With

\[ W = \begin{pmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{pmatrix}. \]

the subspace \( \ker W \), i.e., the so-called unobservable subspace, has the following properties

- is \( A \)-invariant, \( (v \in \text{im } R \implies Av \in \text{im } R) \),
- is contained in \( \ker C \),
- is the largest \( A \)-invariant subspace that is contained \( \ker C \).

Based on \( \ker W \), there is a nonsingular matrix \( S \) such that

\[ S^{-1}AS = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}, \quad CS = \begin{bmatrix}
0 & C_2
\end{bmatrix}, \]

and \( (C_2, A_{22}) \) is observable. Indeed, take a basis of \( \ker W \) and extend it to a basis of \( \mathbb{R}^n \). Make \( S \) from the obtained basis.
Detectability

Let \((\mathbb{R}, \mathbb{R}^q, \mathcal{B})\) be a time-invariant dynamical system. Assume that trajectories in \(\mathcal{B}\) are partitioned as \((w_1, w_2)\) with \(w_i \in \mathbb{R}^{q_i}, i = 1, 2,\) and \(q_1 + q_2 = q.\)

We say that \(w_2\) is detectable from \(w_1\) if for all trajectories of the form \((w_1, w_2), (w_1, w'_2) \in \mathcal{B},\) it follows that
\[
\lim_{t \to \infty} w_2(t) - w'_2(t) = 0.
\]
Hence, \(w_1\) determines the asymptotic behavior of \(w_2.\)

Let \(R_1(\xi) \in \mathbb{R}^{g \times q_1}[\xi]\) and \(R_2(\xi) \in \mathbb{R}^{g \times q_2}[\xi]\) and let \(\mathcal{B}\) be the behavior defined by
\[
R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2.
\]
Detectability of state space models

- Then the variable $w_2$ is detectable from $w_1$ if and only if
  \[ \text{rank} \ R_2(\lambda) = q_2 \]
  for all $\lambda \in \mathbb{C}_+$.  
- Detectability (of $x$ from $(u, y)$) for the i/s/o system
  \[
  \dot{x} = Ax + Bu, \quad y = Cx + Du
  \]
is equivalent to the condition that
  \[ \text{rank} \ \begin{pmatrix} \lambda I_n - A \\ C \end{pmatrix} = n \]
  for all $\lambda \in \mathbb{C}_+$.  

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Kalman decomposition

Consider the $i/s/o$ system

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du. \]

Based on $\text{im } R$, the controllable subspace, and $\ker W$, the unobservable subspace, there is a nonsingular matrix $S$ such that (see Figure 5.3)

\[
S^{-1}AS = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
0 & A_{22} & A_{23} & A_{24} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix}, \quad S^{-1}B = \begin{bmatrix}
B_1 \\
B_2 \\
0 \\
0
\end{bmatrix},
\]

\[
CS = \begin{bmatrix}
0 & C_2 & 0 & C_4
\end{bmatrix},
\]

with \((\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix})\) controllable and

\((\begin{bmatrix} C_2 & C_4 \\ A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix})\) observable.
Polynomial tests

Consider the SISO system \( \frac{d}{dt} x = Ax + bu, \ y = cx. \)

- Define \( p(\xi) \in \mathbb{R}[\xi] \) and the row \( r(\xi) \in \mathbb{R}^{1 \times n}[\xi] \) by

\[
p(\xi) = \det(\xi I_n - A), \quad r(\xi) = p(\xi) c(\xi I_n - A)^{-1},
\]

and write \( r(\xi) = (r_1(\xi), r_2(\xi), \cdots, r_n(\xi)). \)

Then, \((c, A)\) is observable if and only if the polynomials \( p(\xi), r_1(\xi), r_2(\xi), \cdots, r_n(\xi) \) have no common roots.

- Define the column \( s(\xi) \in \mathbb{R}^{n \times 1}[\xi] \) by

\[
s(\xi) = (\xi I_n - A)^{-1} bp(\xi)
\]

and write \( s(\xi) = (s_1(\xi), s_2(\xi), \cdots, s_n(\xi))^\top. \)

Then, \((A, b)\) is controllable if and only if the polynomials \( p(\xi), s_1(\xi), s_2(\xi), \cdots, s_n(\xi) \) have no common roots.
Chapter 6: Elimination and state space

We can consider various types of behavior.

- **Full behavior**

  \[ \mathcal{B}_f = \{(w, l) \in C^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^d) | R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell\} \]

- **Manifest behavior**

  \[ \mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) | \exists \ell \in C^\infty(\mathbb{R}, \mathbb{R}^d) : (w, \ell) \in \mathcal{B}_f\} \]

How to go from one to the other?

The answer lies in elimination of latent variables.

- Hence, we consider

  \[ \mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) | \exists \ell \in C^\infty(\mathbb{R}, \mathbb{R}^d) : R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell\} \]

  versus

  \[ \mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) | R'\left(\frac{d}{dt}\right)w = 0\}. \]
Examples and transformations

Examples

- Three masses attached to walls (figure 6.1).
- Electrical circuit (figure 6.2).

Eliminate latent variables to get description in terms of manifest variables only.

Useful transformations

- Consider two behaviors $B_i, i = 1, 2$, described by

  $$B_i = \{ w | R_i \left( \frac{d}{dt} \right) w = 0 \}, i = 1, 2,$$

- Then $B_1 \subseteq B_2$ if and only if there is a polynomial matrix $V(\xi)$ such that $V(\xi)R_1(\xi) = R_2(\xi)$.
- When $R_1(\xi)$ and $R_2(\xi)$ have the same number of rows. Then $B_1 = B_2$ if and only if there is a unimodular matrix $V(\xi)$ such that $V(\xi)R_1(\xi) = R_2(\xi)$.
Elimination procedure

- Consider a latent variable model $R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell$.
- Do row compression of $M(\xi)$ by means of a unimodular matrix $U(\xi)$:

  $$U(\xi)R(\xi) = \begin{bmatrix} R'(\xi) \\ R''(\xi) \end{bmatrix}, \quad U(\xi)M(\xi) = \begin{bmatrix} 0 \\ M'''(\xi) \end{bmatrix},$$

  with $M'''(\xi)$ full row rank.
- Then the latent variable model also described by

  $$R'(\frac{d}{dt})w = 0, \quad R''(\frac{d}{dt})w = M''(\frac{d}{dt})\ell.$$

- By the full rank condition on $M'''(\xi)$ the latent variable $\ell$ can be shown to depend on $w$ and to not restrict $w$. Hence, the evolution of $w$ is described by

  $$R'(\frac{d}{dt})w = 0.$$
Example

Consider \( R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \) with

\[
R(\xi) = \begin{bmatrix} 2 + \xi^2 & 0 \\ 1 & 1 \\ 0 & 2 + \xi^2 \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} 1 \\ 2 + \xi^2 \\ 1 \end{bmatrix}.
\]

Then

\[
U(\xi)R(\xi) = \begin{bmatrix} 2 + \xi^2 & 0 \\ \frac{3 + 4\xi^2 + \xi^4}{2 + \xi^2} & -1 \\ -2 - \xi^2 & 2 + \xi^2 \end{bmatrix}, \quad U(\xi)M(\xi) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

with the unimodular matrix \( U(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 2 + \xi^2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \).

So, \( R'(\frac{d}{dt})w = 0 \), with \( R'(\xi) = \begin{bmatrix} 3 + 4\xi^2 + \xi^4 & -1 \\ -2 - \xi^2 & 2 + \xi^2 \end{bmatrix} \).
Consider two scalar i/o systems

\begin{align*}
p_1\left(\frac{d}{dt}\right)y_1 &= q_1\left(\frac{d}{dt}\right)u_1, \\
p_2\left(\frac{d}{dt}\right)y_2 &= q_2\left(\frac{d}{dt}\right)u_2.
\end{align*}

Let the systems be series interconnected, i.e., \( y_1 = u_2 \).

A latent variable model is given by \( R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell \) with

\[
R(\xi) = \begin{bmatrix}
q_1(\xi) & 0 \\
0 & p_2(\xi)
\end{bmatrix},
M(\xi) = \begin{bmatrix}
p_1(\xi) & 0 \\
0 & -1 & q_2(\xi)
\end{bmatrix},
\]

and

\[
w = \begin{bmatrix}
u_1 \\
y_2
\end{bmatrix},
\ell = \begin{bmatrix}
y_1 \\
u_2
\end{bmatrix}.
\]

Elimination of \( \ell \) yields a description in terms of \( u_1 \) and \( y_2 \).

The 'interconnecting' variables \( y_1 \) and \( u_2 \) are eliminated.
Elimination state variables

- Consider an i/s/o system $\frac{d}{dt}x = Ax + bu$, $y = cx$.
- A latent variable model is given by $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ with

$$R(\xi) = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} \xi I_n - A \\ C \end{bmatrix},$$

and

$$w = \begin{bmatrix} u \\ y \end{bmatrix}, \quad \ell = x.$$

- Elimination of $\ell$, the state $x$, yields a description in terms of $u$ and $y$.
- In the resulting description the unobservable mode is eliminated, see book for details. See also next slide for the theorem.
Elimination state variables

**Theorem**
Consider the i/s/o system $\frac{dx}{dt} = Ax + bu$, $y = cx$.
Define the scalar $\bar{p}(\xi) \in \mathbb{R}[\xi]$ and the row $\bar{r}(\xi) \in \mathbb{R}^{1 \times n}[\xi]$ by

$$\bar{p}(\xi) = \det(\xi I_n - A), \quad \bar{r}(\xi) = \bar{p}(\xi)c(\xi I_n - A)^{-1}$$

Write $\bar{r}(\xi) = (\bar{r}_1(\xi), \bar{r}_2(\xi), \cdots, \bar{r}_n(\xi))$ and let $g(\xi)$ be the greatest common divisor of $\bar{p}(\xi), \bar{r}_1(\xi), \bar{r}_2(\xi), \cdots, \bar{r}_n(\xi)$. Define the polynomials

$$p(\xi) = \frac{\bar{p}(\xi)}{g(\xi)}, \quad q(\xi) = \frac{r(\xi)}{g(\xi)}b$$

Then the i/o behavior of the i/s/o representation is given by

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u.$$

Clearly, $(p(\xi), r(\xi))$ are coprime, but $(p(\xi), q(\xi))$ may be not.
Example

Consider

\[ A = \begin{bmatrix} 2 & 4 & -5 \\ -1 & -3 & 15 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}. \]

Then it follows after some computation that

\[ \bar{p}(\xi) = 6 - 5\xi - 2\xi^2 + \xi^3 = (\xi - 3)(\xi + 2)(\xi - 1) \]
\[ \bar{r}(\xi) = [3 - \xi, 6 - 5\xi + \xi^2, -21 + 13\xi - 2\xi^2] \]
\[ = (\xi - 3)[-1, -2 + \xi, 7 - 2\xi]. \]

Hence, \( g(\xi) = \xi - 3, \ p(\xi) = -2 + \xi + \xi^2 \) and \( q(\xi) = 1 - \xi \). So that

\[ -2y + \frac{d}{dt}y + \frac{d^2}{dt^2}y = u - \frac{d}{dt}u. \]

Not controllable because \((p(\xi), q(\xi))\) not coprime!
Realization problem

Consider the scalar i/o system \( p(\frac{d}{dt})y = q(\frac{d}{dt})u \).

Question Does there exist an i/s/o representation of the given i/o system? And if so, how can it be obtained?

Specifically, find natural number \( n \), and real matrices \( A, b, c, d \) of dimensions \( n \times n, n \times 1, 1 \times n, 1 \times 1 \), respectively, such that the given i/o system is also described by

\[
\frac{d}{dt}x = Ax + bu, \ y = cx + du.
\]

for some state (latent variable) \( x \in \mathbb{R}^n \),

The problem is called the realization problem.

A lot of literature is available.

Can also be solved in multi-input/multi-output cases.
Realization problem (special case, \(d=0\))

Consider the scalar i/o system \(p(\frac{d}{dt})y = q(\frac{d}{dt})u\). Assume that

\[
p(\xi) = \xi^n + p_{n-1}\xi^{n-1} + \cdots + p_1\xi + p_0
\]
\[
q(\xi) = q_{n-1}\xi^{n-1} + \cdots + q_1\xi + q_0
\]

Then a realization is given by the following matrices

\[
A = \begin{bmatrix}
0 & \cdots & \cdots & 0 & -p_0 \\
1 & \vdots & \vdots & \vdots & \vdots \\
0 & \ddots & \ddots & \vdots & \vdots \\
& \ddots & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & -p_{n-1}
\end{bmatrix}, \\
b = \begin{bmatrix}
q_0 \\
q_1 \\
\vdots \\
\vdots \\
q_{n-1}
\end{bmatrix},
\]
\[
c = \begin{bmatrix}
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}, \\
d = 0.
\]

The obtained realization is **observer canonical** form. It is observable by nature. It is controllable if \((p(\xi), q(\xi))\) are coprime.
Realization problem (special case, d=0)

Consider the scalar i/o system \( p(\frac{d}{dt})y = q(\frac{d}{dt})u \). Assume that

\[
p(\xi) = \xi^n + p_{n-1}\xi^{n-1} + \cdots + p_1\xi + p_0
\]

\[
q(\xi) = q_{n-1}\xi^{n-1} + \cdots + q_1\xi + q_0
\]

Then another realization is given by the following matrices

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \vdots & \vdots & 0 & 1 \\
-p_0 & -p_1 & \cdots & \cdots & -p_{n-1}
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
\vdots \\
0 \\
1
\end{bmatrix},
\]

\[
c = \begin{bmatrix}
q_0 & q_1 & \cdots & \cdots & q_{n-1}
\end{bmatrix},
\]

\( d = 0 \).

The obtained realization is controller canonical form. It is controllable by nature. If \((p(\xi), q(\xi))\) are coprime, then the i/o behavior of the realization is described by \( p(\frac{d}{dt})y = q(\frac{d}{dt})u \).
Representations

Several representations have been treated so far. All systems can be described in the form of:

- \( R'(\frac{d}{dt})w = 0 \), so-called **kernel** representation.
- \( R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \), called latent variable representation.
- \( P(\frac{d}{dt})y = Q(\frac{d}{dt})u \), where \( P^{-1}(\xi)Q(\xi) \) is proper and \( w \) is an appropriate partition of the components of \( w \), it is called an input-output representation.
- \( \frac{d}{dt}x = Ax + bu, y =Cx + Du \), an i/s/o representation, with \( x \) the state, and \( u, y \) appropriate partitions of the components of \( w \).
- \( E \frac{d}{dt}x + Fx + Gw = 0 \), generalized state space representation, with latent variable \( x \).

Are there more representations?
The following representation is called an image representation:

\[ w = M \left( \frac{d}{dt} \right) \ell, \]

with \( \ell \) a latent variable and \( w \) the manifest variable.

**Theorem** Consider a behavior by a kernel representation. It can be described as an image representation if and only if the behavior is controllable.