Systems and Control

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Behaviors

We consider the behavior $\mathcal{B}$ of all $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ functions w satisfying

$$R(\frac{d}{dt})w = 0$$

with $R \in \mathbb{R}^{q \times q}(s)$.

The mathematical model is then the triple $(\mathbb{R}, \mathbb{R}^q, \mathcal{B})$.

The system is called controllable if for any two trajectories $w_1, w_2 \in \mathcal{B}$ there exists a $t_1 \geqslant 0$ and a trajectory $w \in \mathcal{B}$ with the property:

$$w(t) = \begin{cases} w_1(t) & t \leqslant 0 \\ w_2(t - t_1) & t \geqslant t_1 \end{cases}$$

A behavior described by:

$$R(\frac{d}{dt})w = 0$$

is controllable if and only if rank $R(\lambda)$ is the same for all $\lambda \in \mathbb{C}$.

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ be of full row rank and let $\mathcal{B}$ be the behavior defined by

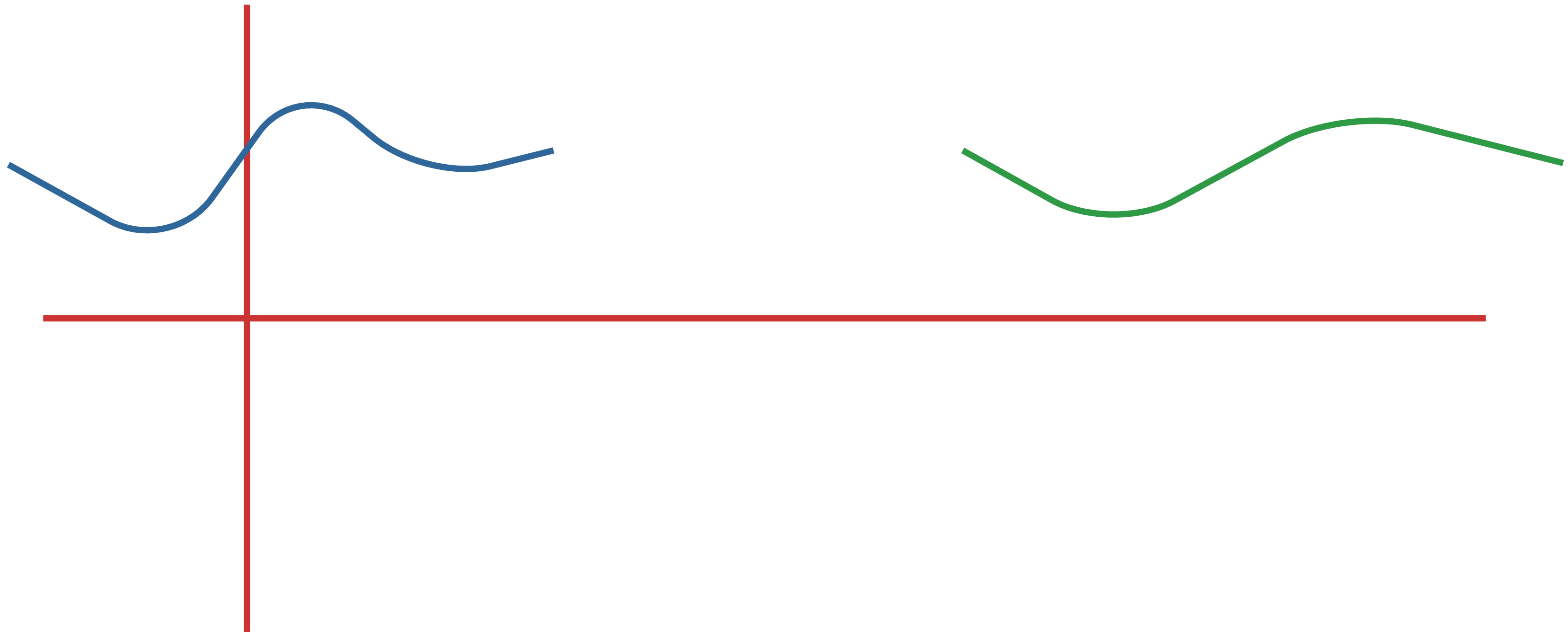
$$R(\frac{d}{dt})w = 0$$

Then there exists subbehaviors $\mathcal{B}_{\text{aut}}$ and $\mathcal{B}_{\text{contr}}$ of $\mathcal{B}$ such that

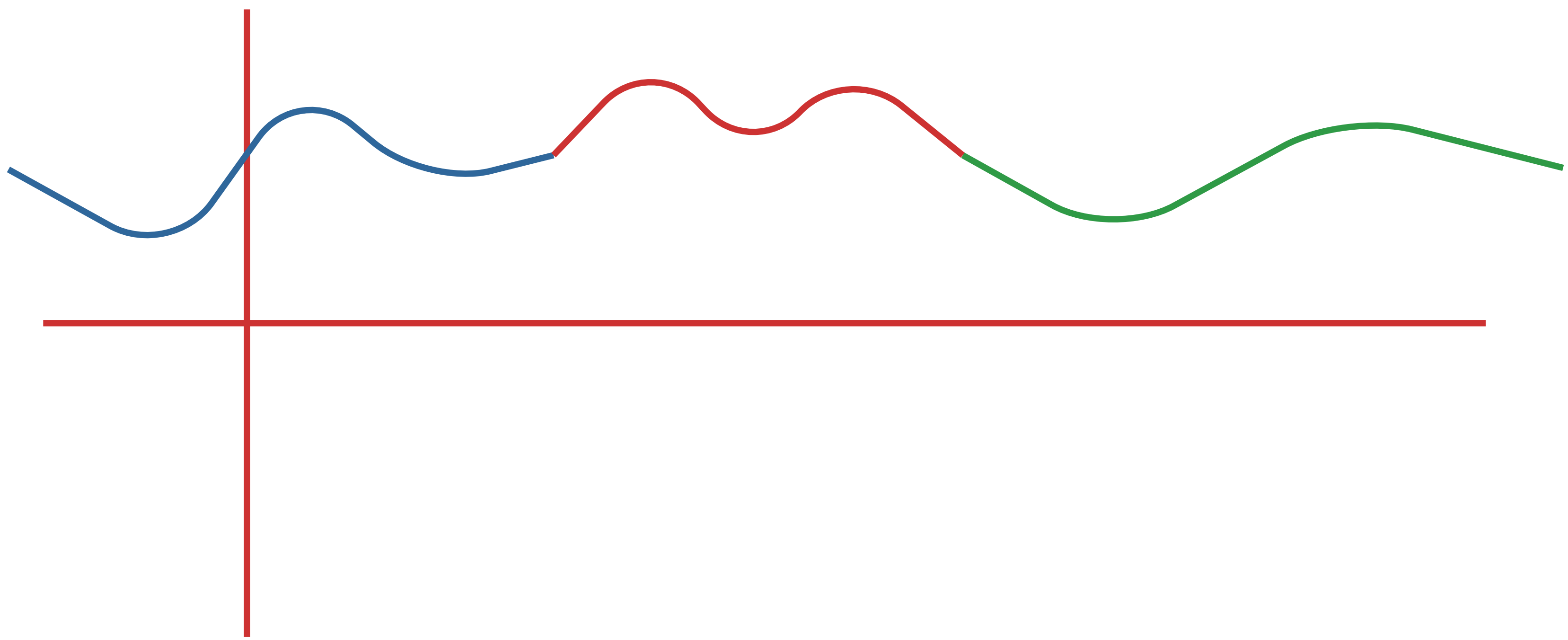
$$\mathcal{B} = \mathcal{B}_{\text{aut}} \oplus \mathcal{B}_{\text{contr}}$$

where $\mathcal{B}_{\text{aut}}$ is autonomous and $\mathcal{B}_{\text{aut}}$ is controllable and the characteristic values of $\mathcal{B}_{\text{aut}}$ are exactly those numbers $\lambda \in \mathbb{C}$ for which rank $R(\lambda) < g$.

Controllability



Controllability



Controllability

Consider the system:

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$
$$y = Cx + Du$$

The system (A, B, C, D) is called controllable if for all $x_0, x_1 \in \mathbb{R}^n$ there exists $t_1 > 0$ and u such that $x(t_1) = x_1$.

Null-controllability

The system (A, B, C, D) is called null-controllable if for all $x_0 \in \mathbb{R}^n$ there exists $t_1 > t_0$ and u such that $x(t_1) = 0$.

Reachability

The system (A, B, C, D) is called reachable if, given $x(t_0) = 0$, there exists for all $x_1 \in \mathbb{R}^n$ a $t_1 > t_0$ and u such that $x(t_1) = x_1$.

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$
$$y = Cx + Du$$

Reachability does not depend on the matrices C and D.

$$R = \left(\begin{array}{c|c|c|c|c} B & AB & A^2B & \cdots & A^{n-1}B \end{array} \right)$$

plays an important role.

R has full row rank if and only if

$$q^{\mathrm{T}} R = 0$$

implies that $q = 0$.

R has full row rank if and only if:

$$q^{\mathrm{T}} e^{At} B = 0$$

for all $t > 0$ implies that $q = 0$.

Let an arbitrary $t > 0$ be given. R has full row rank if and only if the matrix

$$L = \int_0^t e^{A\tau} B B^{\mathrm{T}} \left(e^{A\tau}\right)^{\mathrm{T}} d\tau$$

is invertible.

The input

$$u(\tau) = B^{\mathrm{T}} \left(e^{A(t-\tau)}\right)^{\mathrm{T}} L^{-1} x_1$$

steers the system from initial condition $x(0) = 0$ to $x(t) = x_1$ for arbitrary $x_1 \in \mathbb{R}^n$.

Theorem *The following statements are equivalent:*

1. *The system* (A, B, C, D) *is controllable,*
2. *The system* (A, B, C, D) *is reachable,*
3. *The system* (A, B, C, D) *is null-controllable,*

Theorem *The following statements are equivalent:*

1. *The system* (A, B, C, D) *is controllable,*
2. R *has rank* n,
3. $\operatorname{im} R = \mathbb{R}^n$.

$$\operatorname{im} R = \{x_1 \in \mathbb{R}^n \mid \text{ there exists } t_1 > t_0 \text{ and } u \in \underline{U} \text{ such that } x_1 = x(t_1, 0, u)\}$$

Definition *A linear subspace $V \in \mathbb{R}^n$ is called A-invariant if* $AV \subset V$.

Theorem im R *is the smallest subspace of* $\mathbb{R}^n$ *such that*

- im $B \subset$ im R,
- im R *is* A*-invariant.*

Theorem *The pair* (A, B) *is not controllable if there exists a vector* $q \neq 0$ *such that*

$$q^T A = \lambda q^T, \quad q^T B = 0$$

Theorem *The pair (A, B) is controllable if and only if*

$$\operatorname{rank}\begin{pmatrix}\xi I - A & -B\end{pmatrix} = n \text{ for all } \xi \in \mathbb{C}$$

If (A, B) is not controllable, then there exists an invertible matrix S zodanig dat

$$S^{-1}AS = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad S^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

S is related to a basis transformation $z = S^{-1}x$.

Let $\mathbb{R}, \mathbb{W}_1 \times \mathbb{W}_2, \mathcal{B})$ be a time-invariant dynamical system. Trajectories in $\mathcal{B}$ are partitioned as (w_1, w_2) witb $w_i : \mathbb{R} \to \mathbb{W}_i$, $i = 1, 2$. We say that w_2 is observable from w_1 if for all $(w_1, w_2), (w_1, w_2') \in \mathcal{B}$ implies $w_2 = w_2'$.

Let $R_1(\xi) \in \mathbb{R}^{g \times q_1}$ and $R_2(\xi) \in \mathbb{R}^{g \times q_2}$. Let $\mathcal{B}$ be the behavior defined by

$$R_1(\tfrac{d}{dt})w_1 = R_2(\tfrac{d}{dt})w_2$$

Then the variable w_2 is observable from w_1 if and only if

$$\text{rank}\, R_2(\lambda) = q_2$$

for all $\lambda \in \mathbb{C}$.

Observability

Consider the system:

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

The system (A, B, C, D) is called observable if there exists $t_1 > t_0$ such that $y(t, x_0, u) = y(t, x_1, u)$ for all $t \in [t_0, t_1]$ implies that $x_0 = x_1$.

Observability matrix

$$W = \begin{pmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{pmatrix}$$

Lemma *If* $Wx = 0$ *then, we have* $CA^k x = 0$ *for all* $k \geqslant 0$

Theorem *The following statements are equivalent*

1. *The system* (A, B, C, D) *is observable,*
2. W *has rank* n*,*
3. $\ker W = \{0\}$.

Theorem (A, B) *is controllable if and only if* (A^T, B^T) *is observable.*

(A, C) *is observable if and only if* (A^T, C^T) *is controllable.*

$$\ker W = \{x_1 \in \mathbb{R}^n \mid \text{ for all } u \in \underline{U} \text{ implies } y(t,0,u) = y(t,x_1,u) \text{ for all } t > 0\}$$

Theorem $\ker W$ *is the largest linear subspace of* $\mathbb{R}^n$ *such that*

- $\ker W \subset \ker C$,
- $\ker W$ *is* A*-invariant.*

Theorem *The pair* (A, C) *is not observable if and only if there exists* $q \neq 0$ *such that:*

$$Aq = \lambda q, \quad Cq = 0$$

Theorem *The pair* (A, C) *is observable if and only if*

$$\operatorname{rank}\begin{pmatrix} \xi I - A \\ -C \end{pmatrix} = n \text{ for all } \xi \in \mathbb{C}$$

If (A, C) is not observable then there exists an invertible matrix S such that

$$S^{-1}AS = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \quad CS = \begin{pmatrix} C_1 & 0 \end{pmatrix}$$

S is related to a basis transformation $z = S^{-1}x$.

We consider the behavior $\mathcal{B}$ of all $\mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^{q})$ functions w satisfying

$$R(\frac{d}{dt})w = 0$$

The mathematical model is then the triple $(\mathbb{R}, \mathbb{R}^{q}, \mathcal{B})$.

Consider two behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ described by:

$$\mathcal{B}_1 = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_1(\frac{d}{dt})w = 0 \right\}$$
$$\mathcal{B}_2 = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_2(\frac{d}{dt})w = 0 \right\}$$

Theorem $\mathcal{B}_1 \subset \mathcal{B}_2$ *if and only if there exists a polynomial* V *such that* $VR_1 = R_2$.

Consider two behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ described by:

$$\mathcal{B}_1 = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_1(\frac{d}{dt})w = 0 \right\}$$
$$\mathcal{B}_2 = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_2(\frac{d}{dt})w = 0 \right\}$$

with R_1 and R_2 having the same number of rows.

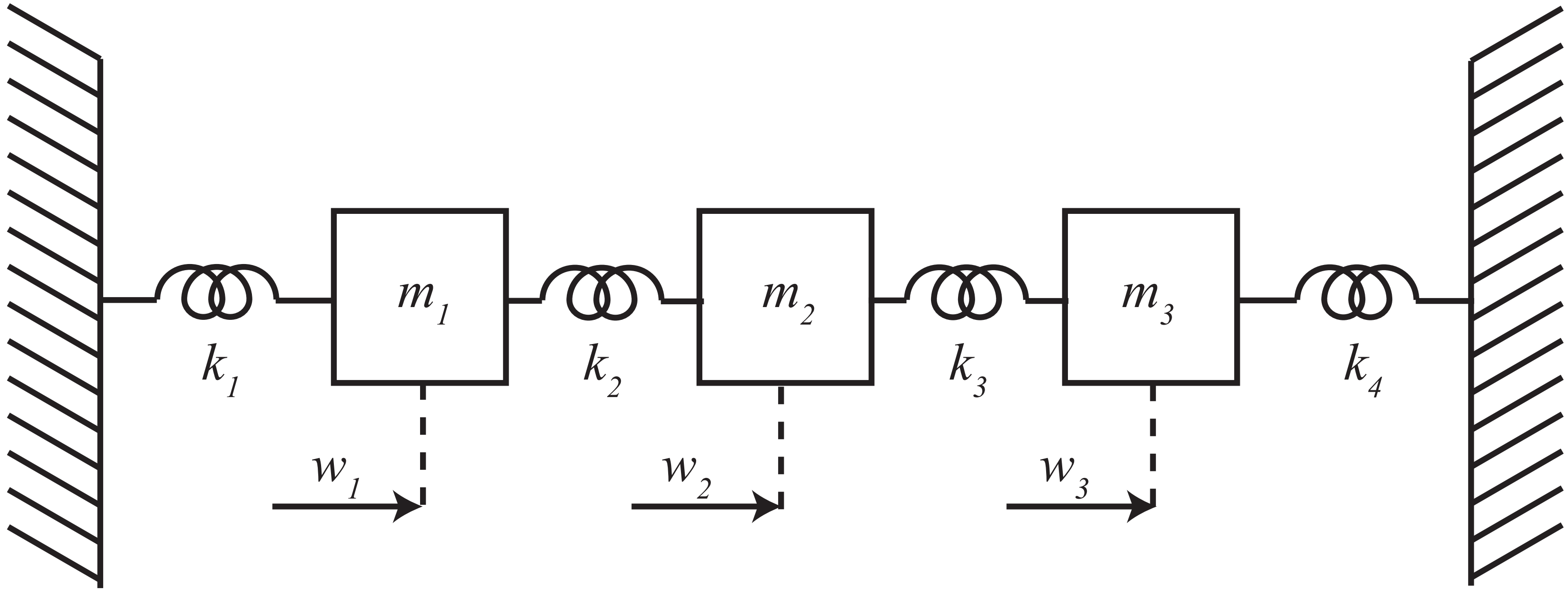
Theorem $\mathcal{B}_1 = \mathcal{B}_2$ *if and only if there exists a unimodular polynomial* V *such that* $VR_1 = R_2$.

Elimination of latent variables

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^q) \mid \exists \ell \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^d) \ \text{ s.t. } \ R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \right\}$$

versus

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^q) \mid R_1(\frac{d}{dt})w = 0 \right\}$$


m_1
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w_1
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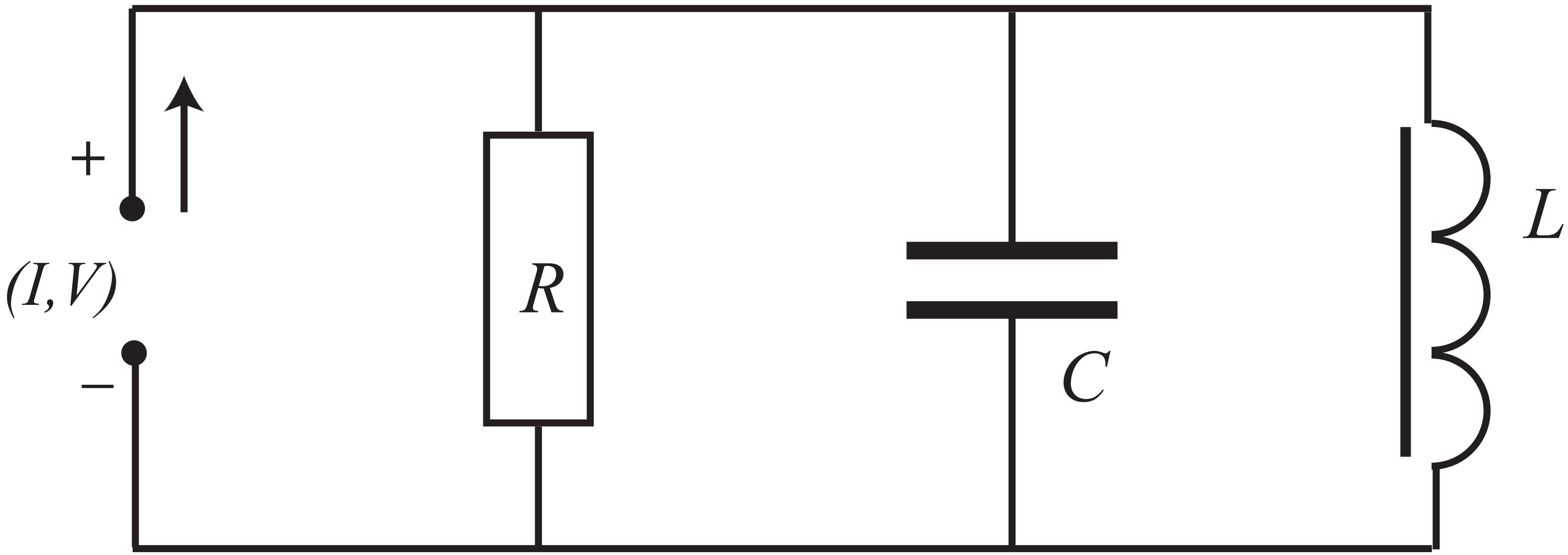
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Full behavior:

$$\mathcal{B}_f = \left\{(w,\ell) \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{q+d}) \mid R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\right\}$$

Manifest behavior:

$$\mathcal{B} = \left\{w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{q}) \mid \exists \ell \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^{d}) \text{ s.t. } (w,\ell) \in \mathcal{B}_f\right\}$$

- The manifest behavior is linear and shift-invariant
- The manifest behavior can be described by a set of linear time-invariant differential equations:

$$\mathcal{B} = \left\{ w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^{q}) \mid R'(\frac{d}{dt})w = 0 \right\}$$

A i/s/o representation of a system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

x consists of the latent variables.

We want a model of the form:

$$R(\frac{d}{dt})u = P(\frac{d}{dt})y$$

We bring the SISO system in the form:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + b_1 u \\ \dot{x}_2 = \qquad\qquad A_{22}x_2 + b_2 u \\ y = \qquad\qquad\quad c_2 x_2 \end{cases}$$

Define

$$\tilde{p}(\xi) = \det(\xi I - A_{22})$$

$$\tilde{r}(\xi) = \tilde{p}(\xi) c_2 (\xi I - A_{22})^{-1}$$

We have:

$$\tilde{r}(\frac{d}{dt}) b_2 u = \tilde{p}(\frac{d}{dt}) y$$

A state space system:

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx + du \end{cases}$$

Define

$$\tilde{p}(\xi) = \det(\xi I - A)$$
$$\tilde{r}(\xi) = \tilde{p}(\xi)c(\xi I - A)^{-1}$$

Let $g(\xi)$ be the common divisor of $\tilde{p}$ and $\tilde{r}$.

Then the behavior of the i/s/o representation is described by:

$$r(\frac{d}{dt})bu = p(\frac{d}{dt})y$$

where

$$p(\xi) = \frac{\widetilde{p}(\xi)}{g(\xi)}, \qquad r(\xi) = \frac{\widetilde{r}(\xi)}{g(\xi)}b + dp(\xi)$$

Let the polynomials p and q be given by:

$$p(\xi) = p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + \xi^n$$
$$q(\xi) = q_0 + q_1\xi + \cdots + q_{n-1}\xi^{n-1} + q_n\xi^n$$

and consider the i/o system described by:

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

Let $q(\xi) = dp(\xi) + \tilde{q}(\xi)$ with $\deg \tilde{q} < \deg p$. We define:

$$A = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \vdots \\ \vdots \\ \tilde{q}_{n-1} \end{pmatrix}$$

and

$$c_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}, \quad d = q_n$$

Let (A_1, c_1) and (A_2, c_2) be observable pairs and assume that A_1 and A_2 have the same characteristic polynomial. Then there exists exactly one nonsingular matrix such that:

$$A_2 = S^{-1} A_1 S, \qquad c_2 = c_1 S$$

Let (A, c) be an observable pair with p the characteristic polyonomial of A where:

$$p(\xi) = p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + \xi^n$$

Then there exists a nonsingular matrix S such that:

$$S^{-1}AS = \begin{pmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{pmatrix}, \quad cS = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$$

Controller canonical form

Let p and q be given:

$$p(\xi) = p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + \xi^n$$
$$q(\xi) = q_0 + q_1\xi + \cdots + q_{n-1}\xi^{n-1} + q_n\xi^n$$

Assume p and q have no common zeros. Hence the i/o behavior:

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

is controllable.

Let $q(\xi) = dp(\xi) + \tilde{q}(\xi)$ with $\deg \tilde{q} < \deg p$. We define:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$c_0 = \begin{pmatrix} \tilde{q}_0 & \tilde{q}_1 & \cdots & \tilde{q}_{n-1} \end{pmatrix}, \quad d = q_n$$

Then the i/o behavior of

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

is equal to the i/s/o behavior of:

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx + du \end{cases}$$

Consider two observable i/s/o representations:

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y = c_1 x_1 + d_1 u \end{cases} \qquad \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 u \\ y = c_2 x_2 + d_2 u \end{cases}$$

The two i/s/o representations define the same input/output behavior if and only if there exists a nonsingular matrix such that:

$$S^{-1} A_1 S = A_2, \quad S^{-1} b_1 = b_2, \quad c_1 S = c_2, \quad d_1 = d_2$$

Assume that:

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 u \\ y = c_1 x_1 + d_1 u \end{cases} \qquad \begin{cases} \dot{x}_2 = A_2 x_2 + b_2 u \\ y = c_2 x_2 + d_2 u \end{cases}$$

define the same i/s/o behavior and that (A_1, c_1) is observable. Then

$$\dim x_1 \leqslant \dim x_2$$

Image representations

Consider the following representations of a behavior

$$\mathcal{B}_\ell = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \exists \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d) \text{ s.t. } R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \right\}$$

We showed that we can equivalently represent the behavior via a kernel representation:

$$\left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R_1(\frac{d}{dt})w = 0 \right\}$$

or a i/s/o representation:

$$\begin{cases} \dot{x} = Ax + bu \\ y = cx + du \end{cases}$$

The following representation is called an image representation:

$$w = M(\frac{d}{dt})\ell$$

Theorem *Consider a behavior described by a kernel representation. It can be described as an image representation if and only if the behavior is controllable.*

We look at an i/s/o system:

$$\begin{cases} \dot{x} = Ax + Bu, \quad x(t) \in \mathbb{R}^n \\ y = Cx + Du \end{cases}$$

with A asymptotically stable.

Can we find a lower order system that describes the input-output behavior approximately?

- Which states are hard to reach using the input u?
- Which states are hard to observe from the output y?

Consider Hilbert spaces (normed vector spaces with an inner product) $\mathcal{V}$ and $\mathcal{W}$.

Let $A : \mathcal{V} \to \mathcal{W}$ be a continuous linear function. Then there exists a continuous linear function A^* such that:

$$\langle x, Ay \rangle = \langle A^* x, y \rangle$$

for all $x \in \mathcal{W}$ and $y \in \mathcal{V}$.

Let $\mathcal{V}_1 \subset \mathcal{V}$. If $x \in \mathcal{V}$ satisfies $x = x_1 + x_2$ with $x_1 \in \mathcal{V}_1$ and $x_2 \in \mathcal{V}_1^{\perp}$ then:

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$$

We have $\operatorname{im} A = (\ker A^*)^{\perp}$.

Consider:

$$V(x_0) = \inf_{u,T} \left\{ \|u\|_2^2 \mid u \in L_2[-T,0], x(-T) = 0, x(0) = x_0 \right\}$$

We have:

$$V(x_0) = x_0' P^{-1} x_0$$

where P is the unique solution of:

$$AP + PA' + BB' = 0$$

Consider:

$$W(x_0) = \|y\|_2^2$$

where $u = 0$ and $x(0) = x_0$ while $y \in L_2[0, \infty)$.

We have:

$$W(x_0) = x_0' Q x_0$$

where Q is the unique solution of:

$$A'Q + QA + C'C = 0$$