A new and simple proof of the equivalence theorem for behaviors*

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Abstract

Two full row rank representations of the same behavior are related through a left unimodular transformation. We present a new and extremely simple and insightful proof for this well-established fact.

Mathematics subject classification: 93A30

1 Introduction

One of the features of the behavioral approach to linear systems theory is the emphasis on trajectories rather than on specific representations of the set of all possible trajectories. If a behavior is specified through a set of linear, constant coefficient differential or difference equations, then the set of all possible representations is completely characterized by the equivalence theorem for behaviors. This theorem states that two full row rank matrices of polynomials that define the same behavior are related by a left unimodular transformation. Proofs of this result may be found in [2, Corollary 2.5] and [1, Theorem 3.6.4]. Both proofs use properties of the behavior, in particular the fact that every behavior admits an input/output structure. A proof in the spirit of the behavioral philosophy would rely on as few properties of the behavior as possible. In [1, Chapter 2] the notion of left unimodular transformation is introduced. It is readily obtained that two polynomial matrices that are related through a left unimodular transformation define the same behavior. It is somewhat unsatisfactory that the proof of the converse is given at the end of the next chapter in which a long and extensive study of autonomous and input-output behaviors is presented.

The proof that we present in the present paper is almost completely algebraic. The only explicit reference to a solution of a differential (or difference) equation that is used is that corresponding to $\frac{dz}{dt} = \lambda z$ or $z(k+1) = \lambda z(k)$. Here, $z$ is just a scalar valued function. Since this new proof is strikingly simple and insightful as compared to the ones just mentioned and since the result in itself is central in the behavioral theory, it appears appropriate to present it here.

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*To appear in Systems & Control Letters
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2 The simple proof

2.1 Lemma
Let \( R(\xi) \in \mathbb{R}^{g \times q}[\xi] \) and \( r(\xi) \in \mathbb{R}^{1 \times q}[\xi] \). If \( w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^q) \), and \( R(\frac{d}{d\xi})w = 0 \) imply \( r(\frac{d}{d\xi})w = 0 \), then there exist \( \alpha(\xi) \in \mathbb{R}^{1 \times q}[\xi] \) such that \( r(\xi) = \alpha(\xi)R(\xi) \).

Proof Without loss of generality we may assume that \( R(\xi) \) is of full row rank and hence \( g \leq q \). Choose unimodular matrices \( U(\xi) \) and \( V(\xi) \) such that \( U(\xi)R(\xi)V(\xi) = [D(\xi) \ 0] := \tilde{R}(\xi) \) with \( D(\xi) = \text{diag}(d_1(\xi), \ldots, d_q(\xi)) \). For instance, \( \tilde{R}(\xi) \) could be chosen to be the Smith form, see, e.g., [1], of \( R(\xi) \).

Define \( \tilde{r}(\xi) := r(\xi)V(\xi) \). Then \( \tilde{R}(\frac{d}{d\xi})\tilde{w} = 0 \) implies \( \tilde{R}(\frac{d}{d\xi})V(\frac{d}{d\xi})\tilde{w} = 0 \) and therefore also \( r(\frac{d}{d\xi})V(\frac{d}{d\xi})\tilde{w} = 0 \), which in turn implies \( \tilde{r}(\frac{d}{d\xi})\tilde{w} = 0 \). Since the components \( \tilde{w}_{g+1}, \ldots, \tilde{w}_q \) are unrestricted by \( \tilde{R}(\frac{d}{d\xi})\tilde{w} = 0 \), it follows that \( \tilde{r}_{g+1} = \ldots = \tilde{r}_q = 0 \). Assume that for some \( 1 \leq i \leq g \) \( d_i(\xi) \) does not divide \( \tilde{r}_i(\xi) \). Then there exists \( \lambda \in \mathbb{C} \) such that \( d_i(\lambda) = 0 \) and \( \tilde{r}_i(\lambda) \neq 0 \). Define \( \tilde{w} = [0, \ldots, 0, e^{i\lambda}, 0, \ldots, 0] \), Then \( \tilde{R}(\frac{d}{d\xi})\tilde{w} = 0 \) and \( \tilde{r}(\frac{d}{d\xi})\tilde{w} \neq 0 \). This is a contradiction and therefore there exists a polynomial vector \( \tilde{\alpha}(\xi) \) such that \( \tilde{r}(\xi) = \tilde{\alpha}(\xi)\tilde{R}(\xi) \). Hence \( r(\xi) = \tilde{r}(\xi)V(\xi)^{-1} = \tilde{\alpha}(\xi)U(\xi)U(\xi)^{-1}\tilde{R}(\xi)V(\xi)^{-1} = \alpha(\xi)R(\xi) \). Where \( \alpha(\xi) = \tilde{\alpha}(\xi)U(\xi) \). \( \square \)

2.2 Corollary
Let \( R_i(\xi) \in \mathbb{R}^{g \times q}[\xi] \) be of full row rank, \( i = 1, 2 \). Define \( B_i = \{ w \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^q) \mid R_i(\frac{d}{d\xi})w = 0 \} \). If \( B_1 \neq B_2 \), then \( g_1 = g_2 \) and there exists a unimodular matrix \( U(\xi) \) such that \( R_2(\xi) = U(\xi)R_1(\xi) \).

Proof From Lemma 2.1 it follows that there exist polynomial matrices \( F(\xi), G(\xi) \) such that \( R_1(\xi) = F(\xi)R_2(\xi) \) and \( R_2(\xi) = G(\xi)R_1(\xi) \). This implies that \( R_1(\xi) = F(\xi)G(\xi)R_1(\xi) \). Since \( R_1(\xi) \) is of full row rank it follows that \( F(\xi)G(\xi) = I \). Similarly \( G(\xi)F(\xi) = I \). This can be the case only if \( F(\xi) \) and \( G(\xi) \) are unimodular. \( \square \)

References
