The Behavioral Description of Linear Time-invariant Systems

Jan Willem Polderman

Department of Applied Mathematics
University of Twente

General Information

- Please register for the course at `disc.tudelft.nl`
- Eight lectures:
  - 1-4: Jan Willem Polderman (University of Twente)
  - 5-8: Kanat Camlibel (University of Groningen)
- Four homework sets. Homework I can be downloaded and is due on January 30th.
- Web site: search for homepage of Jan Willem Polderman. The site contains presentations, homework etc.
- Book: Polderman & Willems. Available as e-book from Springer.
- Please prepare all lectures by reading the slides and watching the video's at
  `https://vimeo.com/album/3878060/`

Definition

A *mathematical model* is a pair $(\mathbb{U}, \mathfrak{B})$ with $\mathbb{U}$ a set, called the *universum*—its elements are called *outcomes*—and $\mathfrak{B}$ a subset of $\mathbb{U}$, called the *behavior*.

Example

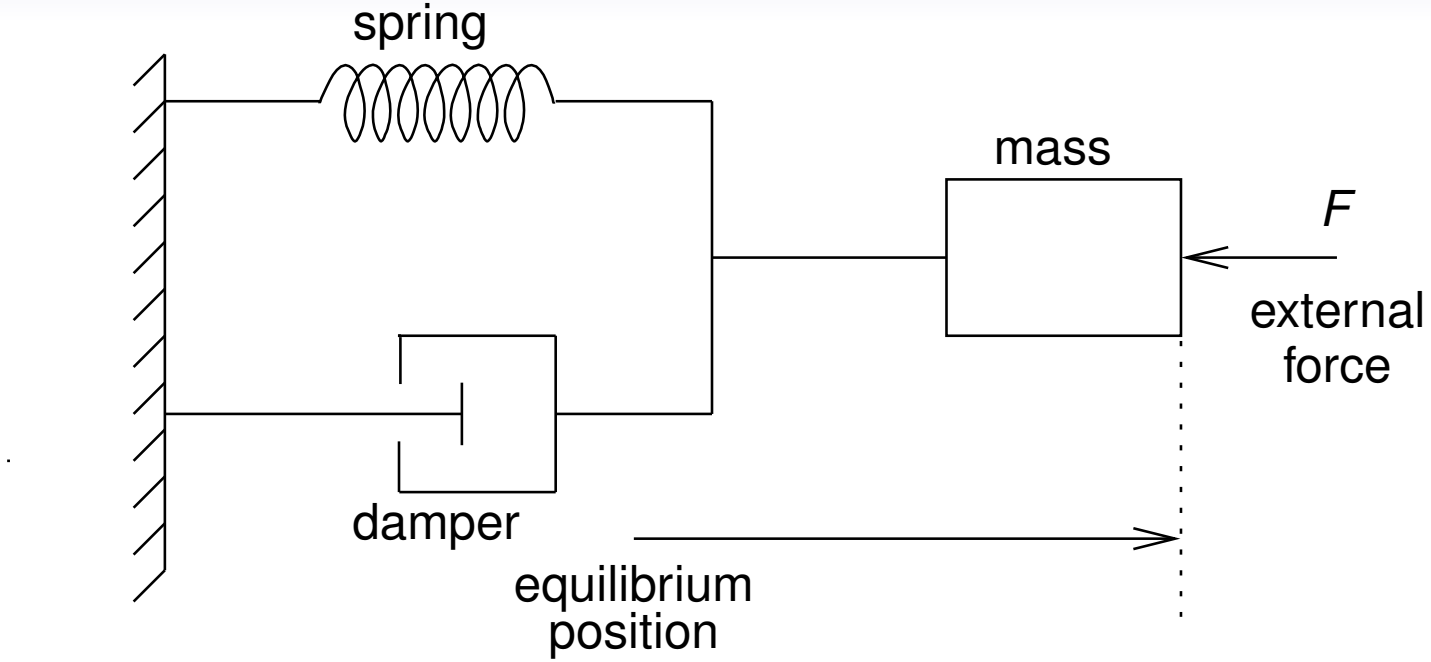
$\mathbb{U} = \{\text{ice, water, steam}\} \times [-273, \infty)$ and $\mathfrak{B} = (\{\text{ice}\} \times [-273, 0]) \cup (\{\text{water}\} \times [0, 100]) \cup (\{\text{steam}\} \times [100, \infty))$.

Definition

A *dynamical system* Σ is defined as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}),$$

with $\mathbb{T}$ a subset of $\mathbb{R}$, called the *time axis*, $\mathbb{W}$ a set called the *signal space*, and $\mathfrak{B}$ a subset of $\mathbb{W}^{\mathbb{T}}$ called the *behavior* ($\mathbb{W}^{\mathbb{T}}$ is standard mathematical notation for the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$).



$$M\frac{d^2}{dt^2}y + d\frac{d}{dt}y + ky = F.$$

$$\underbrace{\begin{bmatrix} M(\frac{d}{dt})^2 + d\frac{d}{dt} + k & -1 \end{bmatrix}}_{R(\frac{d}{dt})} \underbrace{\begin{bmatrix} y \\ F \end{bmatrix}}_{w} = 0$$

Generally

$$R(\frac{d}{dt})w = 0 \quad R(\xi) \in \mathbb{R}^{g \times q}[\xi].$$

Number of equation: g.
Number of variables: q.

Definition

A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be *linear* if $\mathbb{W}$ is a vector space (over a field $\mathbb{F}$: for the purposes of this book, think of it as $\mathbb{R}$ or $\mathbb{C}$), and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$ (which is a vector space in the obvious way by pointwise addition and multiplication by a scalar).

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Definition

A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$ is said to be *time-invariant* if $\sigma^t\mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{T}$ (σ^t denotes the *backward t-shift*: $(\sigma^t f)(t') := f(t' + t)$). If $\mathbb{T} = \mathbb{Z}$, then this condition is equivalent to $\sigma\mathfrak{B} = \mathfrak{B}$. If $\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{R}_+$, then time-invariance requires $\sigma^t\mathfrak{B} \subseteq \mathfrak{B}$ for all $t \in \mathbb{T}$. In this course we almost exclusively deal with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and therefore we may as well think of time-invariance as $\sigma^t\mathfrak{B} = \mathfrak{B}$. The condition $\sigma^t\mathfrak{B} = \mathfrak{B}$ is often called *shift-invariance* of $\mathfrak{B}$.

Basic definitions and the calculus of representations

Recall that a dynamical system is defined by a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T}$ the time axis, $\mathbb{W}$ the signal space and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ the behavior. We specialize to the case where:

- $\mathbb{T} = \mathbb{R}$
- $\mathbb{W} = \mathbb{R}^q$

Furthermore we restrict the attention to dynamical systems defined by linear, constant-coefficient, differential equations:

$$R(\frac{d}{dt})w = 0.$$

Definition

This equation defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, with $\mathfrak{B}$ defined as

$$\mathfrak{B} := \{w \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) \mid w \text{ satisfies } R(\tfrac{d}{dt})w = 0 \text{ weakly.}\}$$

Definition

A function $w : \mathbb{R} \to \mathbb{R}^q$ is called *locally integrable* if for all $a, b \in \mathbb{R}$,

$$\int_a^b \|w(t)\| dt < \infty.$$

Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^q$: if $v \in \mathbb{R}^q$, then $\|v\| = \sqrt{\sum_{i=1}^q v_i^2}$. The space of locally integrable functions $w : \mathbb{R} \to \mathbb{R}^q$ is denoted by $\mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$.

Example

Let $R(\xi) \in \mathbb{R}^{2\times 3}[\xi]$ be given by

$$R(\xi) = \begin{bmatrix} \xi^3 & -2+\xi & 3 \\ -1+\xi^2 & 1+\xi+\xi^2 & \xi \end{bmatrix}.$$

The multivariable differential equation $R(\frac{d}{dt})w = 0$ is

$$\frac{d^3}{dt^3}w_1 - 2w_2 + \frac{d}{dt}w_2 + 3w_3 = 0,$$

$$-w_1 + \frac{d^2}{dt^2}w_1 + w_2 + \frac{d}{dt}w_2 + \frac{d^2}{dt^2}w_2 + \frac{d}{dt}w_3 = 0.$$

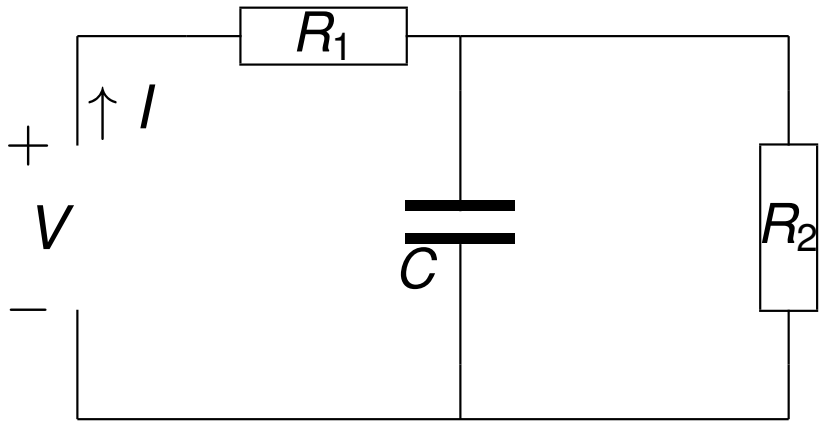
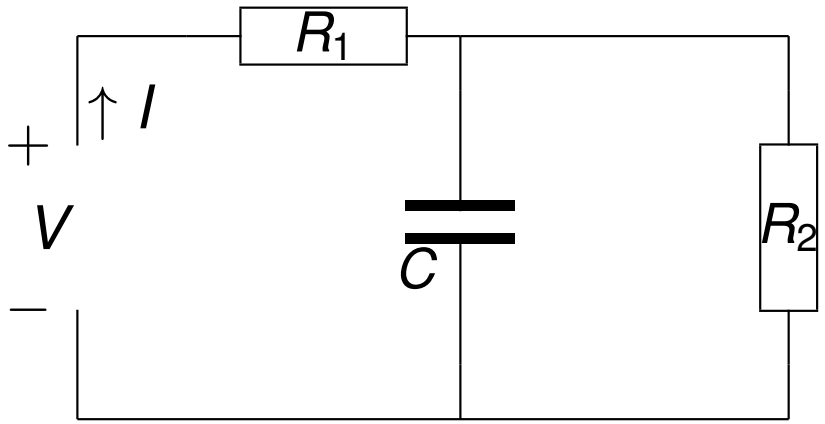


Figure: Electrical circuit.



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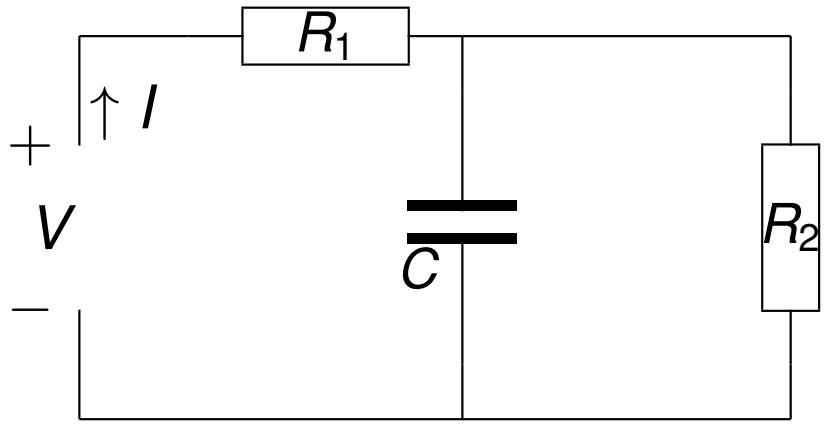
With $C = 1$ and $R_1 = R_2 = 1$ we obtain:

$$V + \frac{d}{dt}V = 2I + \frac{d}{dt}I.$$



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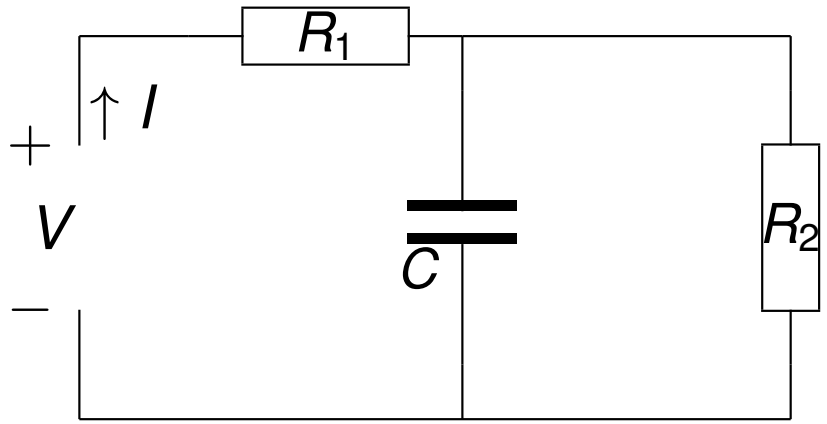

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With $C = 1$ and $R_1 = R_2 = 1$ we obtain:

$$V + \frac{d}{dt}V = 2I + \frac{d}{dt}I.$$

Take $V(t) = \underline{1}(t)$ then: $I(t) = (\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2t})\underline{1}(t) + k\mathrm{e}^{-2t}, \quad k \in \mathbb{R}$

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Notice that $V(t)$ nor $I(t)$ are differentiable. What then is the interpretation of solution?

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1. *In the sense of distributions?*
2. *In the sense of 'delta function calculus'?*
3. *Just ignore that there could be a problem?*
4. *Integral equations!*

Weak Solutions

Consider once more

$$V + \frac{d}{dt}V = 2I + \frac{d}{dt}I.$$

'Integrate' both sides:

$$\int_0^t V(\tau)d\tau + V(t) = 2\int_0^t I(\tau)d\tau + I(t) + d_0.$$

Substituting non-smooth functions in this equation is no longer a problem. In fact:
Smooth solutions of the differential equation are also solutions of the integral equation and vice versa.

Let $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ and consider

$$R(\frac{d}{dt})w = 0.$$

Define the *integral operator* acting on $\mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$ by

$$(\int w)(t) := \int_0^t w(\tau)d\tau, \qquad (\int^{k+1} w)(t) := \int_0^t (\int^k w)(\tau)d\tau,$$

Let $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L, \quad R_L \neq 0 \in \mathbb{R}^{g\times q}[\xi]$ be given. Define $R^*(\xi)$ as

$$R^*(\xi) := \xi^L R(\frac{1}{\xi}) = R_L + R_{L-1}\xi + \cdots + R_1\xi^{L-1} + R_0\xi^L.$$

and consider

$$(R^*(\textstyle\int)w)(t) = c_0 + c_1 t + \cdots + c_{L-1} t^{L-1}, \quad c_i \in \mathbb{R}^g.$$

We call $w \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$ a *weak* solution of the system of differential equations if there exist constant vectors $c_i \in \mathbb{R}^g$ such that the integral equation is satisfied for almost all $t \in \mathbb{R}$. For $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ we define

$$\mathfrak{B} = \{w \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0, \quad \text{weakly}\}$$

Remark

Weak solution does not mean strong solution for almost all t!

Theorem

- *Every strong solution is also weak.*
- *Every weak solution that is sufficiently smooth on some interval I is a strong solution on that interval.*

Theorem

$\mathfrak{B}$ *is closed.*
$\mathfrak{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ *is dense in* $\mathfrak{B}$*.*
$\mathfrak{B}_1 \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) = \mathfrak{B}_2 \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ *implies* $\mathfrak{B}_1 = \mathfrak{B}_2$

Theorem

$\mathfrak{B}$ *is linear.*
$\mathfrak{B}$ *is time-invariant.*

Definition (Equivalent differential equations)

Let $R_i(\xi) \in \mathbb{R}^{g_i \times q}[\xi]$, $i = 1, 2$. The differential equations

$$R_1(\frac{d}{dt})w = 0 \quad \text{and} \quad R_2(\frac{d}{dt})w = 0$$

are said to be *equivalent* if they define the same dynamical system. In other words, equivalence means that w is a solution of

$$R_1(\frac{d}{dt})w = 0$$

if and only if it is also a solution of

$$R_2(\frac{d}{dt})w = 0.$$

Theorem

Let $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ and $U(\xi) \in \mathbb{R}^{g\times g}[\xi]$. Define $R'(\xi) := U(\xi)R(\xi)$. Denote the behaviors corresponding to $R(\xi)$ and $R'(\xi)$ by $\mathfrak{B}$ and $\mathfrak{B}'$ respectively. Then:

1. $\mathfrak{B} \subset \mathfrak{B}'$.
2. *If in addition, $U^{-1}(\xi)$ exists and if $U^{-1}(\xi) \in \mathbb{R}^{g\times g}[\xi]$, then $\mathfrak{B} = \mathfrak{B}'$.*

Proof.

1. Choose $w \in \mathfrak{B}$. Since $R(\frac{d}{dt})w = 0$, we also have that $U(\frac{d}{dt})R(\frac{d}{dt})w = 0$. It follows that then also $w \in \mathfrak{B}'$.
2. By 1, it suffices to prove that $\mathfrak{B}' \subset \mathfrak{B}$. Since $U^{-1}(\xi)$ is a well-defined polynomial matrix, this follows by just applying Part 1 to $U^{-1}(\xi)R(\xi)$ and $R(\xi)$. □

The converse is also true:

Theorem

Let $R_i(\xi) \in \mathbb{R}^{g_i \times q}[\xi]$ define the behaviors $\mathfrak{B}_i$, $i = 1, 2$. If $\mathfrak{B}_1 \subset \mathfrak{B}_2$ then there exists matrix $F(\xi) \in \mathbb{R}^{g_1 \times g_2}[\xi]$ such that

$$R_2(\xi) = F(\xi) R_1(\xi).$$

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Remark

The statement relates two systems of differential equations that have the same solution set. It appears logical that to prove the statement, we need to know something about solutions of systems of differential equations. By transforming the matrix in diagonal form, the core of the proof relies on scalar differential equations only.

Corollary

Let $R_i(\xi) \in \mathbb{R}^{g\times q}[\xi]$, $i = 1, 2$, define the same behavior, then there exists a matrix $U(\xi) \in \mathbb{R}^{g\times g}[\xi]$ such that

$$R_2(\xi) = U(\xi)R_1(\xi),$$

and $U(\xi)$ has a polynomial inverse.

Polynomial matrices with a polynomial inverse play a very important role. Hence the following definition.

Definition (Unimodular matrix)

Let $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Then $U(\xi)$ is said to be a *unimodular* polynomial matrix if there exists a polynomial matrix $V(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that $V(\xi)U(\xi) = I$. Equivalently, if $\det U(\xi)$ is equal to a nonzero constant.

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Example

$$U(\xi) = \begin{bmatrix} -1 + 2\xi - \xi^2 & -2 + \xi \\ 2 + \xi^3 & 1 - \xi^2 \end{bmatrix},$$

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$$\det(U(\xi) = 3$$

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Example

$$U(\xi)^{-1} = \frac{1}{3}\begin{bmatrix} 1-\xi^2 & 2-\xi \\ -2-\xi^3 & -1+2\xi-\xi^2 \end{bmatrix}$$

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Identity matrix with two rows interchanged.

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2.

$$\operatorname{diag}(1, \ldots, 1, \underset{\substack{\uparrow \\ i\text{th place}}}{\alpha}, 1, \ldots, 1), \quad \alpha \neq 0$$

There are three types of *elementary* unimodular matrices:
3.

$$
\begin{bmatrix}
1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & \xi^d & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1
\end{bmatrix}
\longleftarrow i\text{th row.}
$$

$\uparrow$

jth column

Theorem

$U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ is unimodular if and only if it is a finite product of elementary unimodular matrices.

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Remark

The relevance of this theorem is that whenever statements are made about unimodular matrices it is often convenient to prove the statement for elementary unimodular matrices and subsequently that the property to which the statement refers carries over to products of unimodular matrices.

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Proof.

The proof is based on the observation that the inverse of a given unimodular matrix may be transformed into the identity matrix by means of successive pre-multiplication with elementary unimodular matrices. □

Theorem

Let $r_1(\xi), \ldots, r_k(\xi) \in \mathbb{R}[\xi]$ *and assume that* $r_1(\xi), \ldots, r_k(\xi)$ *have no common factor. Then there exists a unimodular matrix* $U(\xi) \in \mathbb{R}^{k \times k}[\xi]$ *such that the last row of* $U(\xi)$ *equals* $[r_1(\xi), \ldots, r_k(\xi)]$.

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Proof.

Based on excessive use of the Euclidean algorithm (division with remainder). □

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Corollary (Bezout)

Let $r_1(\xi), \ldots, r_k(\xi) \in \mathbb{R}[\xi]$. Assume that the greatest common divisor of $r_1(\xi), \ldots, r_k(\xi)$ is 1, i.e., the k polynomials have no common factor, i.e., they are coprime. Then there exist polynomials $a_1(\xi), \ldots, a_k(\xi) \in \mathbb{R}[\xi]$ such that

$$r_1(\xi)a_1(\xi) + \cdots + r_k(\xi)a_k(\xi) = 1.$$

Proof.

$$\begin{bmatrix} 0 & \cdots 0 & 1 \end{bmatrix} U(\xi) = \begin{bmatrix} r_1(\xi) \cdots r_k(\xi) \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} 0 & \cdots 0 & 1 \end{bmatrix} = \begin{bmatrix} r_1(\xi) \cdots r_k(\xi) \end{bmatrix} U(\xi)^{-1}$$

Left unimodular transformations can be used to bring the polynomial matrix in a more suitable form, like the *upper triangular form.*

Theorem (Upper triangular form)

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$*. There exists a unimodular matrix* $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$ *such that* $U(\xi)R(\xi) = T(\xi)$ *and* $T_{ij}(\xi) = 0$ *for* $i = 1, \ldots, g,\ j < i$*.*

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Proof.

Division with remainder. □

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Theorem

Let $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ *and* $V(\xi) \in \mathbb{R}^{q\times q}[\xi]$. *Denote the behaviors defined by* $R(\frac{d}{dt})w = 0$ *and* $R(\frac{d}{dt})V(\frac{d}{dt})w = 0$ *by* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *respectively. If* $V(\xi)$ *is unimodular, then* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *are isomorphic as vector spaces.*

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Theorem (Smith form, square case)

Let $R(\xi) \in \mathbb{R}^{g\times g}[\xi]$. There exist unimodular matrices $U(\xi), V(\xi) \in \mathbb{R}^{g\times g}$ such that

1. $U(\xi)R(\xi)V(\xi) = \operatorname{diag}(d_1(\xi), \ldots, d_g(\xi))$.
2. *There exist (scalar) polynomials $q_i(\xi)$ such that $d_{i+1}(\xi) = q_i(\xi)d_i(\xi)$, $i = 1, \ldots, g-1$.*

Example

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Let $\mathfrak{B}$ denote the behavior represented by $R(\frac{d}{dt})w = 0$. By pre-multiplication with a suitable unimodular matrix and deleting the resulting zero row, we find that $\mathfrak{B}$ is thus also represented by

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Intuitively, it is clear that no further reduction is possible. In more complicated situations, we would like to have a precise criterion on the basis of which we can conclude that we are dealing with the minimal number of equations.

Definition (Independence)

The polynomial *vectors* $r_1(\xi), \ldots, r_k(\xi)$ are said to be *independent* over $\mathbb{R}[\xi]$ if

$$\sum_{j=1}^{k} a_j(\xi) r_j(\xi) = 0 \quad \Leftrightarrow \quad a_1(\xi) = \cdots = a_k(\xi) = 0.$$

Here $a_1(\xi), \ldots, a_k(\xi)$ are scalar polynomials.

Definition (Row rank and column rank)

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. The *row rank* (*column rank*) of $R(\xi)$ is defined as the maximal number of independent rows (columns). We say that $R(\xi)$ has *full row rank* (*full column rank*) if the row rank (column rank) equals the number of rows (columns) in $R(\xi)$.

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Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. Denote the row rank and the column rank of $R(\xi)$ by k_r and k_c respectively.

1. *The integers k_r, k_c are invariant with respect to pre- and postmultiplication by unimodular matrices.*
2. *$k_r = k_c$, and hence we can speak about the* rank *of $R(\xi)$.*
3. *There exists a $k_r \times k_r$ submatrix of $R(\xi)$ with nonzero determinant.*

Proof.

Property 1. is easy to see for elementary unimodular matrices.

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Property 2. follows by transforming into Smith form, for which the statement is obvious, and using Property 1.
Property 3. follows likewise. □

Theorem

Every behavior $\mathfrak{B}$ defined by $R(\frac{d}{dt})w = 0$, $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ admits an equivalent full row rank representation, that is, there exists a representation $\tilde{R}(\frac{d}{dt})w = 0$ of $\mathfrak{B}$ with $\tilde{R}(\xi) \in \mathbb{R}^{\tilde{g}\times q}$ and $\operatorname{rank}\mathbb{R} = \tilde{g}$.

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Proof.

Find suitable unimodular $U(\xi)$ so that $U(\xi)R(\xi)$ has a zero row. Delete that zero row and repeat until no further reduction is possible. The resulting matrix $\tilde{R}(\xi)$ has full row rank. □

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Remark

It appears obvious that it is desirable to transform a given matrix $R(\xi)$ into full row rank form so as to reduce redundancy and the number of equations.
A further reduction of the number of equations could perhaps be possible by starting from some other matrix that defines the same behavior. This leads to the notion of minimality.

Definition (Minimality)

Let the behavior $\mathfrak{B}$ be defined by $R(\frac{d}{dt})w = 0$, $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$. The representation $R(\frac{d}{dt})w = 0$ is called *minimal* if every other representation has at least g rows, that is, if $w \in \mathfrak{B}$ if and only if $R'(\frac{d}{dt})w = 0$ for some $R'(\xi) \in \mathbb{R}^{g' \times q}[\xi]$ implies $g' \geq g$.

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If $R(\xi)$ is a minimal representation of $\mathfrak{B}$ if and only if $R(\xi)$ has full row rank.

Proof.

Easy part: minimality implies full row rank.
Not so easy part: full row rank implies minimality. □

Time Domain Description of Linear Systems

Definition

A behavior $\mathfrak{B}$ is called *autonomous* if for all $w_1, w_2 \in \mathfrak{B}$

$$w_1(t) = w_2(t) \text{ for } t \leq 0 \quad \Rightarrow \quad w_1(t) = w_2(t) \text{ for almost all } t.$$

In words: the future of every trajectory is completely determined by its past.

Theorem

Let $P(\xi) \in \mathbb{R}[\xi]$ *be a monic polynomial and let* $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$, *be the distinct roots of* $P(\xi)$ *of multiplicity* n_i*:* $P(\xi) = \prod_{k=1}^{N} (\xi - \lambda_k)^{n_k}$.
The corresponding behavior $\mathfrak{B}$ *defined by* $P(\frac{d}{dt})w = 0$ *is autonomous*

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The corresponding behavior $\mathfrak{B}$ defined by $P(\frac{d}{dt})w = 0$ is autonomous and is a finite-dimensional subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ of dimension $n = \deg P(\xi)$.

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The corresponding behavior $\mathfrak{B}$ defined by $P(\frac{d}{dt})w = 0$ is autonomous and is a finite-dimensional subspace of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ of dimension $n = \deg P(\xi)$. Moreover, $w \in \mathfrak{B}$ if and only if it is of the form

$$w(t) = \sum_{k=1}^{N} r_k(t) e^{\lambda_k t}$$

with $r_k(\xi) \in \mathbb{C}[\xi]$ an arbitrary polynomial of degree less than n_k.

The polynomial $P(\xi)$ is called the *characteristic polynomial* of $\mathfrak{B}$, and the roots of $P(\xi)$ are called the *characteristic values.*

Theorem

Let $P(\xi) \in \mathbb{R}^{q\times q}[\xi]$ *and let* $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, N$, *be the distinct roots of* $\det P(\xi)$ *of multiplicity* n_i*:* $\det P(\xi) = c\prod_{k=1}^{N}(\xi - \lambda_k)^{n_k}$ *for some nonzero constant* c*.*

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$$w(t) = \sum_{i=1}^{N}\sum_{j=0}^{n_i-1} B_{ij}t^j e^{\lambda_i t},$$

where the vectors $B_{ij} \in \mathbb{C}^q$ *satisfy the relations*

$$\sum_{j=\ell}^{n_i-1}\begin{pmatrix} j \\ \ell \end{pmatrix} P^{(j-\ell)}(\lambda_i)B_{ij} = 0, \quad i = 1, \ldots, N;\ \ell = 0, \ldots, n_i-1.$$

The polynomial $\prod_{k=1}^{N}(\xi - \lambda_k)^{n_k}$ is called the *characteristic polynomial* of the autonomous behavior $\mathfrak{B}$. The roots of $\det P(\xi)$, $\lambda_1, \cdots, \lambda_N$, are called the characteristic values of the behavior. For the case that $P(\xi) = I\xi - A$ for some matrix A, the characteristic values are just the eigenvalues of A.

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and higher order analogies.

Definition

Let $\mathfrak{B}$ be a behavior with signal space $\mathbb{R}^q$. Partition the signal space as $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ and $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ correspondingly as $w = \mathrm{col}(w_1, w_2)$ ($w_1 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ and $w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p)$). This partition is called an *input/output partition* if:

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If these hold, then w_1 is called an *input variable* and w_2 is called an *output variable*.

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Theorem

Let $\mathfrak{B}$ be the behavior defined by

$$\mathfrak{B} = \{w = \mathrm{col}(u, y) \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^p) \mid P(\frac{d}{dt})y = Q(\frac{d}{dt})u\},$$

and let $u \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$. Let $P^{-1}(\xi)Q(\xi)$ be proper and its partial fraction expansion be given by

$$P^{-1}(\xi)Q(\xi) = A_0 + \sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{A_{ij}}{(\xi - \lambda_i)^j}.$$

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Define y by

$$y(t) := A_0 u(t) + \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} \int_0^t \frac{(t-\tau)^{j-1}}{(j-1)!} e^{\lambda_i (t-\tau)} u(\tau) d\tau, \quad t \in \mathbb{R}.$$

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Then $(u, y) \in \mathfrak{B}$.

Corollary

Let $\mathfrak{B}$ be as in the previous theorem. Then (u, y) defines a proper input/output partition.

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This motivates:

Definition

A dynamical system represented by $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$ with $P(\xi) \in \mathbb{R}^{p\times p}[\xi]$ and $Q(\xi) \in \mathbb{R}^{p\times m}[\xi]$ is said to be in *input/output form* if it satisfies:

- $\det P(\xi) \neq 0$.
- $P^{-1}(\xi)Q(\xi)$ is a matrix of proper rational functions. (By Cramer's rule, the entries of $P^{-1}(\xi)Q(\xi)$ are rational functions.)

The following result expresses that the past of the output does not restrict the future of the input.

Theorem

Let $(u, y) \in \mathfrak{B}$ *as before. Let* $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ *be such that* $\tilde{u}(t) = u(t)$, $t \leq t_0$, *for some* $t_0 \in \mathbb{R}$. *Then there exists* $\tilde{y}$ *such that* $(\tilde{u}, \tilde{y}) \in \mathfrak{B}$ *and* $\tilde{y}(t) = y(t)$, $t \leq t_0$.

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Let $(u, y) \in \mathfrak{B}$ as before. Let $\tilde{u} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ be such that $\tilde{u}(t) = u(t)$, $t \leq t_0$, for some $t_0 \in \mathbb{R}$. Then there exists $\tilde{y}$ such that $(\tilde{u}, \tilde{y}) \in \mathfrak{B}$ and $\tilde{y}(t) = y(t)$, $t \leq t_0$.

Proof.

By time invariance we may assume that $t_0 = 0$. Since $(u, y) \in \mathfrak{B}$, it follows that

$$y(t) = y_h(t) + A_0 u(t) + \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} \int_0^t \frac{(t-\tau)^{(j-1)}}{(j-1)!} e^{\lambda_i (t-\tau)} u(\tau) d\tau,$$

where y_h satisfies $P(\frac{d}{dt}) y_h = 0$. Now simply define $\tilde{y}$ as before, but with u replaced by $\tilde{u}$. Since $\tilde{u}(t) = u(t)$ for $t \leq 0$, it follows that $\tilde{y}(t) = y(t)$ for $t \leq 0$. □

Remark

The corollary implies non-anticipation. Indeed, the past of the output is not restricted by the future of the input. Nor is the future of the input restricted by the past of the output. This implies that y does not anticipate u, or simply that the relation between the input and the output is non-anticipating.

Theorem

Let $R(\xi) \in \mathbb{R}^{g \times q}[\xi]$ *be of full row rank. If* $g < q$*, then there exists a choice of columns of* $R(\xi)$: $c_{i_1}(\xi), \ldots, c_{i_g}(\xi) \in \mathbb{R}^{g \times q}[\xi]$ *such that*

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Proof.

Choose $R_1(\xi)$ as a $g \times g$ nonsingular submatrix of $R(\xi)$ such that the degree of its determinant is maximal among the $g \times g$ submatrices of $R(\xi)$. Since $R(\xi)$ has full row rank, we know that $\det R_1(\xi)$ is not the zero polynomial. Denote the matrix formed by the remaining columns of $R(\xi)$ by $R_2(\xi)$. It follows from Cramer's rule that $R_1^{-1}(\xi)R_2(\xi)$ is proper. □

Corollary

Let $R(\frac{d}{dt})w = 0$, $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$, *be a full row rank representation of the behavior* $\mathfrak{B}$. $\mathfrak{B}$ *admits an i/o representation with input* $u = \mathrm{col}(w_{i_{g+1}}, \ldots, w_{i_q})$ *and output* $y = \mathrm{col}(w_{i_1}, \ldots, w_{i_g})$.