Design Methods for Control Systems

Class 8

DISC Course – 2010

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Outline

1. The Structured Singular Value
2. Robust designs using μ and H_∞
   - Mixed sensitivity design
   - H_∞ optimal control design
   - μ control design
3. Design examples
   - Robust stability analysis
   - Design of suspension system

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Non-singularities and robust control

Fundamental algebra problem in robust control

Given $M \in \mathbb{C}^{p\times m}$ and $\mathbb{S} \subset \mathbb{C}^{m\times p}$. Decide whether

$$\det(I - M\Delta) \neq 0 \quad \text{for all } \Delta \in \mathbb{S} \qquad \text{(NS)}$$

- Last week: $\mathbb{S} = \{\Delta \in \mathbb{C}^{m\times p} \mid \sigma_{\max}(\Delta) \leq 1\}$. Then (NS) holds if and only if $\sigma_{\max}(M) < 1$
- So, if $\sigma_{\max}(\mathbb{S}) \leq 1$ we infer that $\sigma_{\max}(M) < 1$ is sufficient for (NS). This small gain condition is easy to check, but conservative.
- Goal: develop more refined and computationally verifiable conditions that take structure of $\mathbb{S}$ into account.

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Structured uncertainties

Definition

Let $\mathbb{D}$ be the set of structured uncertainties of the form

$$\Delta = \text{diag}(p_1 I, \ldots, p_{n_r} I, \delta_1(s) I, \ldots, \delta_{n_d}(s) I, \Delta_1(s), \ldots, \Delta_{n_f}(s))$$

Here,

- $p_1, \ldots, p_{n_r}$ are n_r repeated real parametric uncertainties
- $\delta_1, \ldots, \delta_{n_d}$ are n_d repeated scalar dynamic uncertainties
- $\Delta_1, \ldots, \Delta_{n_f}$ are n_f full unstructured dynamic uncertainties

All identity matrices and full blocks may have different dimensions.

All sizes are measured in H_∞ norm: $\|\Delta\|_\infty \leq 1$.

Structured value sets

Introduce the set $\mathbb{S}$ of all block diagonal complex matrices

$$\text{diag}(p_1 I, \ldots, p_{n_r} I, \delta_1 I, \ldots, \delta_{n_d} I, \Delta_1, \ldots, \Delta_{n_f})$$

with blocks compatible to those of $\mathbb{D}$ and where

- $p_1, \ldots, p_{n_r}$ are real numbers
- $\delta_1, \ldots, \delta_{n_d}$ are complex numbers
- $\Delta_1, \ldots, \Delta_{n_f}$ are complex matrices

Structured uncertainties are described in terms of their frequency response values:

> $\mathbb{D}$ equals the set of all stable transfer matrices Δ whose frequency response $\Delta(j\omega) \in \mathbb{S}$ for all $\omega \in [0, \infty]$.

Structured singular value

Definition

The structured singular value (SSV) of a matrix $M \in \mathbb{C}^{p \times m}$ with respect to the uncertainty set $\mathbb{S}$ is

$$\frac{1}{\mu_{\mathbb{S}}(M)} := \sup \{ r \mid \det(I - M\Delta) \neq 0 \text{ for all } \Delta \in \mathbb{S} \text{ with } \|\Delta\| \leq r \}$$

We set $\mu_{\mathbb{S}}(M) = 0$ if the supremum is unbounded.

Important consequences:

- SSV upperbound: We have $\mu_{\mathbb{S}}(M) \leq \gamma_+$ if and only if

$$\det(I - M\Delta) \neq 0 \text{ for all } \Delta \in \mathbb{S} \text{ with } \|\Delta\| \leq \frac{1}{\gamma_+}$$

- SSV lowerbound: We have $\mu_{\mathbb{S}}(M) \geq \gamma_-$ if and only if

$$\det(I - M\Delta) = 0 \text{ for some } \Delta \in \mathbb{S} \text{ with } \|\Delta\| \leq \frac{1}{\gamma_-}$$

Structured Singular Value Theorem

Theorem

- *Suppose γ_+ is strict SSV upper bound for all frequencies:*

$$\mu_{\mathbb{S}}(M(j\omega)) < \gamma_+ \quad \textit{for all } \omega \in [0, \infty]$$

Then the inverse $(I - M\Delta)^{-1}$ exists and is stable for all structured uncertainties $\Delta \in \mathbb{D}$ with $\|\Delta\|_\infty \leq \frac{1}{\gamma_+}$. such that $I - M\Delta$ has no stable inverse.

- *Suppose γ_- is strict SSV lower bound for some frequency:*

$$\mu_{\mathbb{S}}(M(j\omega)) > \gamma_- \quad \textit{for some } \omega \in [0, \infty]$$

Then there exists $\Delta \in \mathbb{D}$ with $\|\Delta\|_\infty \leq \frac{1}{\gamma_-}$ such that $I - M\Delta$ has no stable inverse.

Comments

Other formulations:

- The following statement is correct:

$$(I - M\Delta)^{-1} \quad \text{is stable for all } \Delta \in \mathbb{D} \text{ with } \|\Delta\|_\infty < 1$$

  if and only if

$$\mu_{\mathbb{S}}(M(j\omega)) \; \leq \; 1 \quad \text{for all } \omega \in [0, \infty].$$

- The following statement is wrong:

$$(I - M\Delta)^{-1} \quad \text{is stable for all } \Delta \in \mathbb{D} \text{ with } \|\Delta\|_\infty \leq 1$$

  if and only if

$$\mu_{\mathbb{S}}(M(j\omega)) \; < \; 1 \quad \text{for all } \omega \in [0, \infty].$$

Example

Suppose that for some $\omega_0 \in \mathbb{R}$ we have $M = M(j\omega) = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$.

Find smallest Δ for which $\det(I - M\Delta) = 0$ where

1. Δ is unstructured
2. Δ is structured as $\Delta = \mathrm{diag}(\delta_1, \delta_2)$ with $\delta_i \in \mathbb{R}$.

---

1. Unstructured

Take SVD $M = U\Sigma V^*$ and set

$$\Delta := \frac{1}{\sigma_1} v_1 u_1^*$$

then $\det(I - M\Delta) = 0$ and $\|\Delta\| = \frac{1}{\sigma_1} = \frac{1}{\|M\|} = 0.3162$.

Example

Suppose that for some $\omega_0 \in \mathbb{R}$ we have $M = M(j\omega) = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$.

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---

2. Structured

$\det(I - M\delta) = 1 + \delta_2 - 2\delta_1$ gives that $\delta_2 = 2\delta_1 - 1$. Intersect this line with the unit box to infer that

$$\Delta = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

with norm $\|\Delta\| = 0.3333$.

Upperbound on the SSV

Some properties:

- $\mu(\alpha M) = |\alpha|\mu(M)$
- $\mu(M) \leq \sigma_{\max}(M)$
- $\mu(M) = \mu(DMD^{-1})$ for any

$$D \in \begin{pmatrix} \mathbb{R}^{\times} & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C}I & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C}^{\cdot\times\cdot} \end{pmatrix}$$

In particular,

$$\mu(M) \leq \inf_{D} \sigma_{\max}(DMD^{-1}) =: \nu(M)$$

Example: parametric uncertainty

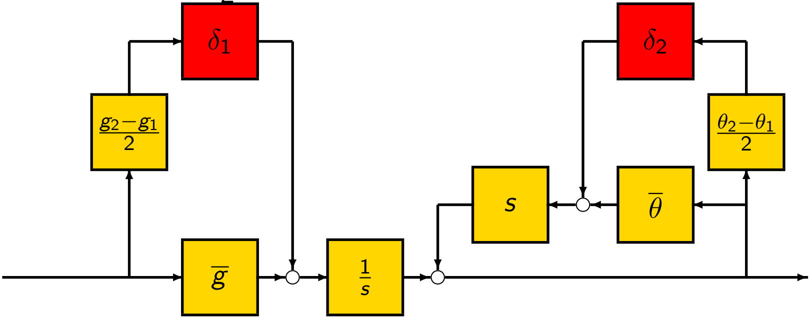
Model:

$$P(s) = \frac{g}{s^2(1+s\theta)}; \qquad g \in [g_1, g_2]; \quad \theta \in [\theta_1, \theta_2]$$

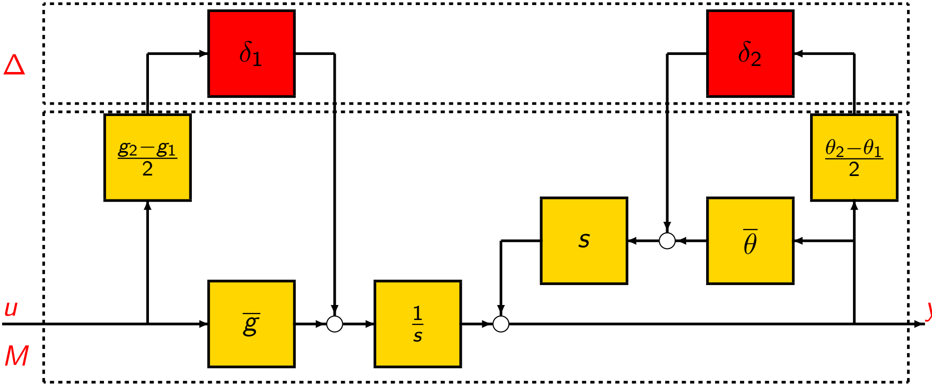
Scaling:

$$g = \overline{g} + \frac{g_2 - g_1}{2}\delta_g, \qquad \delta_1 \in \boldsymbol{\Delta}_1 = \{\delta \in \mathbb{R} \mid |\delta| \leq 1\}$$

$$\theta = \overline{\theta} + \frac{\theta_2 - \theta_1}{2}\delta_2, \qquad \delta_2 \in \boldsymbol{\Delta}_2 = \{\delta \in \mathbb{R} \mid |\delta| \leq 1\}$$



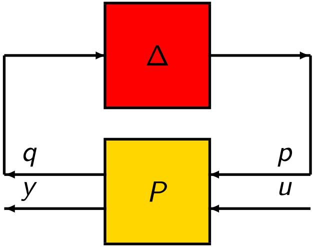
Example: parametric uncertainty



$$\delta_1 \in \Delta_1 := \{\delta \in \mathbb{R} \mid |\delta| \leq 1\}$$

$$\delta_2 \in \Delta_2 := \{\delta \in \mathbb{R} \mid |\delta| \leq 1\}$$

Generalized plant



Here, P and Δ are decomposed as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

Their interconnection defines the linear fractional transformation:

$$\mathcal{F}(\Delta, P) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}$$

that maps $u \mapsto y$.

Example: MIMO dynamic uncertainty

Uncertain system:

$$P_\Delta(s) = P(s) + \begin{pmatrix} W_{11}(s)\Delta_{11} & W_{12}(s)\Delta_{12} \\ W_{21}(s)\Delta_{21} & W_{22}(s)\Delta_{22} \end{pmatrix}$$

with $|\Delta_{11}| < 1$, $|\Delta_{12}| < 1$, $|\Delta_{21}| < 1$, $|\Delta_{22}| < 1$.

Which representation should we take??

$$\begin{pmatrix} W_{11}(s) & 0 & W_{12}(s) & 0 \\ 0 & W_{21}(s) & 0 & W_{22}(s) \end{pmatrix} \begin{pmatrix} \Delta_{11} & 0 \\ \Delta_{21} & 0 \\ 0 & \Delta_{12} \\ 0 & \Delta_{22} \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 & 0 \\ 0 & 0 & \Delta_{12} & 0 \\ 0 & 0 & 0 & \Delta_{22} \end{pmatrix} \begin{pmatrix} W_{11}(s) & 0 \\ W_{21}(s) & 0 \\ 0 & W_{12}(s) \\ 0 & W_{22}(s) \end{pmatrix}$$

Example: Mixed parametric and dynamic uncertainty

Uncertain system:

$$P_\Delta(s) = \begin{pmatrix} \frac{1+W_1\Delta_1}{s+1} & \frac{1+W_2(s)\Delta_2(s)}{s+2} \\ \frac{1+W_3\Delta_1}{s+3} & \frac{1}{s+4} \end{pmatrix}; \qquad \begin{cases} |\Delta_1| < 1 & \text{real parametric} \\ \|\Delta_2\|_\infty < 1 & \text{dynamic} \end{cases}$$

We can write:

$$P_\Delta(s) = \underbrace{\begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+3} & \frac{1}{s+4} \end{pmatrix}}_{P} + \underbrace{\begin{pmatrix} \frac{W_1}{s+1} & 0 & \frac{W_2(s)}{s+2} \\ 0 & \frac{W_3}{s+3} & 0 \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_1 & 0 \\ 0 & 0 & \Delta_2 \end{pmatrix}}_{\Delta} \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{V}$$

$$= \mathcal{F}\left(\Delta, \begin{pmatrix} 0 & V \\ W & P \end{pmatrix}\right)$$

Uncertainty set:

$$\mathbb{D} = \{\Delta = \text{diag}(\Delta_1, \Delta_1, \Delta_2) \in H_\infty \mid |\Delta(j\omega)| < 1 \ \forall\omega\}$$

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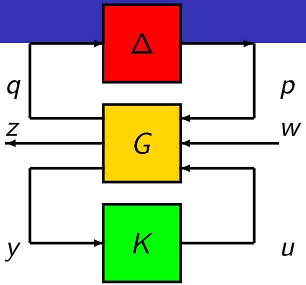
General setting



Problem

Robust stability analysis problem

For given controller K, test whether it robustly stabilizes $\mathcal{F}(\Delta, G)$ against all uncertainties $\Delta \in \mathbb{D}$.

Problem

Robust stability synthesis problem

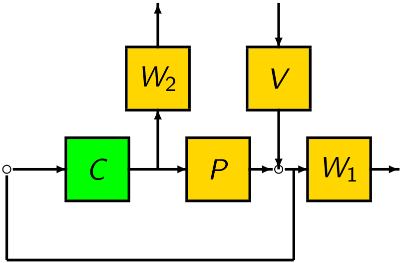
Find a controller K that robustly stabilizes $\mathcal{F}(\Delta, G)$ against all uncertainties $\Delta \in \mathbb{D}$.

Mixed sensitivity design

Mixed sensitivity problem:

$$\min_{C \text{stabilizing}} \left\| \begin{pmatrix} W_1 S V \\ W_2 U V \end{pmatrix} \right\|_\infty$$

where $U = C(I + PC)^{-1}$ is input sensitivity matrix



SISO $\left\| \begin{pmatrix} W_1 S V \\ W_2 U V \end{pmatrix} \right\|_\infty \leq \gamma$ gives

$$|S(j\omega)| \leq \frac{\gamma}{|W_1(j\omega)||V(j\omega)|}, \qquad \omega \in \mathbb{R}$$

$$|U(j\omega)| \leq \frac{\gamma}{|W_2(j\omega)||V(j\omega)|}, \qquad \omega \in \mathbb{R}$$

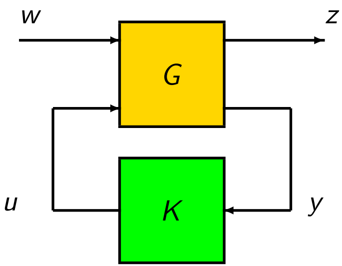
The standard H_∞ design problem

Standard H_∞ problem:

$$\min_{K \text{stabilizing}} \|\mathcal{F}(G, K)\|_\infty$$

where $\mathcal{F}(G, K)$ is closed loop transfer function

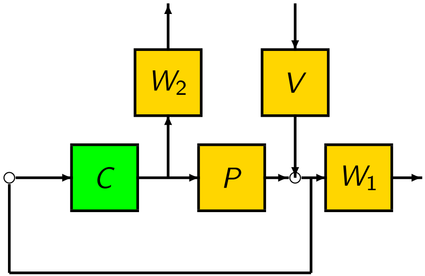
$$\mathcal{F}(G, K) : w \mapsto z$$



Recall linear fractional transformation

$$\mathcal{F}(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

The standard H_∞ design problem



Mixed sensitivity problem can be rephrased as standard H_∞ problem by setting

$$z_1 = W_1 V w + W_1 P u$$
$$z_2 = W_2 u$$
$$y = -Vw - Pu$$

The standard H_∞ design problem

Design

- Finding optimal K is not easy/not possible.
- Finding suboptimal K is easy (today)
  Given $\gamma > 0$ determine K such that $\|\mathcal{F}(G, K)\|_\infty < \gamma$ or tell me that so such K exist
- Then K is rational if G is rational and typically of same order
- Do a line search on γ to find γ_{opt} and K_{opt}.

All this beauty in one Matlab routine:

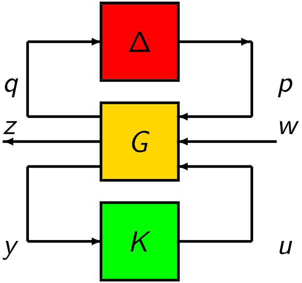
hinfsyn or *hinflmi*

μ controller design

Optimal μ synthesis problem:

$$\min_{K \text{stabilizing}} \sup_{\omega} \mu_{\mathbb{S}}(\mathcal{F}(G, K))$$

- Very hard problem
  Rumors say that it's NP hard
- Need approximate solutions



Here, $\Delta(j\omega) \in \mathbb{S}$ is structured.

$D - K$ iteration

1. Find whatever stabilizing controller K
2. Choose frequency grid ω_m, $m = 1, 2, \ldots$
3. Compute for all grid points

$$\nu(H(j\omega_m)) := \min_{D_m} \sigma_{\max}(D_m H(j\omega) D_m^{-1})$$

4. Find a rational, stable fit $D(s)$ such that

$$D(j\omega_m) = D_m \quad \text{in magnitude}$$

5. Find new controller K that (approximately) minimizes $\|D\mathcal{F}(G, K)D^{-1}\|_\infty$.
6. Return to step 3 (or stop)

All this in Matlab routine

`dksyn`

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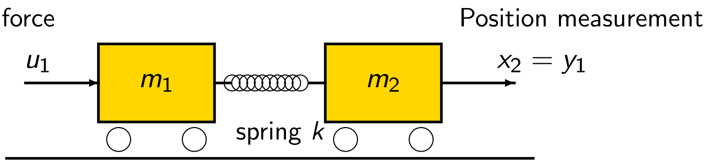
Example of dynamic uncertainty



Model:

$$m_1 \ddot{x}_1 = u_1 - f, \quad m_2 \ddot{x}_2 = f, \quad f = k(x_1 - x_2), \quad y_1 = x_2$$

Controller:

$$K(s) = \frac{100(s+1)^3}{(s/1000+1)^3}$$

stabilizes nominal $(m_1 = m_2 = k = 1)$ system.

Uncertainty: spring is unknown but stable **LTI system** k with

$$|k(j\omega) - 1| \;\leq\; 0.2 \quad \text{for all } \omega \in [0, 10],$$

while the error increases by 20dB per decade up to 60dB for $\omega \geq 10$

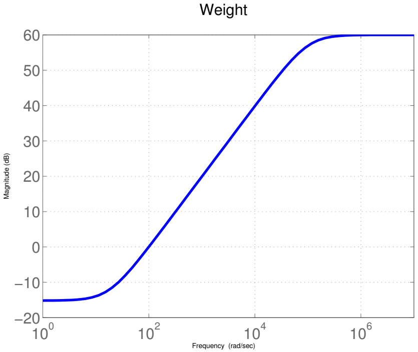
Example of dynamic uncertainty

With stable weight

$$W(s) = \frac{100 - s + 17670}{s + 101500}$$

we allow for all stable $k(s)$ with

$$|k(j\omega) - 1| \leq |W(j\omega)| \forall \omega \in [0, \infty].$$



This corresponds to the set of all $k(s)$ that can be described as

$$k(s) = 1 + W(s)\delta(s) \quad \text{with } \|\delta\|_\infty \leq 1$$

The weight W models the frequency dependence of the uncertainty

Example of dynamic uncertainty

Useful Matlab commands:

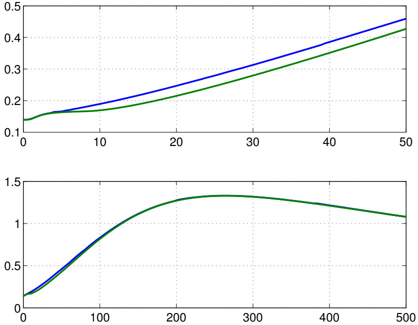
- **LTI system representations**
  zpk,ss,ltisys,ltiss,ltitf,nd2ss,tf2ss
- **uncertain system representations**
  - ultidyn,uss uncertain LTI system
  - ureal uncertain real parameter
  - makeweight creates weighing matrices
- **system interconnections**
  sysic, lft, connect,
- **Uncertainty representation/extraction**
  - lftdata represents uncertain objects as LFT's (very useful!!)
  - frd creates frequency response data objects (for μ analysis)
- μ **and** H_∞ **controller synthesis**
  - mussv computes SSV on frequency grid
  - sigma computes USV on frequency grid
  - hinfsyn,h2syn,dksyn H_∞, H_2 and μ controller synthesis routines

Example of dynamic uncertainty

```
% uncertain system components
m1=ureal('m1',1,'percent',20);
m2=ureal('m2',1,'percent',20);
W=makeweight(0.174,100,1e3);
Delta=ultidyn('Delta',[1 1],'Bound',1); k=1+W*Delta;
% interconnect components
s=zpk('s');G1=1/(m1*s^2);G2=1/(m2*s^2);
systemnames='G1 G2 k';
inputvar='[u1]';outputvar='[G2]';
input_to_G1='[u1-k]'; input_to_G2='[k]';
input_to_k='[G1-G2]'; G=sysic;
systemnames='G';
inputvar='[d;r;u]'; outputvar='[G+d-r;r-d-G]';
input_to_G='[u]'; olsys=sysic;
% connect the controller
clsys=lft(olsys,K);
```

Example of dynamic uncertainty

```
[M,Delta]=lftdata(clsys);
omega=linspace(0,50,100);
M0=frd(M,omega);
blk=[1 1;-1 0;-1 0];
mu=mussv(M0,blk);
plot(mu);
bo=squeeze(mu.res);
[max(bo(1,:))  max(bo(2,:))]
ans = 1.3304 1.3289
bounds = 1./ans
bounds = 0.7517 0.7525
```



- $(I - M\Delta)^{-1}$ stable for all structured Δ with $\|\Delta\|_\infty \leq 0.751$
- $(I - M\Delta)^{-1}$ unstable for some structured Δ with $\|\Delta\|_\infty \leq 0.753$

Example of dynamic uncertainty

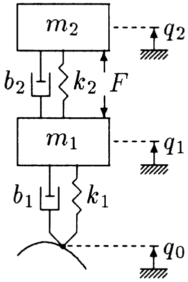
The robust stability margin r^* is the largest r such that K stabilizes the uncertain interconnection for all $\delta(s)$ with $\|\delta\|_\infty \leq r$.

```
[margin,des,report,info]=robuststab(clsys);
margin = UpperBound:  0.8866
     LowerBound:  0.8866
     DestabilizingFrequency:  201.5071
```

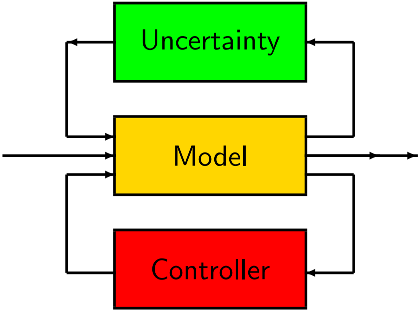
- Conclude that K **does not** robustly stabilize the uncertain interconnction for stable $\delta(s)$ bounded by 1.
- But K **is** robustly stabilizing for $\|\delta\|_\infty \leq 0.88 \approx r^*$
- The computed margin is tight: There does exist a stable δ with $\|\delta\|_\infty \leq 0.89$ for which K is not stabilizing any more.

Design example: Active suspension systems

Mass-spring-damper system



Control configuration



Design example: Active suspension systems

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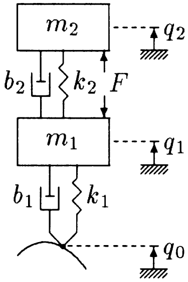
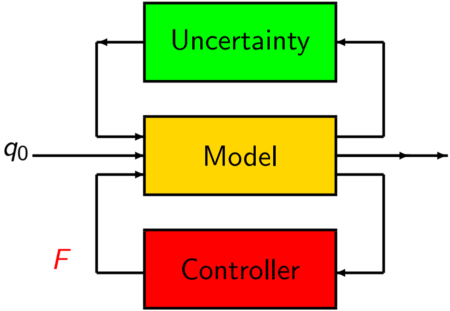
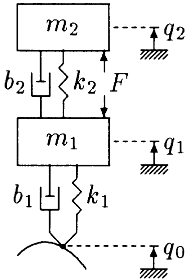


Control configuration

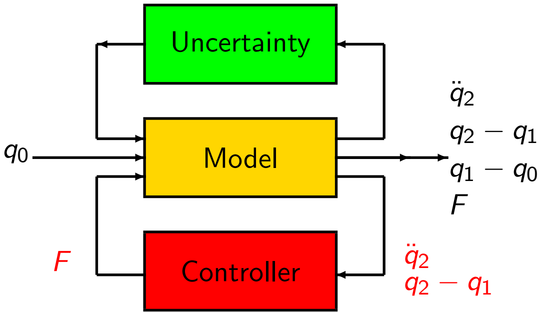


Inputs:
(q_0, F)

Mass-spring-damper system



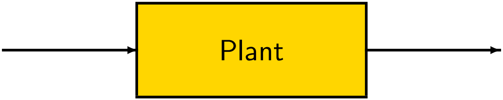
Control configuration



Inputs: (q_0, F) $\mapsto$ Outputs: $(\ddot{q}_2, q_2 - q_1, q_1 - q_0, F, \ddot{q}_2, q_2 - q_1)$

The model

$$\begin{pmatrix} q_0 \\ F \end{pmatrix}$$



$$\begin{pmatrix} \ddot{q}_2 \\ q_2 - q_1 \\ q_1 - q_0 \end{pmatrix}$$

Described by differential equations

$$0 = m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) - F$$
$$0 = m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + k_2(q_1 - q_2) + k_1(q_1 - q_0) + b_1(\dot{q}_1 - \dot{q}_0) + F.$$

Described by state space equations

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_2}{m_2} \end{pmatrix} x + \begin{pmatrix} \frac{b1}{m_1} & 0 \\ 0 & 0 \\ \frac{-b_1^2 - b_1 b_2}{m_1^2} + \frac{k_1}{m_1} & -\frac{1}{m_1} \\ \frac{b_1 b_2}{m_1 m_2} & \frac{1}{m_2} \end{pmatrix} u$$

$$y = \begin{pmatrix} k_2/m_2 & -k_2/m_2 & b_2/m_2 & -b_2/m_2 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 b_2/m_1 m_2 & 1/m_2 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} u$$

The model

Described by differential equations

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Physical specifications

	m_1	m_2	k_1	k_2	b_1	b_2
unloaded	1.5×10^3	1.5×10^3	5.0×10^6	5.0×10^5	1.5×10^{-3}	50×10^{3}
loaded	1.5×10^3	1.0×10^4	5.0×10^6	5.0×10^5	1.5×10^{-3}	50×10^{3}

Control specifications

- Suppress accelerations $\ddot{q}_2$ between 0.5Hz and 5Hz. (car-sickness).
- Suspension deflection $q_2(t) - q_1(t)$ can be at most 0.15 [m].
- For sufficient road grip of the tires it is necessary that

$$-0.08 \leq q_1(t) - q_0(t) \leq 0.02 \quad \text{(in [m])}.$$

- Robustly stable against uncertainty in mass m_2
- Robust performance at different velocities and different roads.

Physical and control specifications

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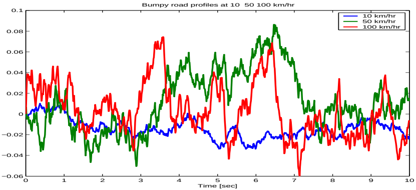
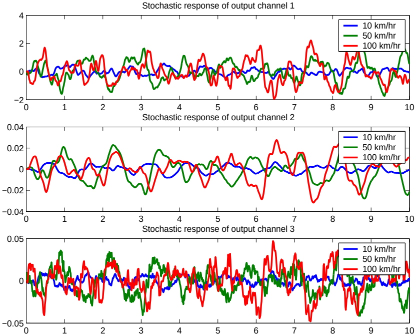
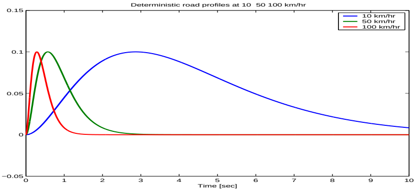
Nominal model responses (uncontrolled)

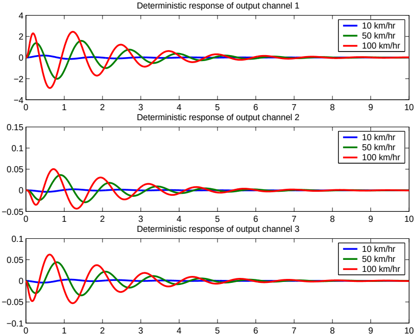
stochastic

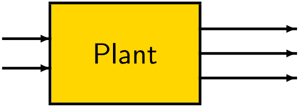


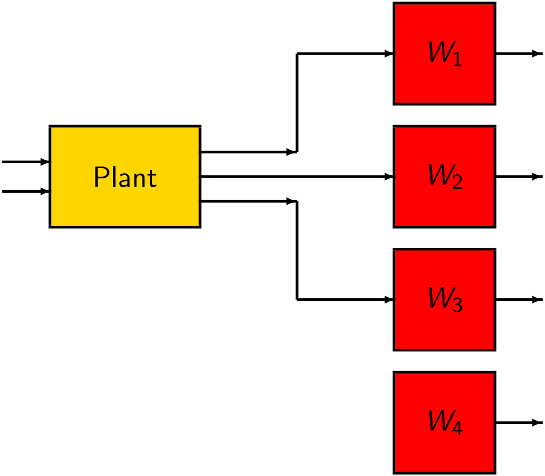


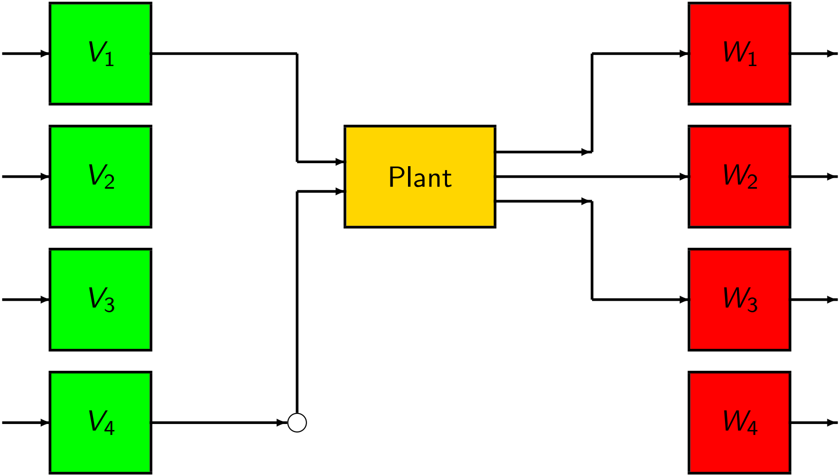
deterministic

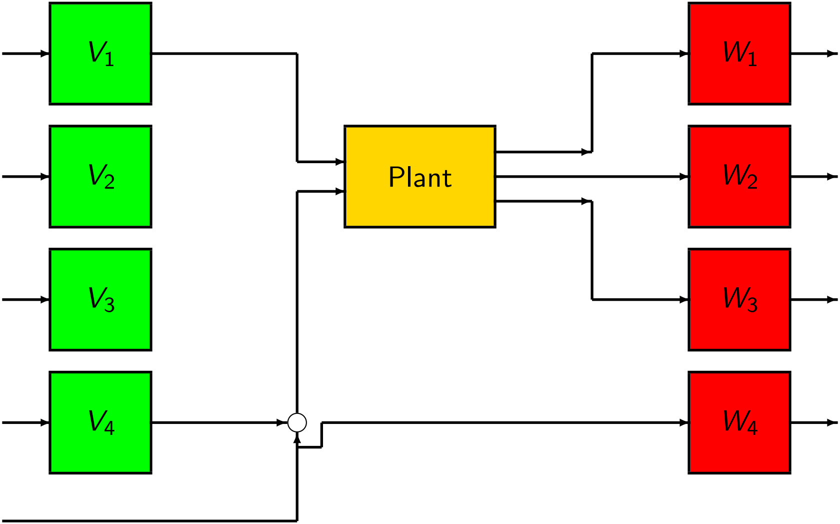




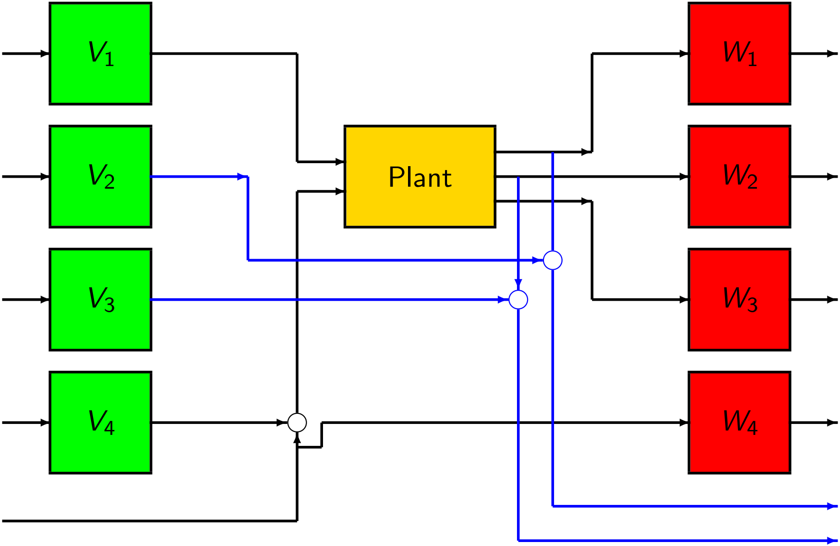

Plant

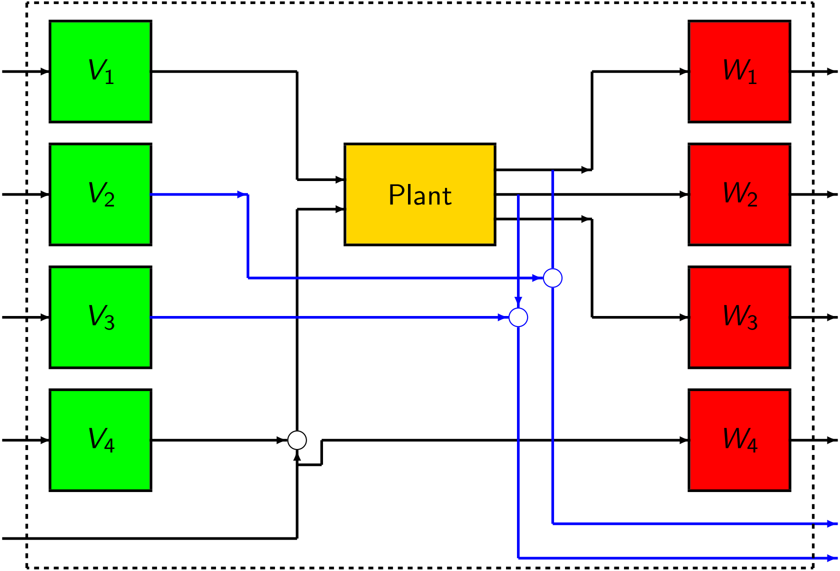

Plant
W_1
W_2
W_3
W_4


V_1
V_2
V_3
V_4
Plant
W_1
W_2
W_3
W_4



V_1
V_2
V_3
V_4
Plant
W_1
W_2
W_3
W_4

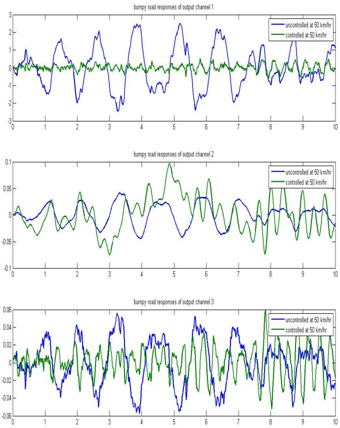

V_1
V_2
V_3
V_4
Plant
W_1
W_2
W_3
W_4



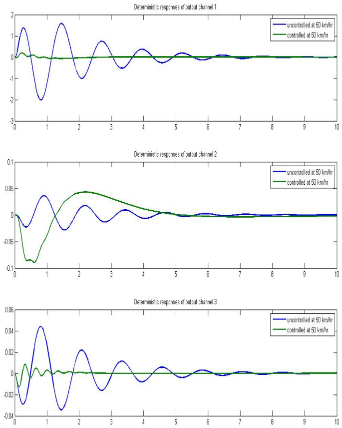
V_1
V_2
V_3
V_4
Plant
W_1
W_2
W_3
W_4

Simulation controlled system

stochastic



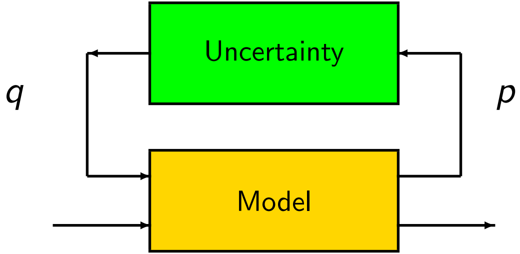
deterministic



Uncontrolled (blue) and controlled (green) response to 'stochastic' and 'deterministic' roads at velocity of 50km/hr

Parametric uncertainty

Assume mass m_2 is varying 20%. Can this be written in the form:



If so, what are inputs and outputs of the uncertainty block???

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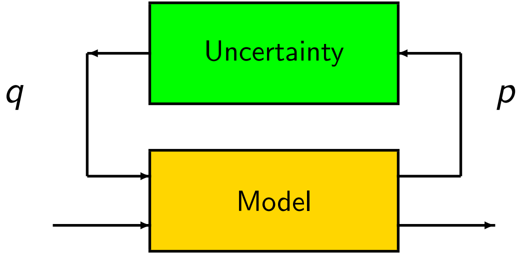


If so, what are inputs and outputs of the uncertainty block???

Pulling out the uncertainty . . .

Note that, with $\delta = 1/m_2$ we have

$$A(m_2) = A_0 + \delta A_1; \qquad B(m_2) = B_0 + \delta B_1$$
$$C(m_2) = C_0 + \delta C_1; \qquad D(m_2) = D_0 + \delta D_1$$

so that

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

where

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad q_1 = x, \quad q_2 = u$$

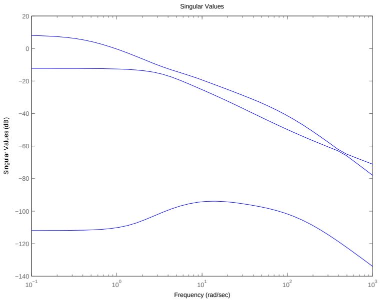
and

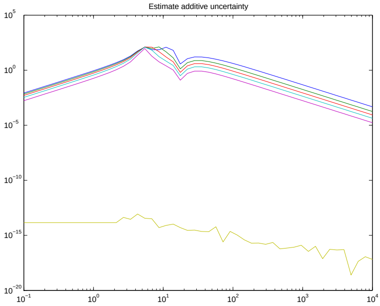
$$\frac{1}{1.0 \times 10^4} \leq \delta \leq \frac{1}{1.5 \times 10^3}$$

Dynamic uncertainty (unstructured)

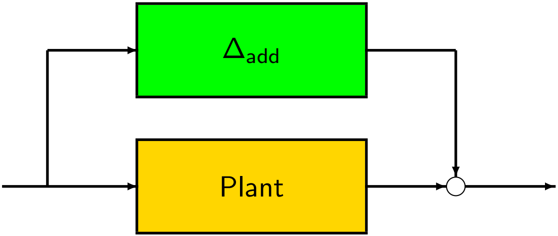
Set $m_2 = 1.0 \times 10^4$ as nominal plant (maximal load) and assume $P_\Delta = P + \Delta$

Frequency response and additive error





Consider additive uncertainty



where Δ_{add} is partitioned as

$$\Delta_{\text{add}} = \begin{pmatrix} \Delta_{q_0 \mapsto \ddot{q}_2} & \Delta_{F \mapsto \ddot{q}_2} \\ \Delta_{q_0 \mapsto q_2 - q_1} & \Delta_{F \mapsto q_2 - q_1} \\ \Delta_{q_0 \mapsto q_1 - q_0} & \Delta_{F \mapsto q_1 - q_0} \end{pmatrix}$$

Robustness analysis

Use the augmented plant to infer that

- $\Delta_{F \mapsto \ddot{q}_2} = V_2 \Delta_1 W_4$
- $\Delta_{F \mapsto q_2 - q_1} = V_3 \Delta_2 W_4$

(the other 4 transfer functions in Δ_{add} can not be pulled out from our augmented plant configuration).

The system is robustly stable against all perturbations

$$\Delta_1 \in H_\infty \quad \text{for which} \quad \|\Delta_1\|_{H_\infty} \leq \frac{1}{\gamma}$$
$$\Delta_2 \in H_\infty \quad \text{for which} \quad \|\Delta_2\|_{H_\infty} \leq \frac{1}{\gamma}$$

where γ is the minimal achievable H_∞ norm of the closed-loop system that interconnects P_{aug} with a stabilizing controller C.