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LQ, LQG.... DMCS

1.1 Analysis $\mathcal{H}_2$

There are many ways to introduce LQ, LQG and variations. This in part comes from the fact that the norm, the cost function, has signal intepretations as well as system interpretation. We choose to define it as a system norm and later interpret it as a signal norm.

Definition 1.1.1 ($\mathcal{H}_2$-norm SISO system). Consider an LTI system with impulse response $h(t)$ and transfer matrix $H(s)$. The $\mathcal{H}_2$-*norm* of the system is defined

$$\|H\|_{\mathcal{H}_2} := \sqrt{\operatorname{tr}\int_{-\infty}^{\infty} h'(t)h(t)\,\mathrm{d}t}$$

□

If the norm is finite then by Parseval's identity we may alternative express it as

$$\|H\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\operatorname{tr}\int_{-\infty}^{\infty}[H(\mathrm{i}\omega)]'H(\mathrm{i}\omega)\,\mathrm{d}\omega}.$$

Example 1.1.2. The system with transfer function $H(s) = 1/(s+a)$ (with $a > 0$) has impulse response $h(t) = \mathrm{e}^{-at}\mathbb{1}(t)$ and so we have that

$$\|H\|_{\mathcal{H}_2}^2 = \int_0^{\infty} \mathrm{e}^{-2at}\,\mathrm{d}t = \frac{1}{2a}$$

Therefore $\|H\|_{\mathcal{H}_2} = \frac{1}{\sqrt{2a}}$. The norm can also be computed through its frequency response,

$$\|H\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\mathrm{i}\omega+a|^2}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2+a^2}\,\mathrm{d}\omega = \frac{1}{2\pi}\frac{1}{a}\arctan(\tfrac{\omega}{a})\big|_{-\infty}^{\infty} = \frac{1}{2a}.$$

□

An implication is that for rational transfer matrices the norm is finite only if the transfer matrix is strictly proper. Such systems have a state space realization without direct feedthrough term

$$\begin{cases} \dot{x} = Ax + Bw \\ z = Cx \end{cases} \tag{1.1}$$

Computation of the norm given a state space realization is particularly attractive. It involves a linear equation only.

Theorem 1.1.3 (Computation via Gramians). *Consider the system* (1.1) *with A stable. Then*

$$\|H\|_{\mathcal{H}_2}^2 = \mathrm{tr}(CXC') = \mathrm{tr}(B'YB)$$

where X and Y are the controllabilty and observability Gramians, which may be obtained from the linear Lyapunov equations

$$AX + XA' + BB' = 0, \qquad A'Y + YA + C'C = 0.$$

□

Example 1.1.4. Continuing with the previous example, a realization of $H(s) = 1/(s+a)$ is $A = -a,\ B = C = 0$. Then X and Y satisfy $-2aX + 1 = 0$ and $-2aY + 1 = 0$ so that $X = Y = 1/(2a)$ and $\|H\|_{\mathcal{H}_2}^2 = CXC' = B'YB = 1/(2a)$. □

1.1.1 Signal interpretation: deterministic

A deterministic signal interpretation of the $\mathcal{H}_2$-norm is easily derived, especially for SISO systems. Indeed, if we drive a SISO system $z = Hw$ with the Dirac delta function $w(t) := \delta(t)$ then the output is the impulse response $z(t) = h(t)$. So the square of the $\mathcal{H}_2$-norm of the system equals the output energy $\int_{-\infty}^{\infty} z^2(t)\,\mathrm{d}t$ when driven by the delta function.

For MIMO systems there holds the following.

Theorem 1.1.5 (MIMO). *For a $p \times m$ transfer matrix H the square of the $\mathcal{H}_2$-norm equals*

$$\|H\|_{\mathcal{H}_2}^2 = \sum_{i=1,\dots,m} \|H_{*i}\|_{\mathcal{H}_2}^2 = \sum_{z_i = H_{*i}\delta} \int_{-\infty}^{\infty} z_i'(t) z_i(t)\,\mathrm{d}t$$

where H_{*i} is the ith column of H. □

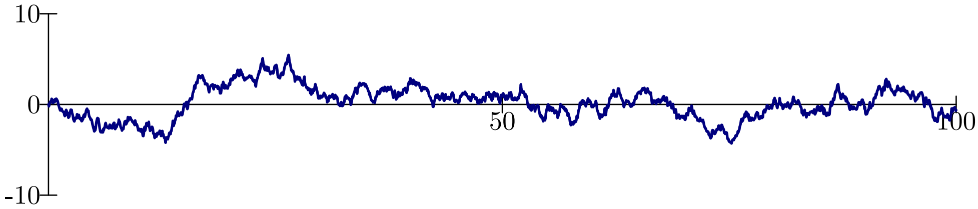

Figure 1.1: A realization of stationary noise

1.1.2 Signal interpretation: stochastic

The $\mathcal{H}_2$-norm has an important stochastic interpretation. One that is useful if stationary noise with known spectral properties affects the system. A problem with stationary noise is that Fourier transformation is not immediately feasible. For instance if we were to restrict the noisy signal $z(t)$ of Fig. 1.1 to

$$z_T(t) = \begin{cases} z(t) & t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

for some choice of T then the Fourier transform of z_T depends critically on this choice of T no matter how large we take it.

If the signal $z(t)$ is stationary—which we colloquially define to mean that its stochastic properties do not vary with time—then the power of the signal,

$$\mathbb{E}z^2(t)$$

does not depend on time and we can estimate this power by avering over time

$$\mathbb{E}z^2(t) \approx \frac{1}{T}\int_0^T z^2(v)\,\mathrm{d}v.$$

Stationarity also implies that the correlation $\mathbb{E}z(t)z^*(t+\tau)$ between $z(t)$ and $z(t+\tau)$ does not depend on time t. The correlation is a measure for the temporal relatedness in the signal and typically it decays to zero as $\tau \to \infty$. If this decay is fast enough—a technical condition—then the Fourier transform of z_T converges in the following sense.

Theorem 1.1.6 (SISO spectral density). *For any stationary signal $z(t)$ with finite power and whose correlation decays exponentially fast[1] the (power) spectral density function[2] equals*

$$\phi_z(\omega) = \lim_{T\to\infty} \mathbb{E}\frac{1}{T}|z_T(\mathrm{i}\omega)|^2 \tag{1.3}$$

where $z_T(t)$ is defined as in (1.2)*. The power $\mathbb{E}z^2(t)$ can be expressed as an integal over frequency,*

$$\mathbb{E}z^2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_z(\omega)\,\mathrm{d}\omega. \tag{1.4}$$

□

Because of (1.4) and the fact that $\phi(\omega) \geq 0$ we can think of the function $\phi_z(\omega)$ as the way that the power of z is distributed over frequency. The theorem shows us how to properly normalize the Fourier transform of $z_T(t)$. The function $\frac{1}{T}|z_T(\mathrm{i}\omega)|^2$ is known as the *periodogram* and it serves as an estimate of $\phi_z(\omega)$. Note that any information about the phase of $z_T(\mathrm{i}\omega)$ is lost in the spectral density. This is in line with the assumed stationarity of the signal, implying that time shifts do not affect its stochastic properties.

It is a classic result that the output $z = Hw$ of a stable LTI system driven by a stationary input w is again stationary and that its spectral density equals

$$\phi_z(\omega) = |H(\mathrm{i}\omega)|^2\phi_w(\omega).$$

This allows us to shape the spectral density function. The important practical implication is this:

> find a stable rational function H such that $|H(\mathrm{i}\omega)|^2$ resembles the (estimated) spectral density of a stationary z then we can think of z as the output of a system $z = Hw$ driven by a process w with unit spectral density, $\phi_w(\omega) = 1$.

A stationary process whose spectral density is 1 is known as unit intensity *white noise*. This is a somewhat illusive process because of its infinite power, but for control system design the importance is foremost that it allows to consider essentially any stationary signal as the output of a system driven by a normalized (namely white) noise process, and that we can compute the power of z via the $\mathcal{H}_2$-norm:

[1] $|\mathbb{E}z(t)z(t+\tau)| < M\mathrm{e}^{-\sigma|\tau|}$ for some $M, \sigma > 0$. This is a weak assumption.

[2] ϕ_z is defined as the Fourier transform of the correlation function, $\phi_z(\omega) = \int_{-\infty}^{\infty} \mathbb{E}z(t)z^*(t+\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau$, but for our purposes we can take (1.3) to be its definition.

Theorem 1.1.7 (power and the $\mathcal{H}_2$-norm). *The power $\mathbb{E}z^2(t)$ of the output of a stable system $z = Hw$ driven by unit intensity white noise input w equals*

$$\mathbb{E}z^2(t) = \|H\|^2_{\mathcal{H}_2}.$$

□

Example 1.1.8. Generating white noise w is problematic because of its infinite power. An approximation can be obtained by taking w piecewise constant on each $[k\varepsilon, (k+1)\varepsilon]$ and independent on disjoint intervals, all with variance $1/\varepsilon$. If ε times the bandwidth of the system H is much less than 1 then the output $z = Hw$ is close to what we want. The data of Fig. 1.1 was generated with this MATLAB code:

```
ep=0.01;
t=0:ep:100;

w=randn(size(t))/sqrt(ep);

A=-.1; B=1; C=1; D=0;               % H(s)=1/(s+0.1), bandwidth=0.1

x=0*w;                              % initialization: x(1)=0
for j=1:(length(t)-1)
  x(j+1)=x(j)+ep*(A*x(j)+B*w(j));  % ensure no smoothing of input
end
z=x;                                % z=x in our case
plot(t,z)
%z=lsim(A,B,C,D,w,t);               % should give similar results
```

The transfer function here is $H(s) = C(sI - A)^{-1}B + D = 1/(s + 0.1)$ so that

$$\|H\|_{\mathcal{H}_2} = \sqrt{5}.$$

The power of z hence is 5 which is consistent with Fig. 1.1. □

MIMO systems and signals. In case the signals z and w have more then one component the above theorem should be modified to:

Theorem 1.1.9 (MIMO spectral density). *For any stationary signal $z(t)$ with finite power and whose correlation $\mathbb{E}z(t)z'(t+\tau) \in \mathbb{R}^{n_z \times n_z}$ decays exponentially fast, the (power) spectral density*

$$\phi_y(\omega) = \lim_{T\to\infty} \mathbb{E}\frac{1}{T} z_T(\mathrm{i}\omega) z'_T(-\mathrm{i}\omega)$$

is well defined and

$$\mathbb{E}z(t)z'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_z(\omega)\,\mathrm{d}\omega.$$

In particular the sum of powers $\mathbb{E}z_i^2(t)$ of all its components quals

$$\sum_i \mathbb{E}z_i^2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\phi_z(\omega)\,\mathrm{d}\omega.$$

□

Here tr denotes the *trace* of the matrix, i.e. the sum of the diagonal entries of the matrix.