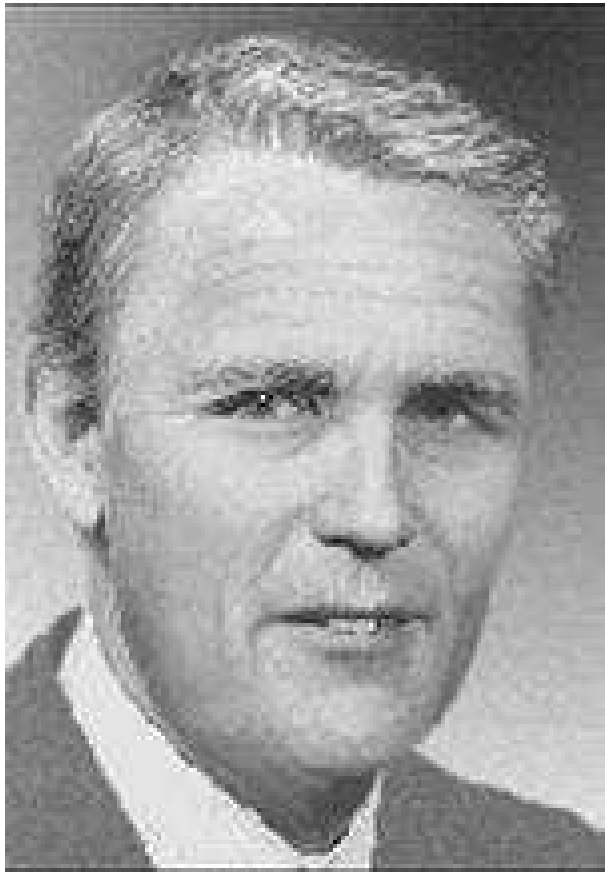
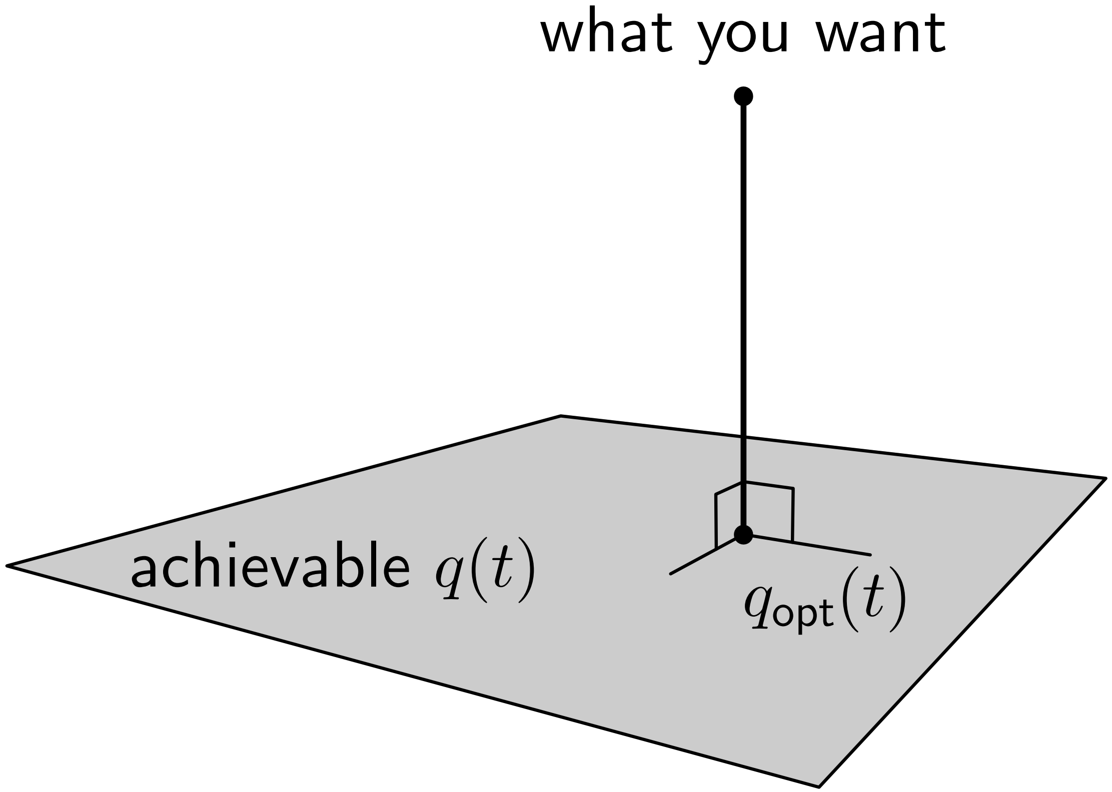
LQ, LQG and $\mathcal{H}_2$ Control System Design (Chapter 4)

1. LQ (state feedback)
2. LQG (output feedback & stochastic)
3. $\mathcal{H}_2$ (general: state/output fb., determ./stoch., time/freq.)
4. MIMO example

R. E. Kalman



LQ, LQG an H_2 are "examples" of the projection theorem:



"minimal cost $\Longleftrightarrow$ orthogonal"

Linear Quadratic (LQ) Theory

Consider

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Dx(t) \end{aligned} \qquad x(0) = x_0 \quad t \geq 0$$

Minimize the weighted sum of state and input energy

$$\mathcal{J} := \int_0^\infty \underbrace{z^\mathsf{T}(t) Q z(t)}_{\text{control error}} + \underbrace{u^\mathsf{T}(t) R u(t)}_{\text{control effort}} \, dt$$

Minimize over input u.

Cost function $\mathcal{J}$

Control error: a nonnegative definite quadratic expression such as

$$z^{\mathsf{T}}(t)Qz(t) := \sum_i q_i z_i^2(t), \qquad q_i > 0$$

Control effort: Similar expression

Trade-off form:

$$\mathcal{J} := \int_0^\infty z^{\mathsf{T}}(t)Qz(t) + \rho\, u^{\mathsf{T}}(t)Ru(t)\, dt$$

ρ large implies low gain, ρ small implies high gain

A sensible initial choice of the weighting matrices is

$$z^{\mathsf{T}}(t)Qz(t) := \sum_i \frac{z_i^2(t)}{\max z_i^2(t)}$$

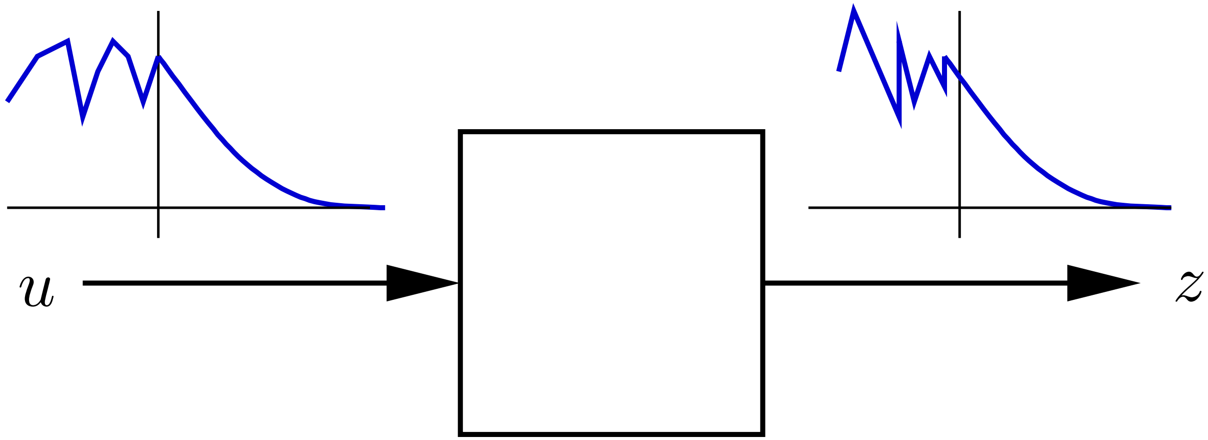
desired max

$$u^{\mathsf{T}}(t)Ru(t) := \rho \sum_i \frac{u_i^2(t)}{\max u_i^2(t)}$$

desired max

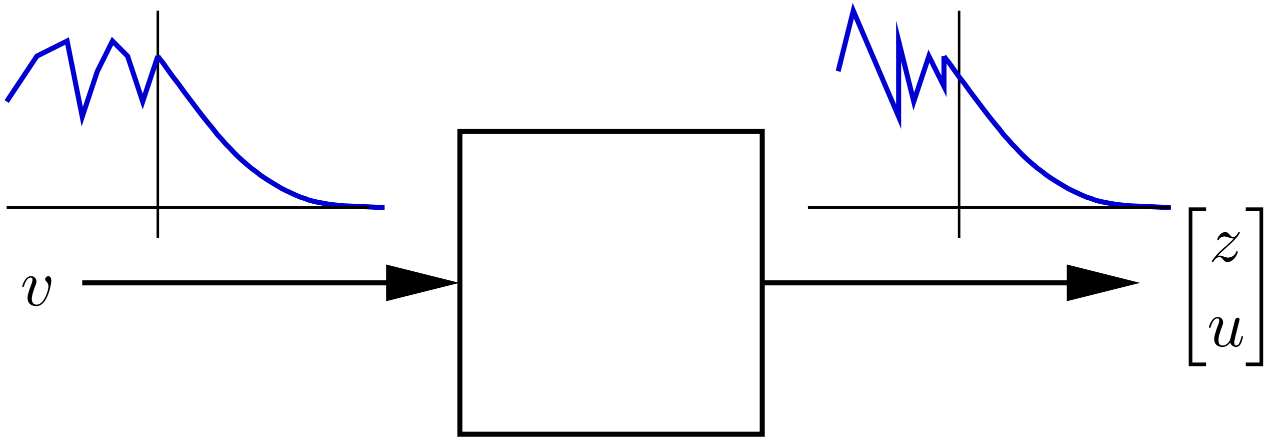
Finetune Q, R by trial and error

Example solution (mixed time/freq.)



Minimize $\int_0^\infty z^2(t) + u^2(t)\,\mathrm{d}t$ for SISO system

$$\begin{cases} \dot{x} = ax + u \\ z = x \qquad\qquad x(0) = x_0 \\ u = \qquad\quad u \end{cases}$$



For $u(t) = -fx(t) + v(t)$ this becomes

$$\begin{cases} \dot{x} = (a - f)x + v \\ z = x \\ u = -fx + v \end{cases} \qquad \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s - (a - f)} \\ \dfrac{-f}{s - (a - f)} + 1 \end{bmatrix} v$$

Is stable if $\boxed{a - f < 0}$

Parseval:

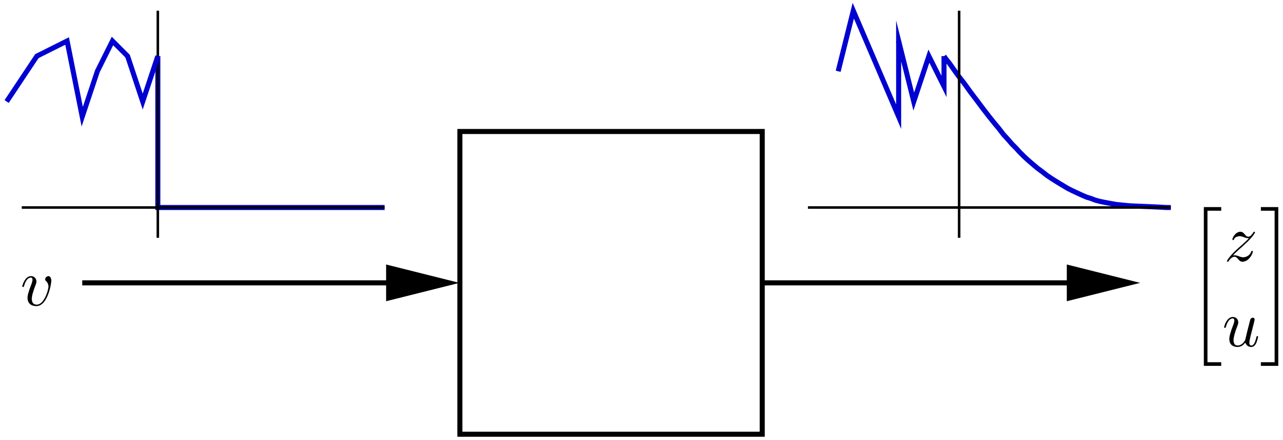
$$\int_{-\infty}^{\infty} z^2(t) + u^2(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |z(\mathrm{j}\omega)|^2 + |u(\mathrm{j}\omega)|^2\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\left|\tfrac{1}{\mathrm{j}\omega-(a-f)}\right|^2 + \left|\tfrac{-f}{\mathrm{j}\omega-(a-f)} + 1\right|^2\right) \times |v(\mathrm{j}\omega)|^2\,\mathrm{d}\omega$$

choose f:
$2af+1-f^2=0$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |v(\mathrm{j}\omega)|^2\,\mathrm{d}\omega$$

$$= \int_{-\infty}^{\infty} v^2(t)\,\mathrm{d}t$$



Cost: $\int_0^\infty z^2(t) + u^2(t) = \int_{-\infty}^\infty - \int_{-\infty}^0 = \int_{-\infty}^\infty v^2(t) -$ given past

Hence optimal is $v(t) = 0$ $\forall t \geq 0$:

$$\dot{x} = (a - f)x$$

$$z = x$$

$$u = -fx \qquad \forall t \geq 0$$

Summary: optimal is state feedback $u(t) = -fx(t)$:

- solve ARE: $2af + 1 - f^2 = 0$
- take the stabilizing solution: $a - f < 0$
- satisfies return difference equality:

$$\left|\frac{1}{\mathrm{j}\omega - (a - f)}\right|^2 + \left|\frac{-f}{\mathrm{j}\omega - (a - f)} + 1\right|^2 = 1$$

- hence also return difference inequality

$$\left|\frac{-f}{\mathrm{j}\omega - (a - f)} + 1\right|^2 \leq 1$$

Message: LQ implies certain frequency domain bounds.

General solution

Optimal solution can be implemented by feedback

$$u(t) = -Fx(t)$$

where

$$F = R^{-1}B^{\mathsf{T}}X$$

in which X is the stabilizing solution of Algebraic Riccati Equation (ARE):

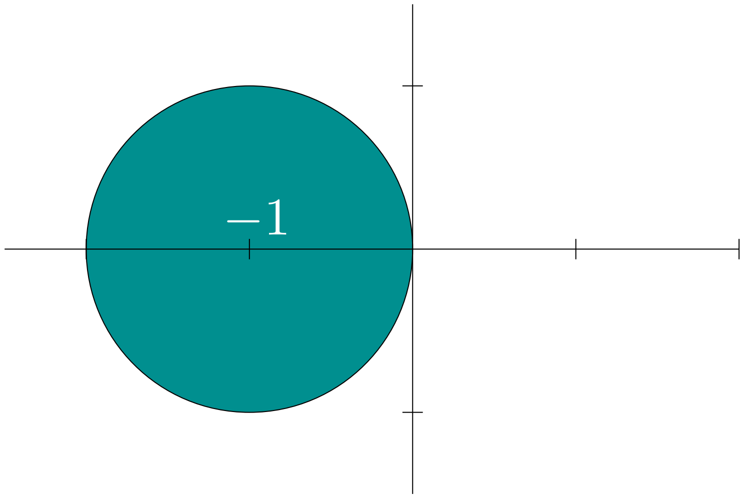
$$\boxed{A^{\mathsf{T}}X + XA + D^{\mathsf{T}}QD - XBR^{-1}B^{\mathsf{T}}X = 0}$$

Properties of the solution

Assume that $\dot{x} = Ax + Bu,\ z = Dx$ is stabilizable and detectable and that Q and R are positive definite. Then

- The ARE has finitely many solutions X
- There is a unique stabilizing X
- There is a unique nonnegative-definite X (the same)
- The input is $u(t) = -Fx(t)$ results in a asymptotically stable closed loop $\dot{x} = (A - BF)x$
- $\mathcal{J} = x^{\mathsf{T}}(0)Xx(0)$

Guaranteed stability margin (SIMO)

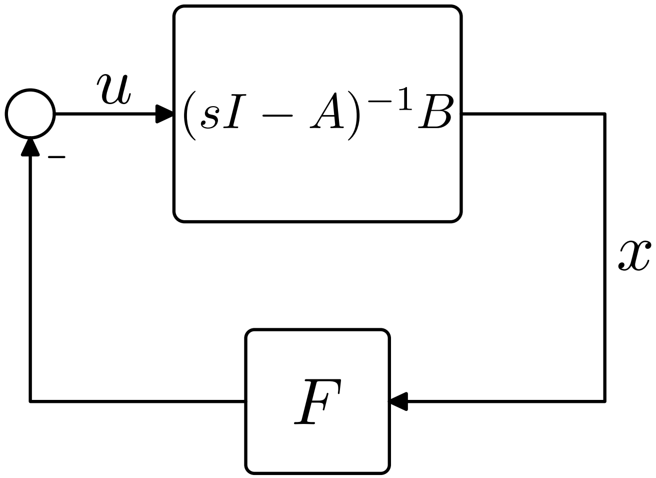


Return difference inequality:

The Nyquist plot of the loop gain $L(s) = F(sI - A)^{-1}B$ stays outside the disc with radius 1

Conclusions:

- (upward) gain margin is ∞
- phase margin is at least 60°
- modulus margin is at least 1



What can be achieved with LQ?

We may achieve

- Good transient response from nonzero initial conditions
- Good robustness

But:

- state feedback is seldom possible
- disturbances?

Stationary processes and white noise

Continuous time white noise process $w(t)$, [pragmatic, take ϵ small]:

- piecewise constant on each ϵ-interval
- mutually independent (per interval)
- identically distributed [needed in continuous time white]
- $\mathbb{E}\, w(t) = 0$ and $\mathrm{var}(w(t)) = 1/\epsilon$

Then

$$\mathbb{E}\, w(t+\tau)w(t) \approx \delta(t)$$

"Easy" to generate (random number generator)

Assume $w(t) = (w_1(t), w_2(t), \dots w_r(t))$ vector of r independent standard white noise processes.

Then filtered

$$z = Hw, \qquad H(s) \text{ stable } q \times r \text{ tr. matrix}$$

is Gaussian and satisfies

$$\text{covariance matrix:} \quad \mathbb{E}\, z(t) z^{\mathsf{T}}(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\mathrm{j}\omega) H^{\mathsf{T}}(-\mathrm{j}\omega)\, \mathrm{d}\omega$$

$$\text{power:} \quad \mathbb{E}\, z(t)^{\mathsf{T}} z(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} H(\mathrm{j}\omega) H^{\mathsf{T}}(-\mathrm{j}\omega)\, \mathrm{d}\omega$$

Covariance matrix is $q \times q$, while power is a number.

Consider $z = Hw$ and windowed $z_T(t) = z(t)\mathbb{1}_{[0,P]}(t)$.
Then the power spectral density exists:

$$\lim_{P\to\infty} \mathbb{E}\,\frac{1}{T}\,z_P(\mathrm{j}\omega)z_P^*(\mathrm{j}\omega) = H(\mathrm{j}\omega)H^{\mathsf{T}}(-\mathrm{j}\omega).$$

Pragmatic converse: given z_T use above to determine H.
Then z can be seen as output of system driven by standard white noise, $z = Hw$.

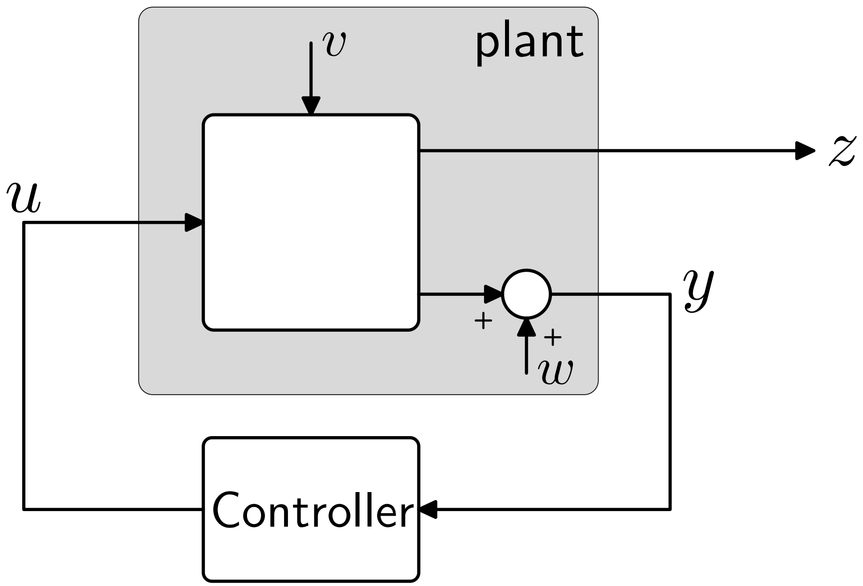
Message: "any" stationary process can be seen as the output of a system driven by white noise.

Linear Quadratic Gaussian (LQG) control

Zero mean, Gaussian, white noise, intensity V

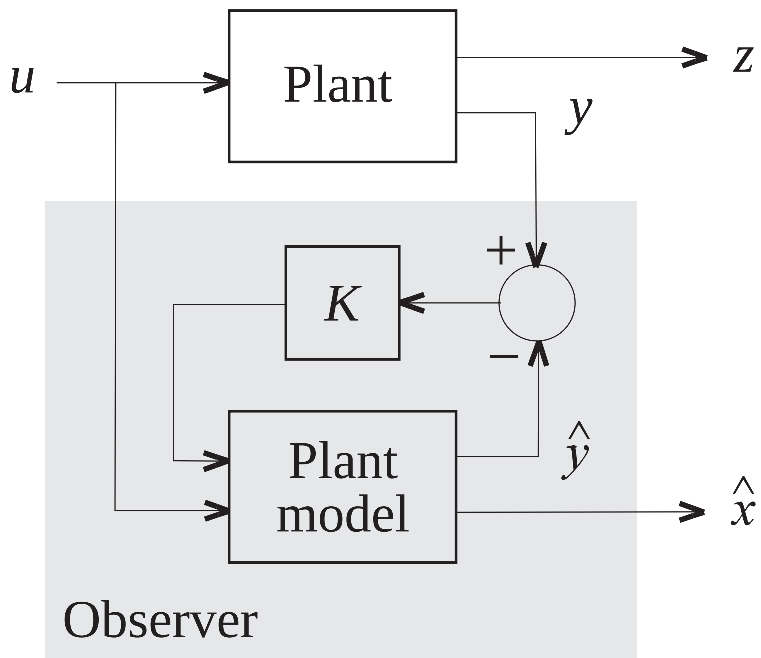
$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Gv(t) \\
y(t) &= Cx(t) + w(t) \\
z(t) &= Dx(t)
\end{aligned}$$

.. white noise, intensity W



All signals in cl. loop stable LTI system are (become) zero mean stationary with certain power:

$$\mathrm{var}(x) = \mathbb{E}\, x^{\mathsf{T}}(t)x(t)$$



Observer:

$$\dot{\hat{x}} = A\hat{x} + Bu \\ + K[y - \underbrace{C\hat{x}}_{\hat{y}}]$$

Define observation error:

$$e(t) := x(t) - \hat{x}(t)$$

Then error equation (if no noise v, w) is

$$\dot{e}(t) = (A - KC)e(t)$$

If $A - KC$ stable then $e(t) \to 0$

In the presence of noise v, w:

$$\dot{e}(t) = (A - KC)e(t) - Gv(t) + Kw(t)$$

If the error system is stable then the error covariance matrix

$$Y(t) := \mathbb{E}\, e(t)e^{\mathsf{T}}(t)$$

converges to a constant matrix Y that satisfies

$$(A - KC)Y + Y(A - KC)^{\mathsf{T}} + GVG^{\mathsf{T}} + KWK^{\mathsf{T}} = 0$$

The variance (power) of $e := x - \hat{x}$ is minimal if $K = YC^{\mathsf{T}}W^{-1}$ where Y is the stabilizing solution Y of the ARE

$$\boxed{AY + YA^{\mathsf{T}} + GVG^{\mathsf{T}} - YC^{\mathsf{T}}W^{-1}CY = 0}$$

Properties of the Kalman filter

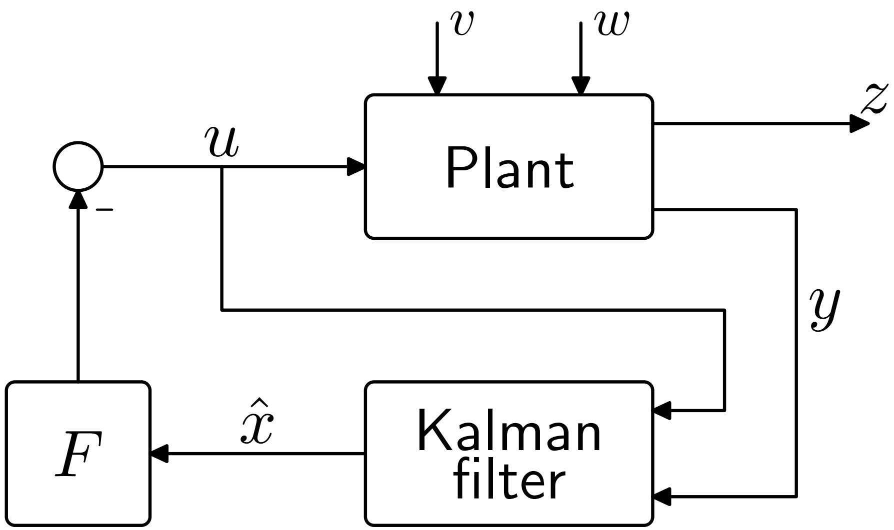
Assume that in $\dot{x} = Ax + Bu + Gv, \; y = Cx + w$ the pair (A, G) is stabilizable and (A, C) detectable, and that noise intensities

$$V\delta(\tau) := \mathbb{E}\, v(t+\tau)v^{\mathsf{T}}(t), \qquad W\delta(\tau) := \mathbb{E}\, w(t+\tau)w^{\mathsf{T}}(t)$$

are positive definite, then:

- The ARE has finitely many solutions
- There is a unique stabilizing Y (and $Y = Y^{\mathsf{T}} \geq 0$)
- The corresponding error system and filter is stable
- $\mathbb{E}\, e(t)e^{\mathsf{T}}(t) = Y$ (once stationary)
- Kalman is optimal causal filter that minimizes $\mathbb{E}\, e^{\mathsf{T}}(t)e(t)$

Optimal output feedback



Separation principle

LQG Control: interconnection of optimal Kalman filter with optimal LQ state feedback law $u = -F\hat{x}$ minimizes

$$\lim_{t \to \infty} \mathbb{E}(z^{\mathsf{T}}(t)Qz(t) + u^{\mathsf{T}}(t)Ru(t))$$

Properties of LQG solution

- The eigenvalues of the interconnected system are those of $A - BF$ and $A - KC$
- Hence the closed loop is stable if both Kalman filter and state feedback are stable

"Exact state reconstruction"

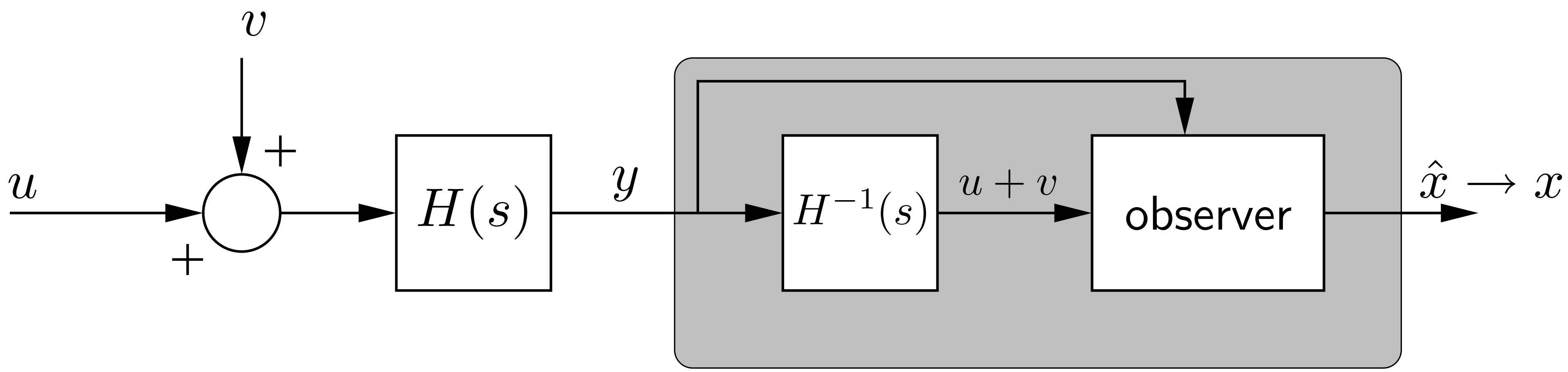
Assumptions:

- The disturbance v is additive to the control input u, that is:

$$\dot{x} = Ax + B(u + v),$$

- The open-loop plant $H(s) := C(sI - A)^{-1}B$ is square: $n_z = n_u$
- The zeros are in the plant are in the left-half complex plane, that is, $H^{-1}(s)$ has all its poles in the left-half complex plane

Then—if there is no measurement noise w—in theory the state may be reconstructed with zero error

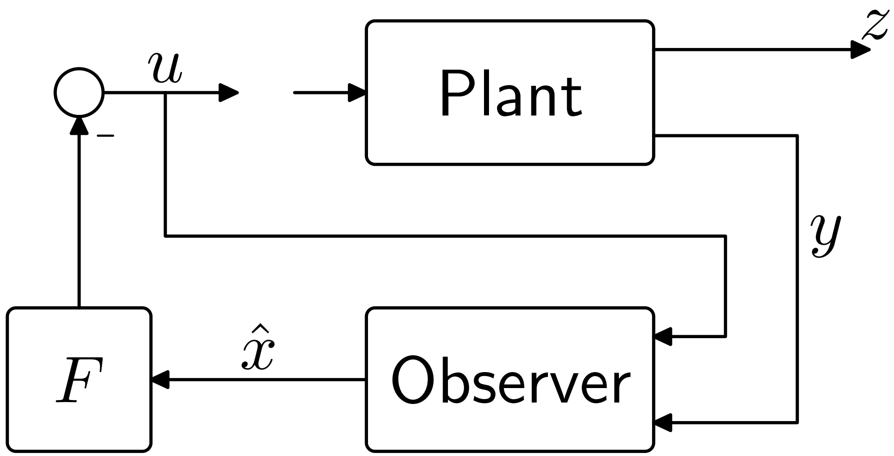


Hence

$$W \to 0 \quad \Longrightarrow \quad Y \to 0,\ K \to \infty$$

Loop transfer recovery

Consider the broken LQG loop



If the conditions for exact state reconstruction hold then

$$W \to 0 \quad \implies \quad L \to F(sI - A)^{-1}B$$

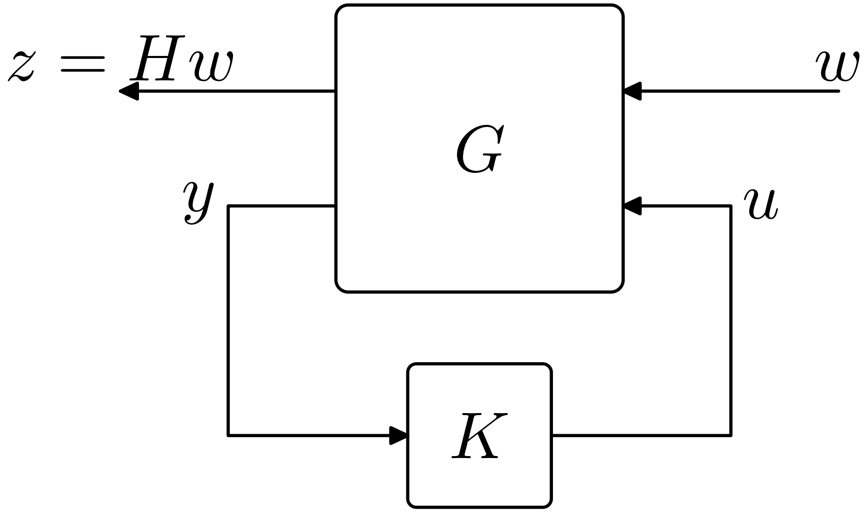
Key assumption: The plant has left-half plane zeros only

Then:

- May approach the state-feedback loop gain arbitrarily closely
- Several of the closed-loop poles and controller poles and zeros approach ∞ as we get closer
- The robustness properties of LQ is recovered

Caveat:
if the loop is broken differently then robustness may not apply

The standard $\mathcal{H}_2$ problem



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

The standard $\mathcal{H}_2$ problem is:

minimize $\|H\|_{\mathcal{H}_2}$ with respect to stabilizing K.

LQ, LQG (deterministic/stochastic/time/freq) are special cases!

$\mathcal{H}_2$-norm

Definition 1 ($\mathcal{H}_2$-norm). Causal systems

$$\|H\|_{\mathcal{H}_2} = \sqrt{\mathrm{tr} \int_{0}^{\infty} h^{\mathsf{T}}(t) h(t)\, \mathrm{d}t}$$

$$= \sqrt{\frac{1}{2\pi} \mathrm{tr} \int_{-\infty}^{\infty} H(\mathrm{j}\omega) H^{\mathsf{T}}(-\mathrm{j}\omega)\, \mathrm{d}\omega}$$

Example 2.

$$\left\| \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 = \|H_{11}\|_{\mathcal{H}_2}^2 + \|H_{12}\|_{\mathcal{H}_2}^2 + \|H_{21}\|_{\mathcal{H}_2}^2 + \|H_{22}\|_{\mathcal{H}_2}^2$$

Rewrite LQ as:

$$\dot{x} = Ax + B_1 w + Bu$$

$$\tilde{z} = \begin{bmatrix} \sqrt{Q}Dx \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{R} \end{bmatrix} u$$

$$y = x$$

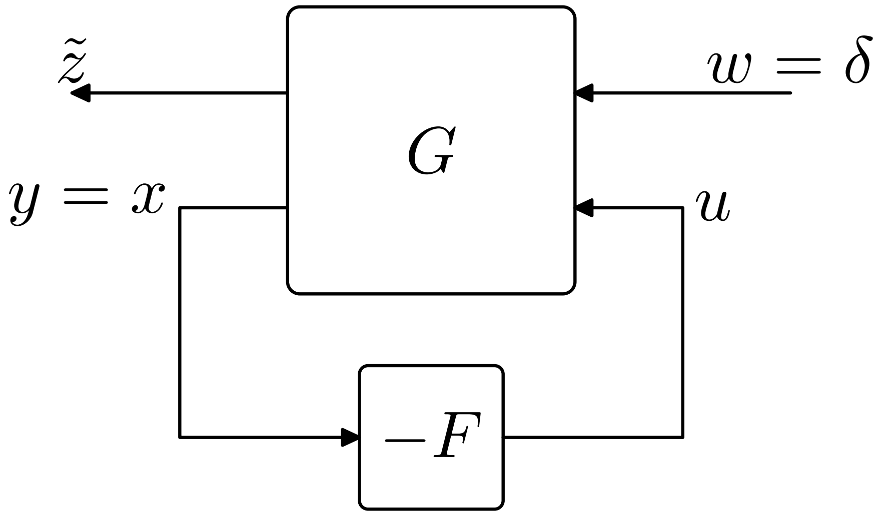
Take

$$w(t) = \delta(t)$$

so that $x(0) = B_1$.
Cost function becomes

$$\mathcal{J} = \int_0^\infty \tilde{z}'(t)\tilde{z}(t)\,\mathrm{d}t$$

This is a standard problem:



$$\mathcal{J} = \int_0^\infty h'(t)h(t)\,\mathrm{d}t$$
$$= \|H\|_{\mathcal{H}_2}^2$$

h is closed loop system impulse response (from w to $\tilde{z}$)

Rewrite LQG as:

$$\dot{x} = Ax + [G\,0]w + Bu$$

$$\tilde{z} = \begin{bmatrix} \sqrt{Q}Dx \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{R} \end{bmatrix} u$$

$$y = Cx + [0\,I]w$$

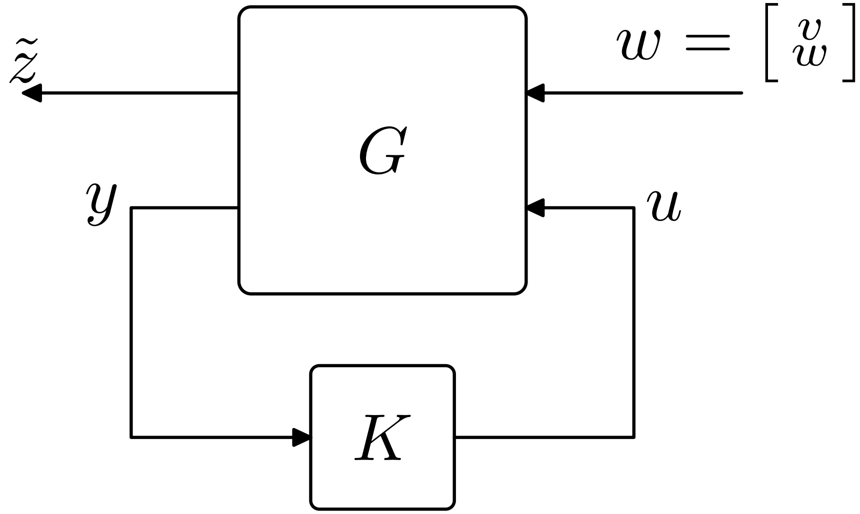
For white input

$$w := \begin{bmatrix} v \\ w \end{bmatrix}, \quad \mathbb{E}\, w(t)w^{\mathsf{T}}(t) = I$$

cost function becomes

$$\mathcal{J} = \mathbb{E}\, \tilde{z}'(t)\tilde{z}(t)$$

This is a standard problem:



$$\mathcal{J} = \frac{1}{2\pi}\,\mathrm{tr} \int_{-\infty}^{\infty} HH'\,\mathrm{d}\omega$$

$$= \|H\|^2_{\mathcal{H}_2}$$

This is the same problem as LQ.
It is nonstochastic.

Solution to standard $\mathcal{H}_2$ problem

Assume G has realization

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x \qquad\qquad\;\; + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

Assume that

- (A, B_2) stabilizable, (A, C_2) detectable
- $D_{12}^{\mathsf{T}} D_{12}$ and $D_{21} D_{21}^{\mathsf{T}}$ nonsingular

- A technicality:

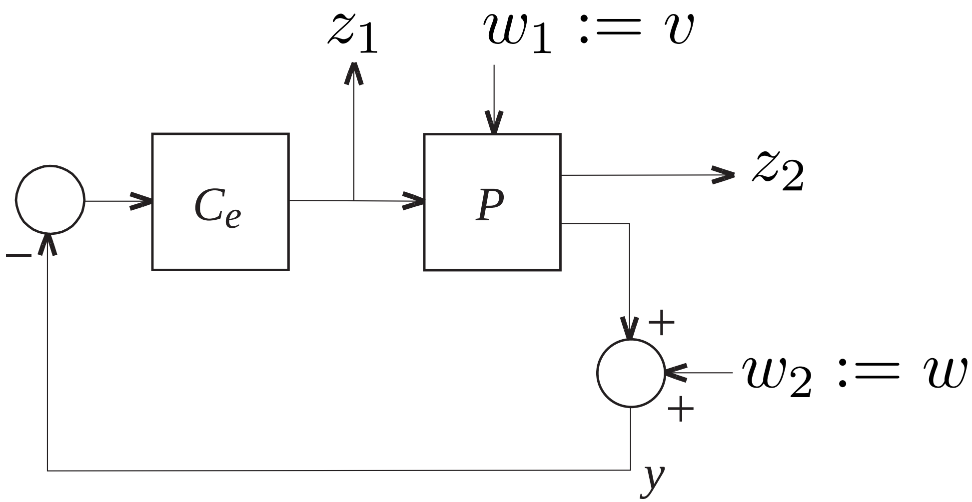
$$\begin{bmatrix} A - sI & B_1 \\ C_2 & D_{21} \end{bmatrix} \qquad \begin{bmatrix} A - sI & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

full rank for $s = \mathrm{j}\omega$

Then

- The problem is essentially an LQG problem
- hence may be solved using solutions of two Riccati equations (observer & state-feedback)
- The controller K may be seen as an observer interconnected with a state-feedback law

$\mathcal{H}_2$-interpretation of the LQG paradigm

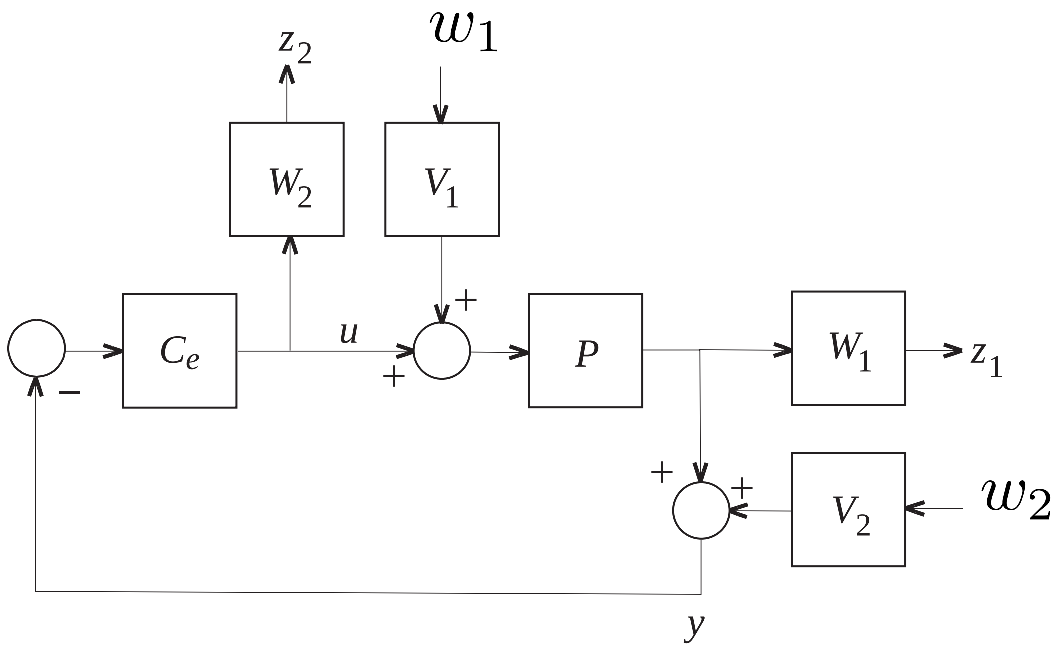


$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = H \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\min_{C_e} \|H\|_{\mathcal{H}_2}$$

Move from signal interpretation to system interpretation:

A more flexible configuration



$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = H \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

V_1, V_2 are (frequency dependent) shaping filters
W_1, W_2 are (frequency dependent) weights

Choose V_1, V_2, W_1, W_2 so that H is a sensible cost function

For the system on previous page:

$$H = \begin{bmatrix} W_1 S P V_1 & W_1 T V_2 \\ W_2 T' V_1 & W_2 U V_2 \end{bmatrix}$$

Then

$$\|H\|_2^2 = \|W_1 S P V_1\|_2^2 + \|W_1 T V_2\|_2^2 + \|W_2 T' V_1\|_2^2 + \|W_2 U V_2\|_2^2$$

This is the sum of squares of norms of all entries of H

It involves the sensitivity functions

$$\begin{aligned} S &= (I + PC_e)^{-1} & T &= (I + PC_e)^{-1} PC_e \\ U &= C_e (I + PC_e)^{-1} & T' &= (I + C_e P)^{-1} C_e P \end{aligned}$$

Design by the LQG method

Suppose

$$\dot{x} = Ax + Bu + Gv, \qquad y = Cx + w, \qquad z = Dx$$

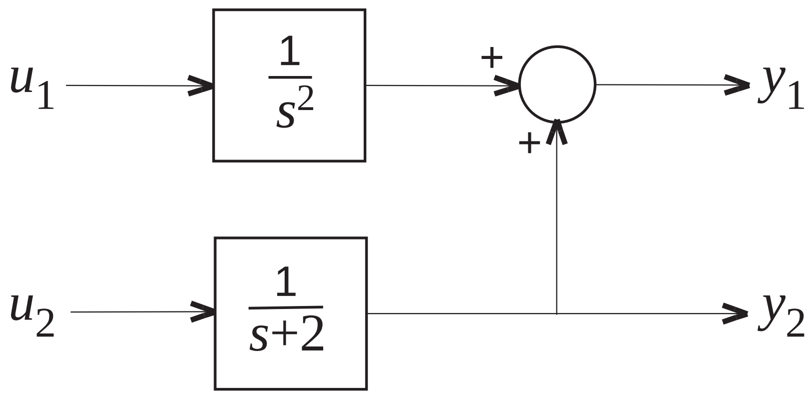
Parameter selection:

- Let $G = B$ so we may apply loop recovery
- Preferably $D = C$—that is: $y = z + w$—otherwise we deal with inferential control which is less robust
- Choose Q and choose $R = \rho R_0$ by scaling arguments as explained
- Likewise choose V and $W = \sigma W_0$ by scaling arguments

Design:

- Determine the state-feedback gain F by varying ρ and tuning Q and R_0.
  - Assess parameter selection by considering the closed-loop pole locations and transient responses
- Design the observer by making σ (hence $W := \sigma W_0$) small enough to achieve loop recovery, and by tuning V and W_0
  - Assess the design by considering the closed-loop pole locations and sensitivity functions

Example



$$P(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

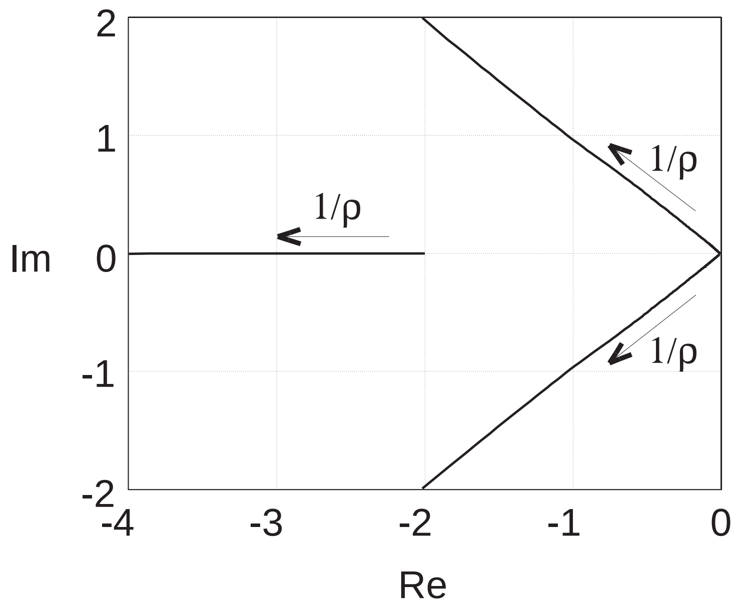
Poles at $0, 0, -2$, no zeros

Required bandwidth: 1 (rad/s) for both channels

Choice of weighting matrices

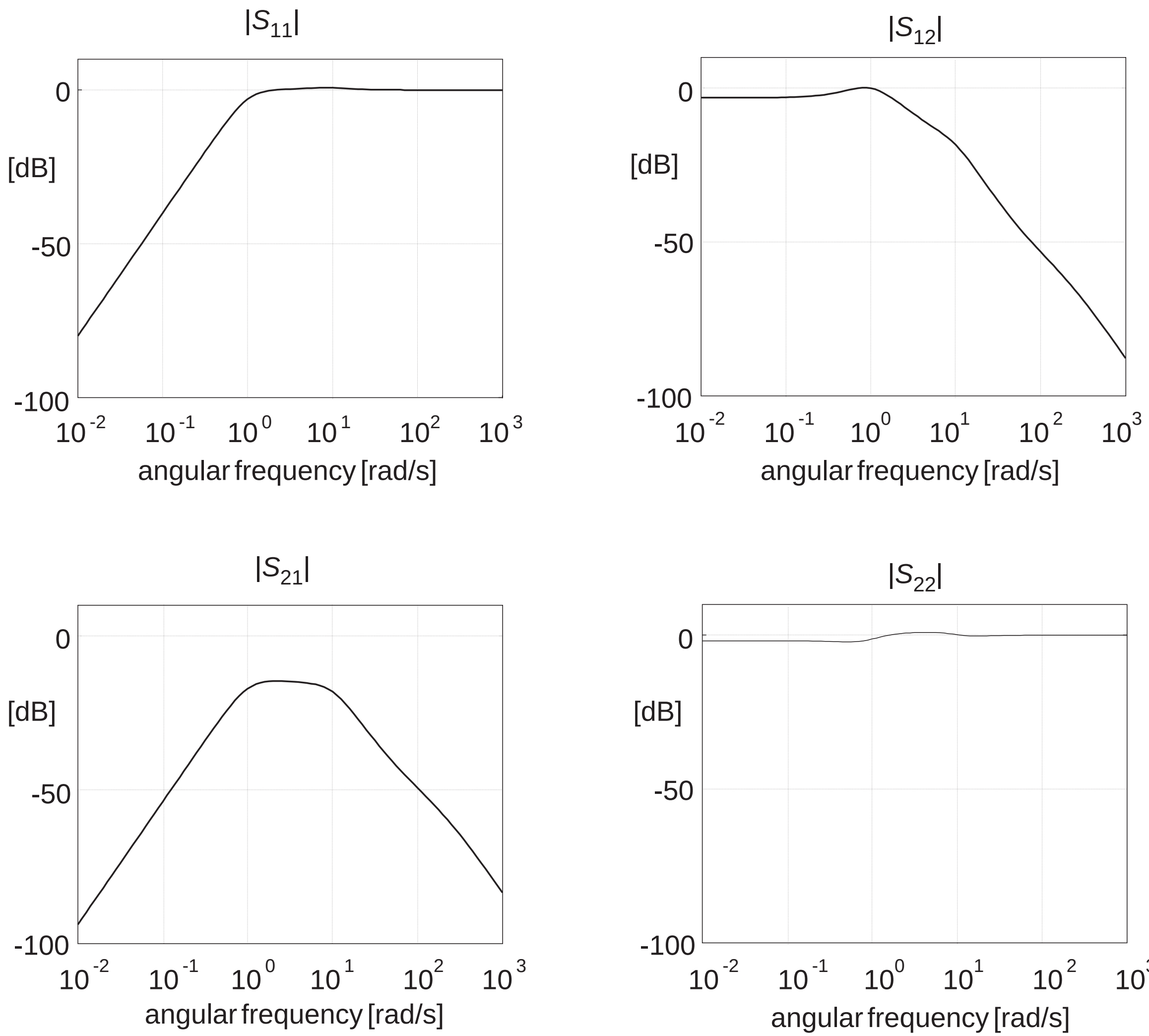
$$Q = I, \qquad R = \rho I, \qquad V = I, \qquad W = \sigma I$$

Loci of "regulator poles" (eigenvalues of $A - BF$):



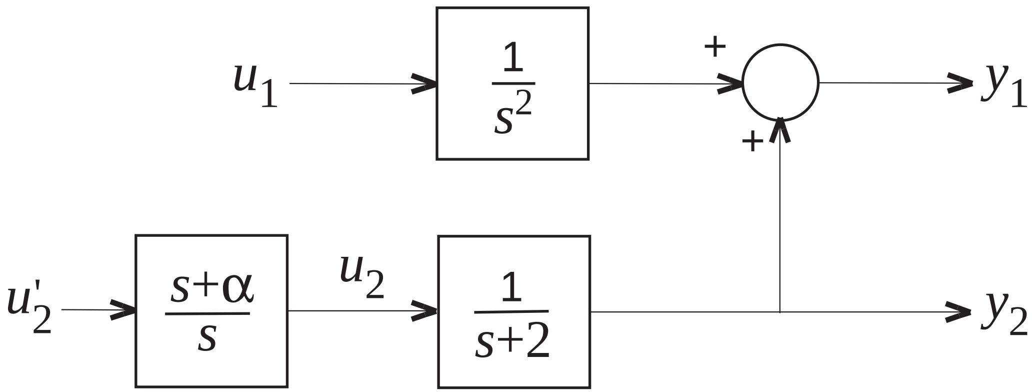
For $\rho = 0.8$ the poles are: -2.5, $-0.72 \pm \mathrm{j}0.70$

For $\sigma = 5 \times 10^{-5}$ the observer poles are: -200, $-7 \pm \mathrm{j}7$

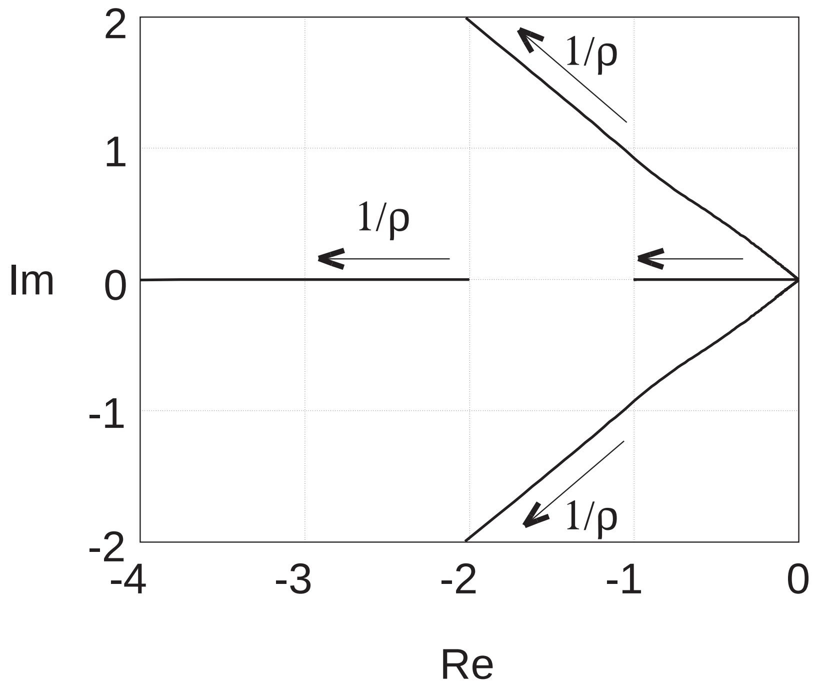


The S_{12} and S_{22} are unsatisfactory (second controller output)

Add integral action in second channel by the integrator-in-the-loop method:

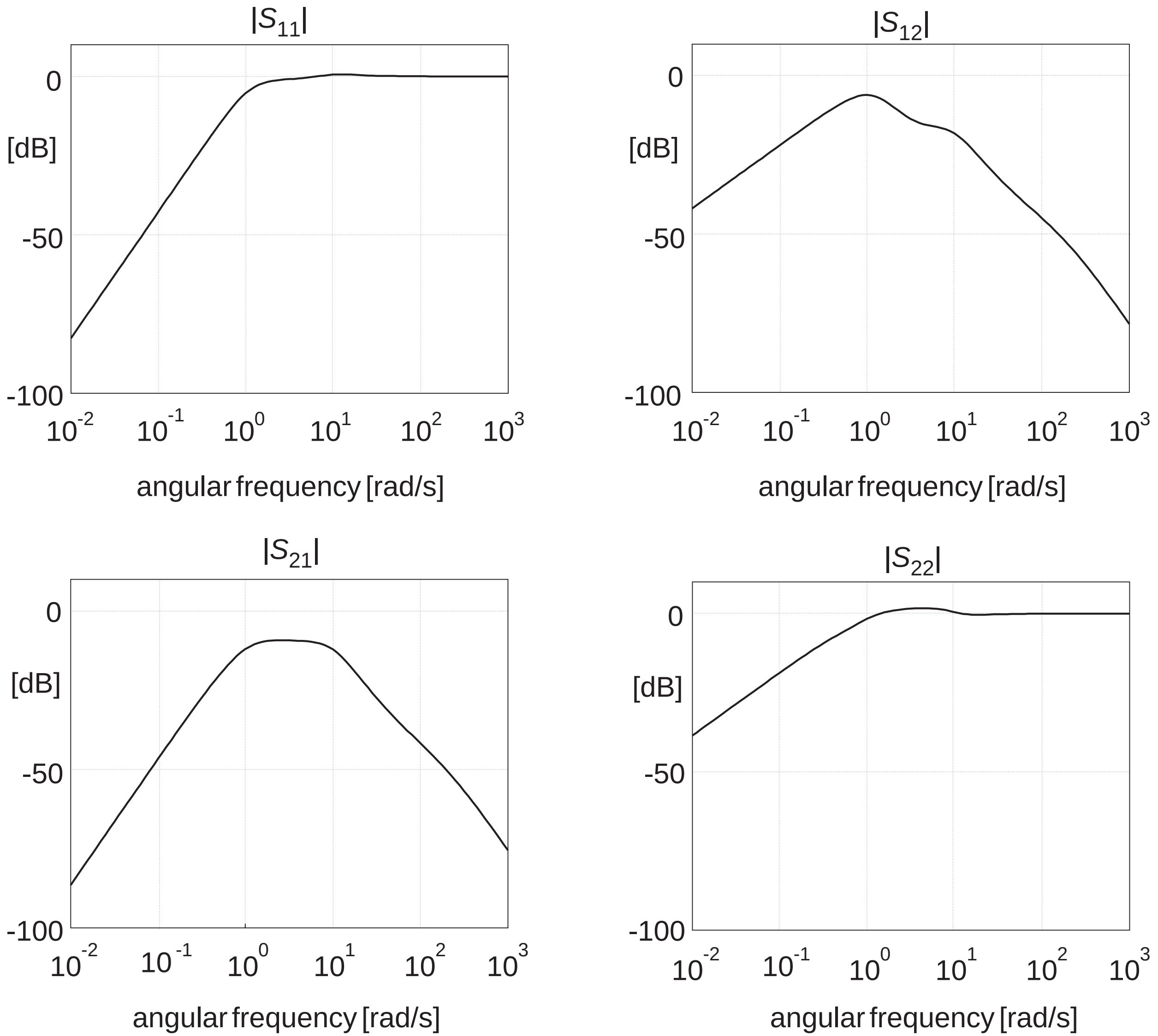


Open-loop poles at $0, 0, 0, -2$ and open-loop zero at -1

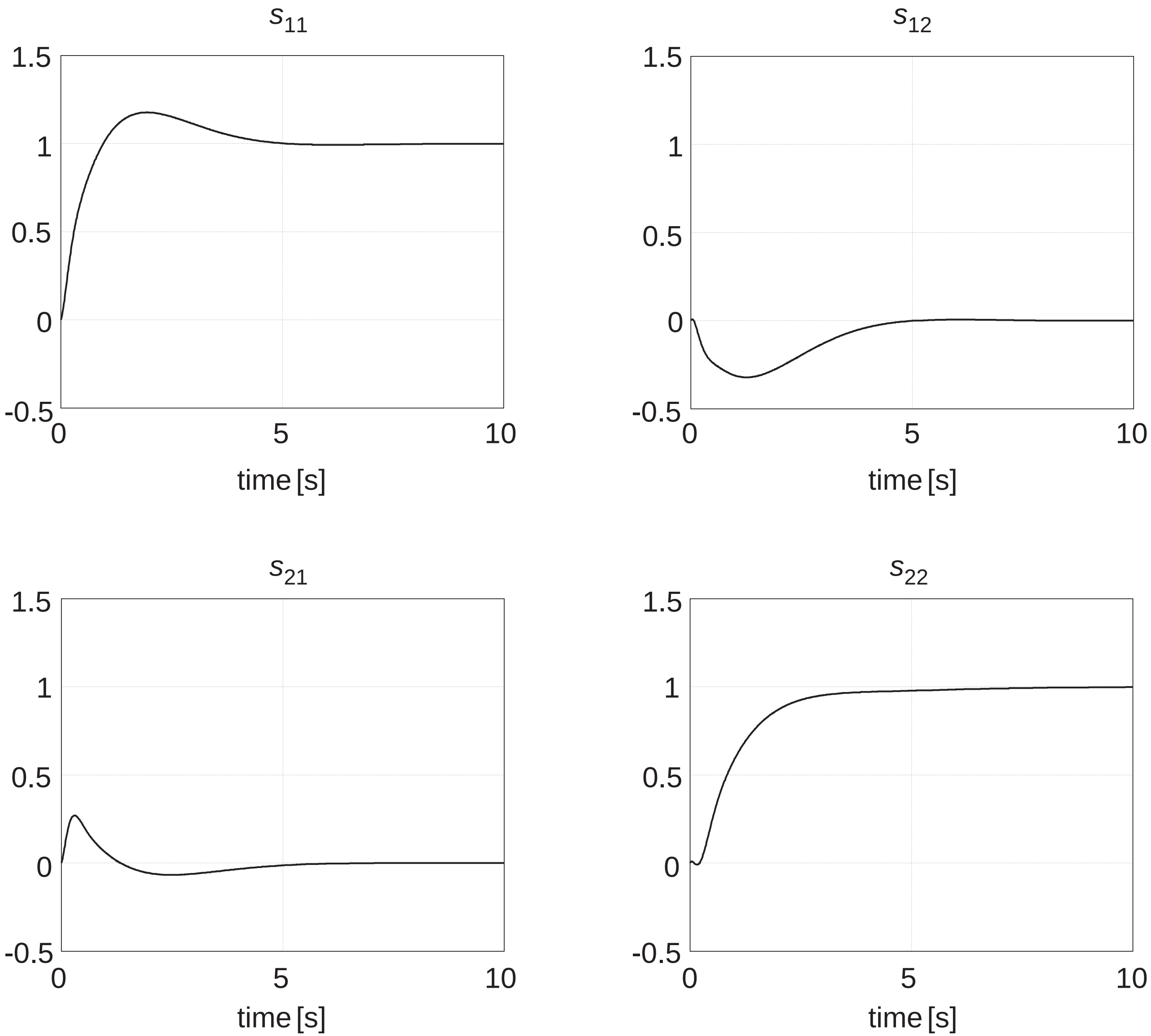


Choose $\rho = 0.5$ and $\sigma = 5 \times 10^{-5}$

Sensitivity matrix:



Step response matrix:



Uncertainty models and Robustness (Chapter 5)

'Will my system remain stable under perturbations?'

'Will my system still perform well under perturbations?'

where

- perturbations = uncertain/unmodeled dynamics
- perturbations = uncertain parameters
- ...

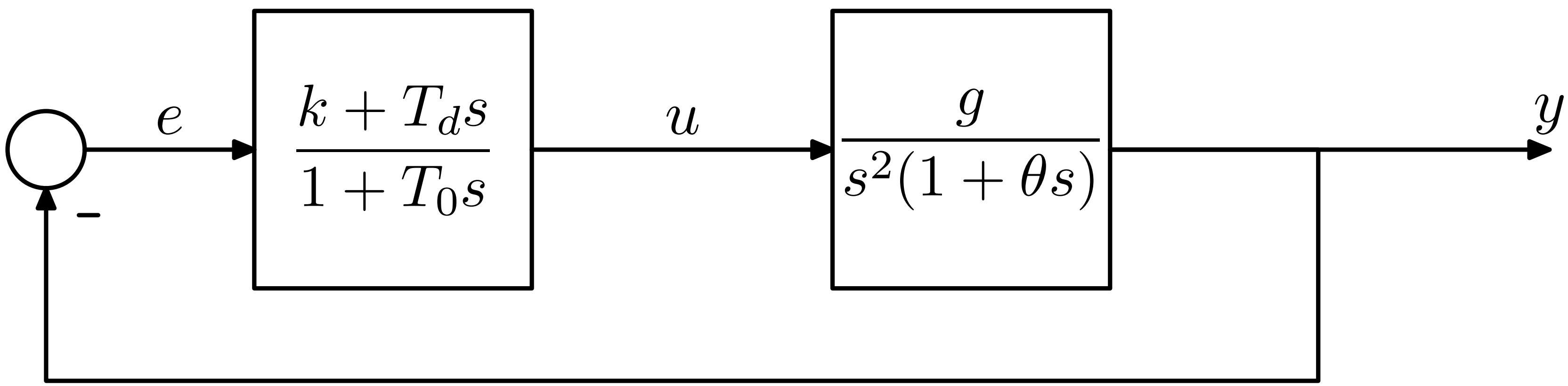
[Chapter 5 = analysis. Chapter 6 = design]

Uncertainty models and Robustness (Chapter 5)

Overview:

- Parametric robustness analysis
- The basic perturbation model (BPM)
- The small gain theorem
- Stability robustness of the BPM
- The $\mathcal{H}_\infty$ norm
- Various perturbation models
- The structured singular value μ
- Combined performance & stability robustness

Parametric uncertainty



Nominal plant $g = g_0 = 1,\ \theta = 0$

Controller $k = 1,\ T_d = \sqrt{2},\ T_0 = 1/10$

Perturbed plant $g \in [\underline{g}, \overline{g}],\ \theta \in [0, \overline{\theta}]$

Parametric uncertainty

Controller is designed for nominal plant.
Question: does it stabilize the perturbed plant?

Closed loop characteristic polynomial

$$\chi(s) = \theta T_0 s^4 + (\theta + T_0)s^3 + s^2 + gT_d s + gk$$

Question becomes: is $\chi(s)$ stable for all $g \in [\underline{g}, \overline{g}],\ \theta \in [0, \overline{\theta}]$?

Routh-Hurwitz criterion

Lemma 3. *A nonconstant polynomial*

$$p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_n, \qquad (p_i \in \mathbb{R},\ p_0 \neq 0)$$

is stable iff

- p_1 *is nonzero,*
- p_0 *and* p_1 *have the same sign*
- $q(s) := p(s) - \frac{p_0}{p_1} s(p_1 s^{n-1} + p_3 s^{n-3} + \cdots)$ *is stable*

Routh-Hurwitz criterion

In terms of coefficients:

$$\chi(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

form Hurwitz tableau with $n+1$ rows

a_0	a_2	a_4	a_6	$\cdots$
a_1	a_3	a_5	a_7	$\cdots$
b_0	b_2	b_4	$\cdots$	
b_1	b_3	b_5	$\cdots$	
$\cdots$				

Routh-Hurwitz criterion

Third row is defined as

$$\begin{bmatrix} b_0 & b_2 & b_4 & \cdots \end{bmatrix} = \begin{bmatrix} a_2 & a_4 & a_6 & \cdots \end{bmatrix} - \frac{a_0}{a_1} \begin{bmatrix} a_3 & a_5 & a_7 & \cdots \end{bmatrix}$$

Other rows similarly defined.

Lemma 4.

$\chi(s)$ *stable* $\iff$ *all elements of first column*
of Tableau have the same sign

Routh-Hurwitz criterion

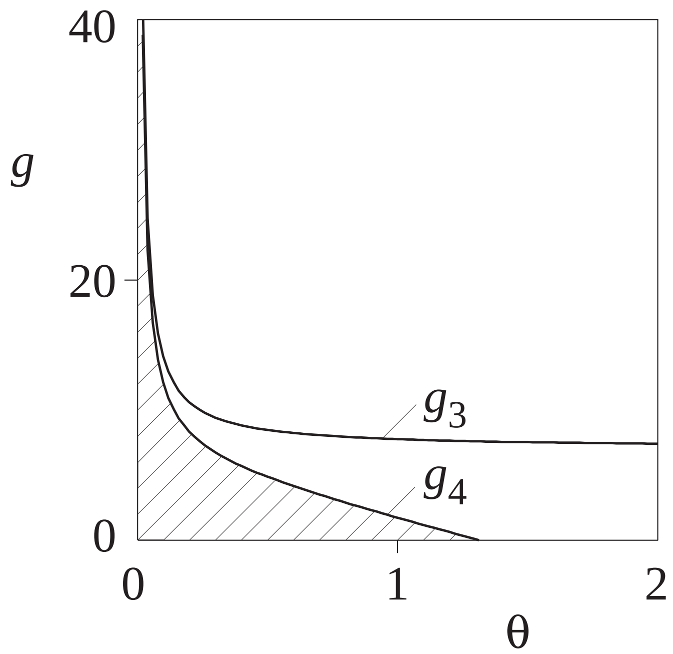
Example 5. Let $\chi(s) = \frac{\theta}{10}s^4 + (\theta + \frac{1}{10})s^3 + 1s^2 + g\sqrt{2}s + g$.

Then

$$
\begin{array}{ccc}
\hline
\dfrac{\theta}{10} & 1 & g \\
\theta + \dfrac{1}{10} & g\sqrt{2} & \\
b_0 & g & \\
b_1 & & \\
g & & \\
\hline
\end{array}
$$

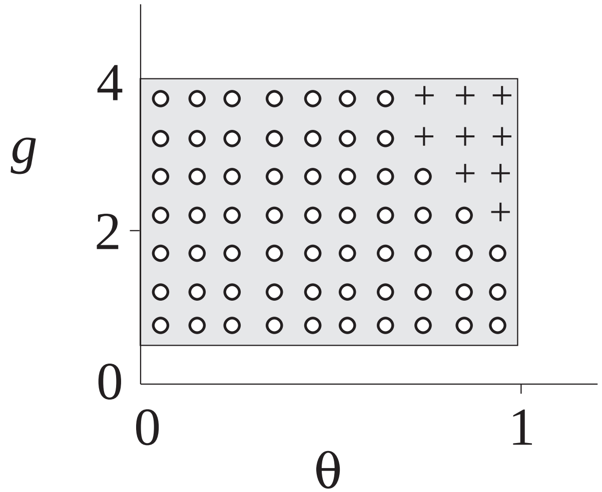
with

$$b_0 = \frac{\theta + \frac{1}{10} - \frac{\sqrt{2}}{10} g\theta}{\theta + \frac{1}{10}}, \qquad b_1 = \frac{(\theta + \frac{1}{10} - \frac{\sqrt{2}}{10} g\theta)\sqrt{2} - (\theta + \frac{1}{10})^2}{\theta + \frac{1}{10} - \frac{\sqrt{2}}{10} g\theta} g.$$



Routh-Hurwitz criterion: numerically easy but analytically difficult

Gridding



- simple
- computationally intensive
- fine grid?

Kharitonov's theorem

All the polynomials

$$\chi(s) = \chi_0 + \chi_1 s + \chi_2 s^2 + \cdots + \chi_n s^n, \qquad \underline{\chi}_i \leq \chi_i \leq \overline{\chi}_i$$

are stable iff the four Kharitonov polynomials are stable

$$\begin{aligned}
k_1(s) &= \underline{\chi}_0 + \underline{\chi}_1 s + \overline{\chi}_2 s^2 + \overline{\chi}_3 s^3 + \underline{\chi}_4 s^4 + \underline{\chi}_5 s^5 + \overline{\chi}_6 s^6 + \cdots \\
k_2(s) &= \overline{\chi}_0 + \overline{\chi}_1 s + \underline{\chi}_2 s^2 + \underline{\chi}_3 s^3 + \overline{\chi}_4 s^4 + \overline{\chi}_5 s^5 + \underline{\chi}_6 s^6 + \cdots \\
k_3(s) &= \overline{\chi}_0 + \underline{\chi}_1 s + \underline{\chi}_2 s^2 + \overline{\chi}_3 s^3 + \overline{\chi}_4 s^4 + \underline{\chi}_5 s^5 + \underline{\chi}_6 s^6 + \cdots \\
k_4(s) &= \underline{\chi}_0 + \overline{\chi}_1 s + \overline{\chi}_2 s^2 + \underline{\chi}_3 s^3 + \underline{\chi}_4 s^4 + \overline{\chi}_5 s^5 + \overline{\chi}_6 s^6 + \cdots
\end{aligned}$$

Kharitonov's theorem

Example 6.

$$\chi(s) = \theta T_0 s^4 + (\theta + T_0)s^3 + s^2 + gT_d s + gk$$

with

$$g \in [\underline{g}, \overline{g}], \quad \theta \in [0, \overline{\theta}] \qquad\qquad (k = 1,\ T_d = \sqrt{2},\ T_0 = 1/10)$$

Then

$$\chi(s) \subset [0, \overline{\theta}/10]s^4 + [10, 10 + \overline{\theta}]s^3 + s^2 + [\underline{g}\sqrt{2}, \overline{g}\sqrt{2}]s + [\underline{g}, \overline{g}]$$

Stability of RHS implies stability of LHS.
Instability of RHS does not imply instability if LHS [conservative]

The edge theorem

- Kharitonov assumes all coefficients χ_k to vary independently
- Generally not the case
- What is generally the case is this:

$$\chi(s) = \phi_0(s) + \sum_{i=1}^{N} a_i \phi_i(s), \qquad a_i \in [\underline{a}_i, \overline{a}_i]$$

for given polynomials $\phi_i(s)$.

The edge theorem

Example 7.

$$
\begin{aligned}
\chi(s) &= \theta T_0 s^4 + (\theta + T_0)s^3 + s^2 + gT_d s + gk \\
&= (T_0 s^3 + s^2) + \theta(T_0 s^4 + s^3) + g(T_d s + k) \\
&= \phi_0(s) + \theta\phi_1(s) + g\phi_2(s)
\end{aligned}
$$

The edge theorem

Theorem 8. *All polynomials*

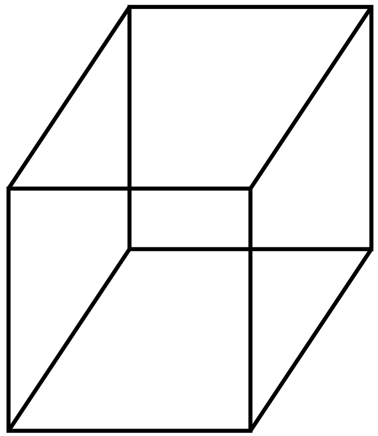
$$\chi(s) = \phi_0(s) + \sum_{i=1}^{N} a_i \phi_i(s), \qquad a_i \in [\underline{a}_i, \overline{a}_i]$$

are stable iff all edges are stable.

Edge:

fix $N-1$ parameters to maximal or minimal value
vary remaining parameter

The edge theorem



Example 9. $N = 3$: There are $N2^{N-1} = 12$ edges.

Edge theorem is practical if we can test stability of edges

Curse of dimension (or not: Sideris 1991)

Testing edges

Edge:

$$p(s) = \lambda p_1(s) + (1 - \lambda)p_2(s), \qquad \lambda \in [0, 1]$$

Lemma 10 (Białas'). *All polynomials are stable iff*

- $p_2(s)$ *is stable*
- *all real eigenvalues of* $P_1 P_2^{-1}$ *are strictly positive*

where P_1 and P_2 are the Hurwitz matrices of $p_1(s)$ and $p_2(s)$.

Testing edges

$$q(s) = q_0 s^n + q_1 s^{n-1} + q_2 s^{n-2} + \cdots + q_n,$$

has Hurwitz matrix defined as

$$Q = \begin{bmatrix} q_1 & q_3 & q_5 & \cdots & \cdots & \cdots \\ q_0 & q_2 & q_4 & \cdots & \cdots & \cdots \\ 0 & q_1 & q_3 & q_5 & \cdots & \cdots \\ 0 & q_0 & q_2 & q_4 & \cdots & \cdots \\ 0 & 0 & q_1 & q_3 & q_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Bummer!

The methods

- Routh-Hurwitz
- Gridding
- Kharitonov
- Edge theorem
- ... many variations..

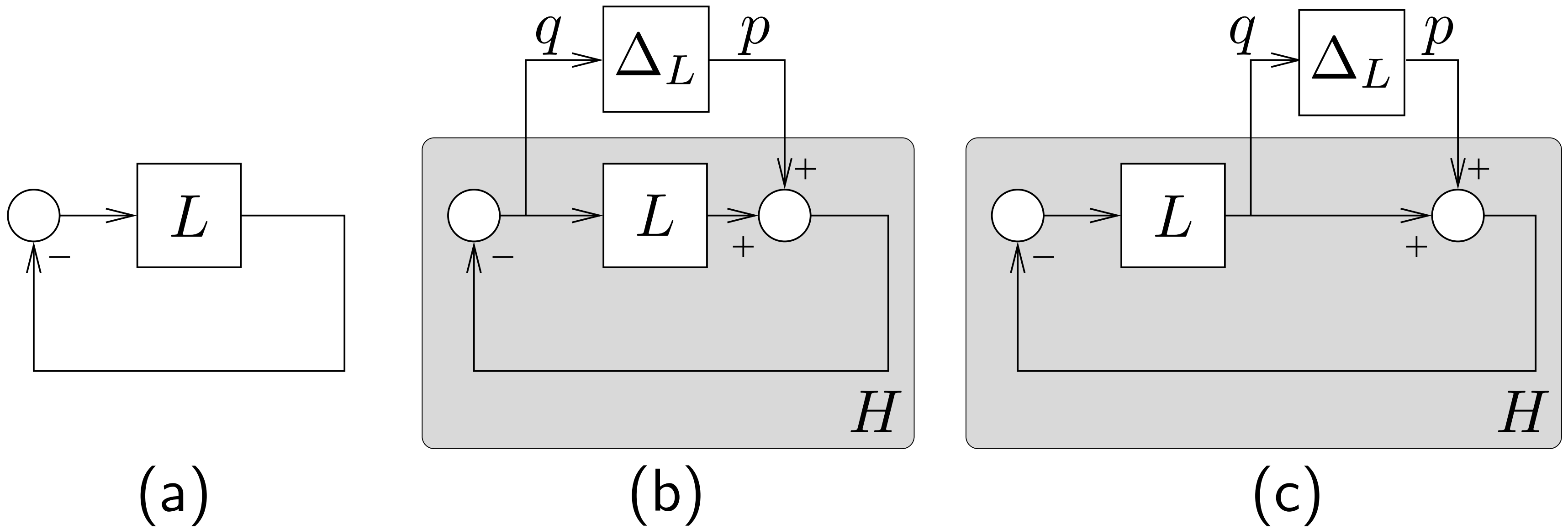
are useful for analysis but not for design

The basic perturbation model (BPM)

- Towards a theory for uncertainties / perturbations, both parametric and dynamic
- .. that is useful for analysis
- .. as well as design

Principle idea (assumption): model uncertainty separate from rest

The basic perturbation model (BPM)



(a) -

(b) Additive uncertainty: $H = -S$

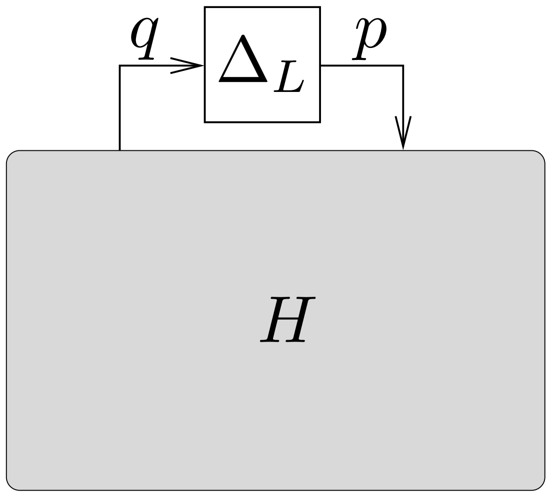
(c) Multiplicative uncertainty: $H = -T$

H is known as the interconnection matrix

The basic perturbation model (BPM)

Claim:

If K stabilizes nominal loop then perturbed loop is internally stable iff 'collapsed' system is internally stable:



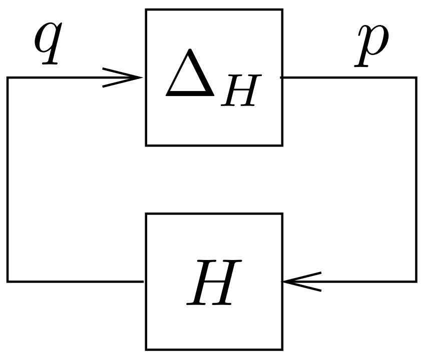
So study only 'collapsed' system

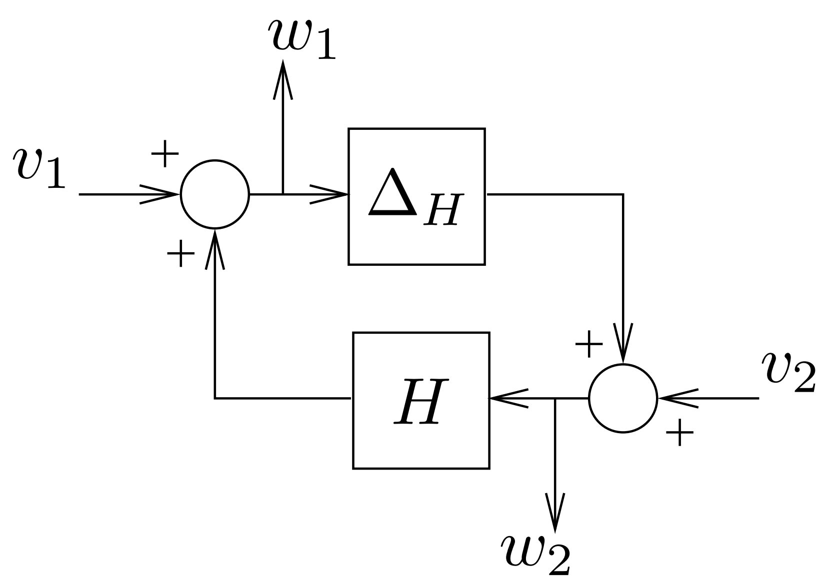
This is a canonical form for studying stability robustness

The basic perturbation model (BPM)

Typical route:

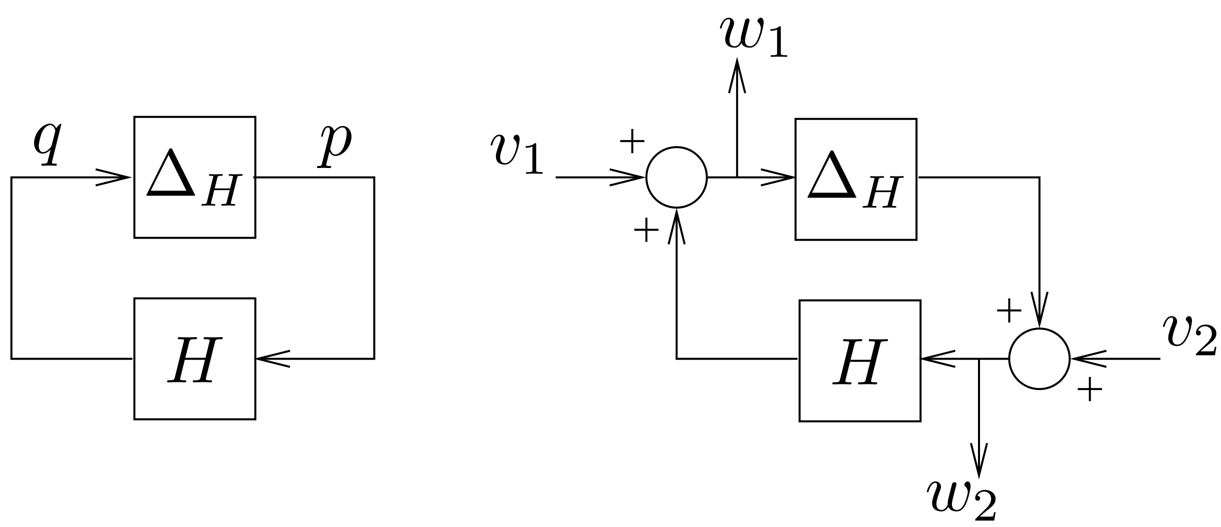
- Choose K to stabilize nominal system
- Then interconnection matrix H is stable
- Then perturbed system is internally stable iff simple BPM is:





(a) Basic perturbation model. (b) Arrangement for internal stability

Stability of the BPM

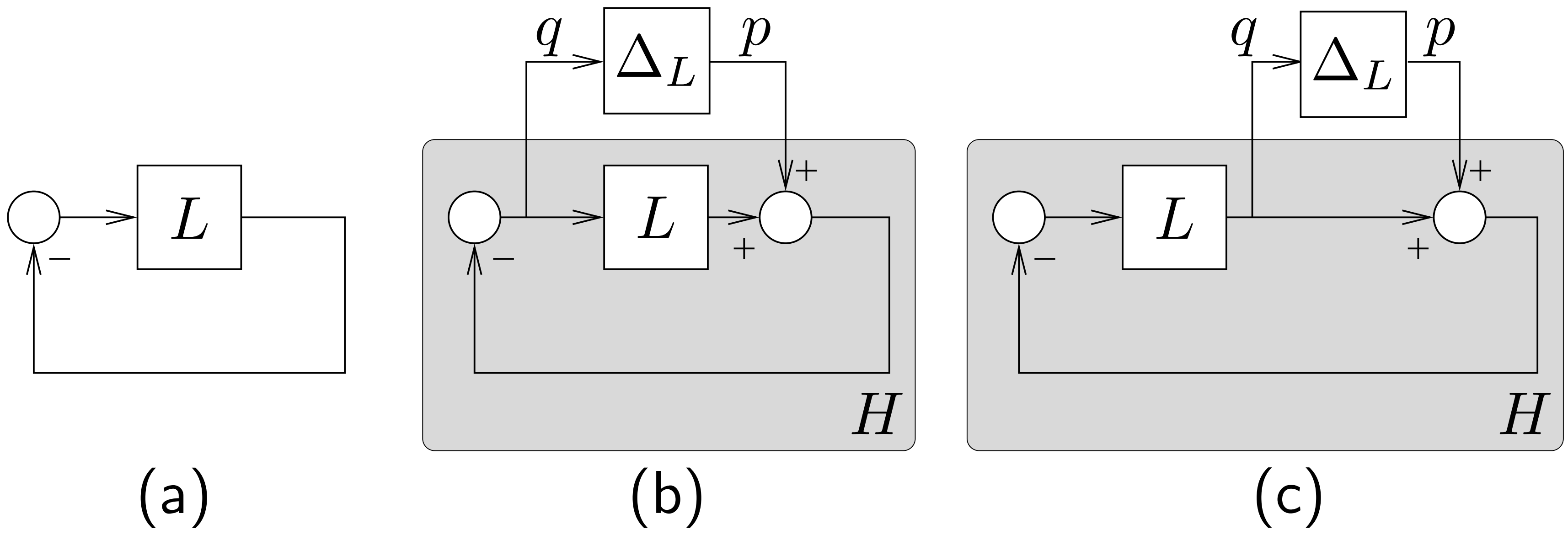


Suppose H and Δ_H are stable. TFAE:

1. BPM is internally stable
2. $(I - H\Delta_H)^{-1}$ is stable
3. $(I - \Delta_H H)^{-1}$ is stable

Stability of the BPM

Example 11.



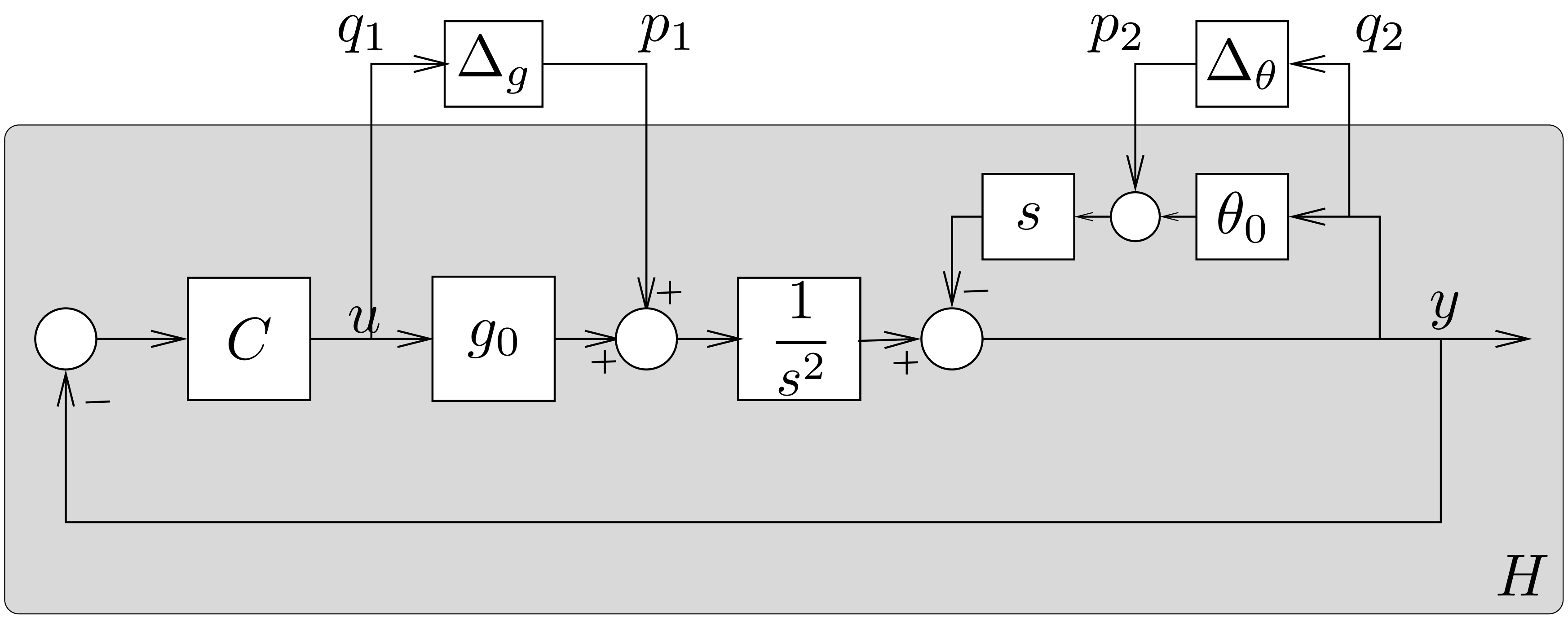
(a) -

(b) stable if $(I + S\Delta_L)^{-1}$ is stable

(c) stable if $(I + T\Delta_L)^{-1}$ is stable

Stability of the BPM

Works for parametric uncertainties

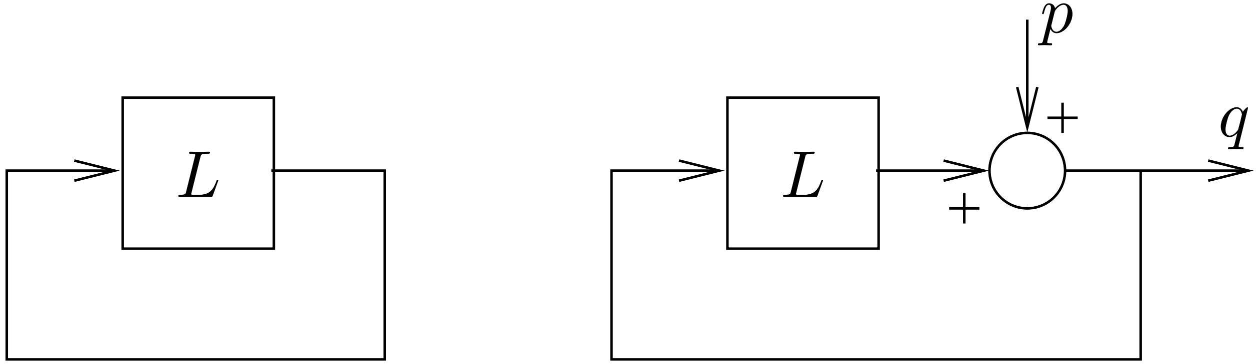


Stability of the BPM

Towards simpler conditions for CL stability

because $(I - H\Delta_H)^{-1}$ is not easy to work with

The small gain theorem



(a) (b)

For SISO we know (Nyquist) that

$$L \text{ stable, } |L(\mathrm{j}\omega)| < 1 \quad \Longrightarrow \quad \text{CL is stable}$$

Generalization to MIMO: 'L is small then CL is stable??'
Need to know what 'small' means for MIMO

The small gain theorem

Induced norm:

$$\|L\| := \sup_u \frac{\|Lu\|_{\mathcal{U}}}{\|u\|_{\mathcal{U}}}$$

with, for instance,

$$\|u\|_{\mathcal{U}} := \sqrt{\int_{-\infty}^{\infty} u^T(t)u(t)dt}$$

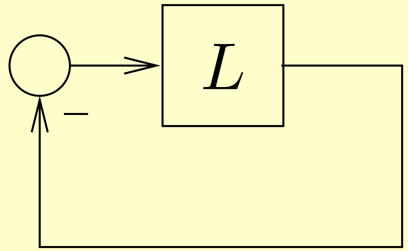
Induced norms have submultiplicative property

$$\|L_1 L_2\| \leq \|L_1\| \|L_2\|$$

The small gain theorem

Theorem 12. *For any induced norm*

$$\|L\| < 1 \quad \Longrightarrow \quad \textit{CL is stable}$$



Proof (sketch)

$$(I - L)^{-1} = I + L + L^2 + L^3 + L^4 + \cdots$$

RHS converges because $\|L^k\| \leq \|L\|^k$ goes to zero exponentially fast

The $\mathcal{H}_\infty$ norm

$$\|L\|_\infty := \sup_u \frac{\|Lu\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$$

$$\|u\|_{\mathcal{L}_2} := \sqrt{\int_{-\infty}^{\infty} u^*(t)u(t)\,dt}$$

Then (if LTI):

$$\|L\|_\infty = \begin{cases} \infty & \text{if unstable} \\ \sup_{\omega\in\mathbb{R}} \overline{\sigma}(L(\mathrm{j}\omega)) & \text{if stable} \end{cases}$$

with

$$\overline{\sigma}(M) = \sqrt{\lambda_{\mathsf{max}}(M^*M)}$$ the largest singular value of M

The $\mathcal{H}_\infty$ norm

So small gain theorem says that

$$\|L\|_\infty < 1 \quad \Longrightarrow \quad \text{CL stable}$$

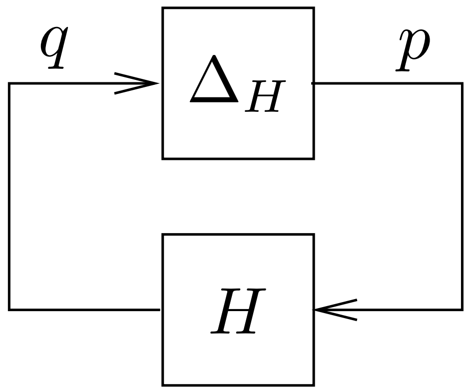
i.e.

$$L \text{ stable and } \sup_{\omega \in \mathbb{R}} \overline{\sigma}(L(\mathrm{j}\omega)) < 1 \quad \Longrightarrow \quad \text{CL stable}$$

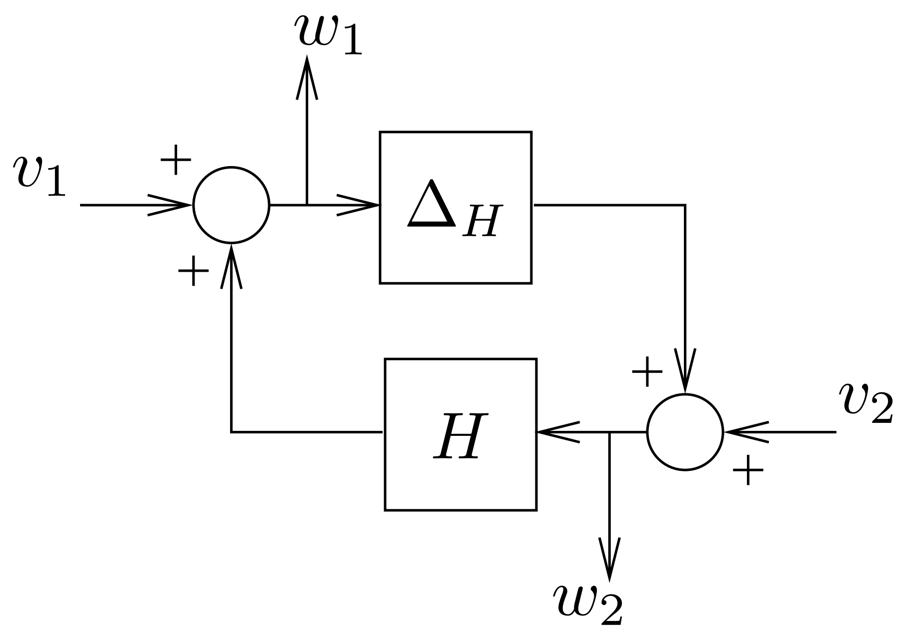
For SISO this is familiar (Nyquist):

$$L \text{ stable and } \sup_{\omega \in \mathbb{R}} |L(\mathrm{j}\omega)| < 1 \quad \Longrightarrow \quad \text{CL stable}$$

Implications small-gain for BPM



(a) Basic perturbation model.



(b) Arrangement for internal stability

.. can take $L = H\Delta_H$ or $L = \Delta_H H$

Implications small-gain for BPM

$$\text{CL stable} \quad \text{if} \quad \|H\Delta_H\|_\infty < 1$$

$$\text{if} \quad \|H\|_\infty \, \|\Delta_H\|_\infty < 1$$

$$\text{if} \quad \|\Delta_H\|_\infty < \frac{1}{\|H\|_\infty}$$

Last one separates known (H) from unknown (Δ)

Implications small-gain for BPM

Yet another sufficient condition:

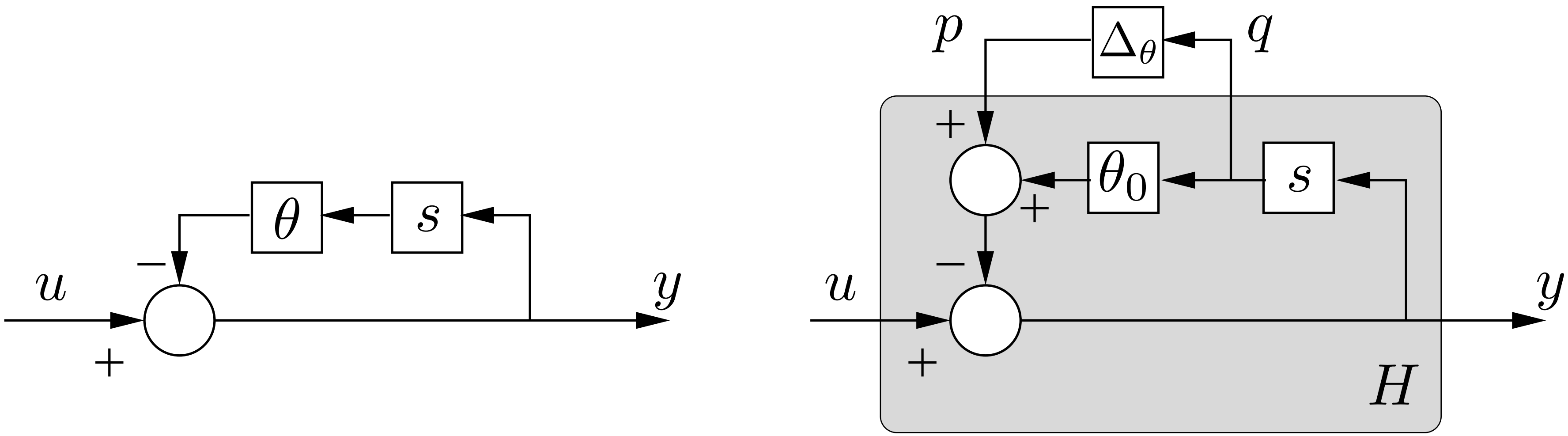
CL is stable for all $\|\Delta_H\|_\infty \leq 1$ if $\|H\|_\infty < 1$

Surprise:

Lemma 13. *The BPM is internally stable for all* Δ_H *with* $\|\Delta_H\|_\infty \leq 1$ *iff* $\|H\|_\infty < 1$

Implications small-gain for BPM

$$P(s) = \frac{1}{1+s\theta}$$



Example 14.

$$
\begin{aligned}
P(s) &= \frac{1}{1+s\theta}, \qquad \theta = \theta_0 + \Delta_\theta, \qquad \theta_0 > 0 \\
H(s) &= \frac{-s}{1+s\theta_0} \\
\|H\|_\infty &= \sup_\omega |H(\mathrm{j}\omega)| = \frac{1}{\theta_0}
\end{aligned}
$$

Small gain condition

$$
\|\Delta_H\|_\infty < \frac{1}{\|H\|_\infty} \quad \Longrightarrow \quad \text{CL stable}
$$

guarantees stability for

$$
\Delta_\theta \in (-\theta_0, \theta_0)
$$

Of course it is stable for every

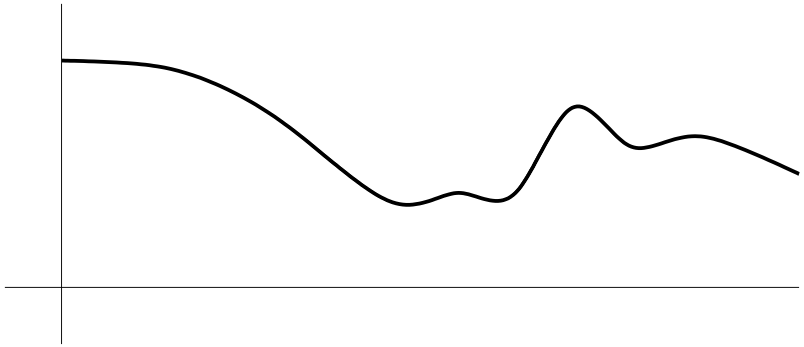
$$\Delta_\theta \in (-\theta_0, \infty)$$

It is also stable for dynamic $\Delta(s) = \dfrac{\theta_0/2}{s+1}$

Implications small-gain for BPM

Example 15 (Dynamic uncertainty)**.**

Suppose (magnitude of) P equals

$P(s) =$ 

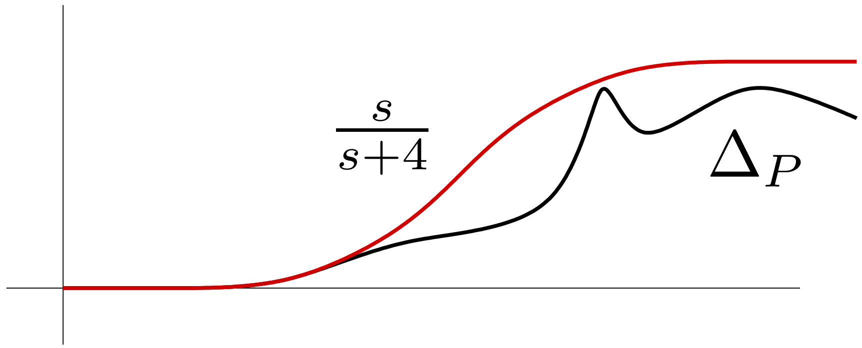
This we model as

$$P(s) = \frac{1}{s+1} + \Delta_P(s)$$

with

Implications small-gain for BPM

$$\Delta_P(s) = P(s) - \frac{1}{s+1} =$$

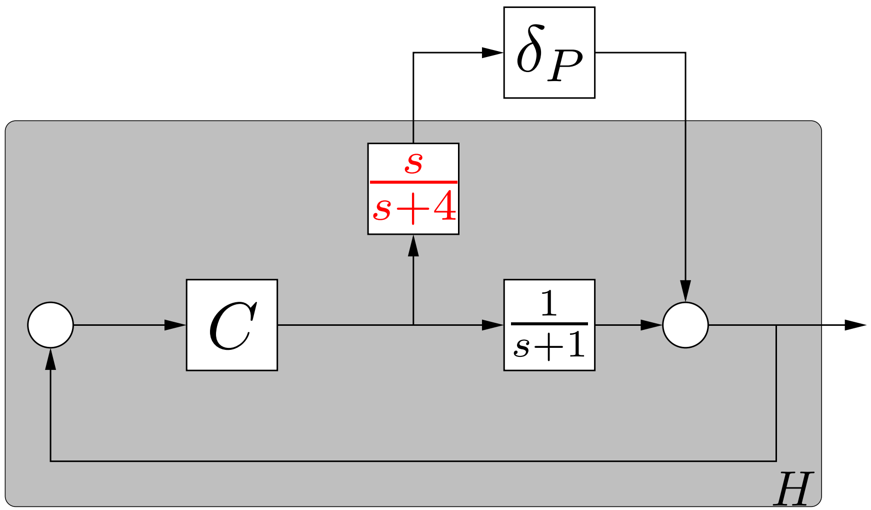


So

$$P(s) \subset \frac{1}{s+1} + \frac{s}{s+4}\delta_P(s), \qquad \|\delta_P\|_\infty \leq 1.$$

Implications small-gain for BPM

Pull out δ_P (not Δ_P)



Then

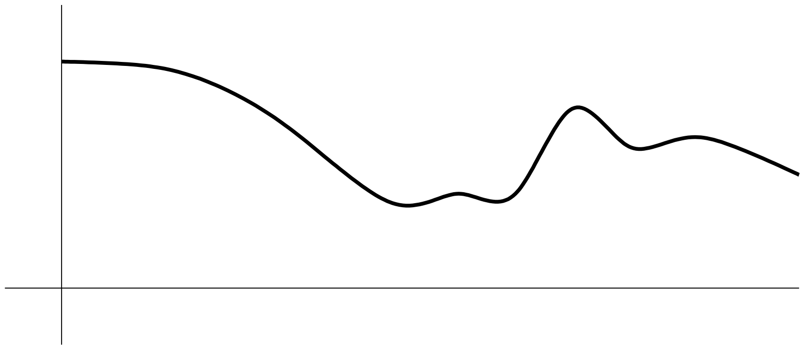
$$H(s) = -\frac{s}{s+4}C(s)S(s)$$

Robust stability is guaranteed if $\| -\frac{s}{s+4}C(s)S(s)\|_\infty < 1$

Various perturbation models

We can model

$$P(s) =$$



as

$$\Delta = P - \frac{1}{s+1}$$

$$\Delta = \frac{P - \frac{1}{s+1}}{\frac{1}{s+1}}$$

- Additive uncertainty: $P(s) = \frac{1}{s+1} + \Delta(s)$
- Multiplicative (proportional): $P(s) = \frac{1}{s+1}(1 + \Delta(s))$

Various perturbation models

Question: Suppose

$$P_\delta(s) = \frac{1}{s + \frac{1}{10} + \delta} \qquad \delta \in [-1, 1]$$

How to model the uncertainty?

- Additive: $\Delta = P_\delta - P_0$. Not wise!
- Proportional (multiplicative) $\Delta = (P_\delta - P_0)/P_0$. Not wise!
- Numerator-denominator perturbations is okay (natural here):

$$P_\delta = \frac{N + V\Delta_N W_2}{D + V\Delta_D W_1}$$

If

$$P_\delta = \frac{N + V\Delta_N W_2}{D + V\Delta_D W_1}$$

then (see exercise)

$$H = -\begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix} D^{-1} V, \qquad \Delta = \begin{bmatrix} \Delta_D & \Delta_N \end{bmatrix}$$

For

$$P_\delta = \frac{N + V\Delta_N W_2}{D + V\Delta_D W_1} = \frac{1}{s + \frac{1}{10} + \delta}$$

we may take

$$N = 1, \quad V = 1, \quad W_2 = \emptyset, \quad D = s + 1/10, \quad W_1 = 1$$

Then

$$H = -W_1 S D^{-1} V = -\frac{1}{s + \frac{1}{10}} S, \qquad\qquad \Delta_D = \delta$$

Nonlinear and/or time-varying perturbations

Stability is guaranteed if

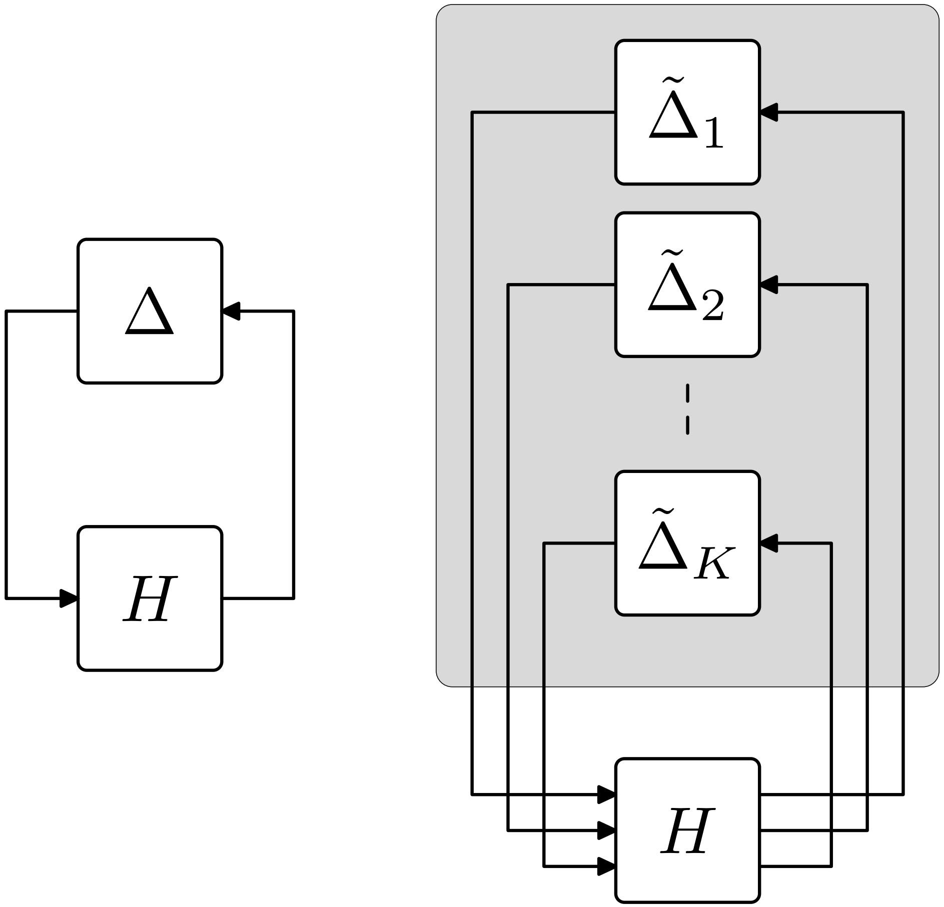
$$\sup_u \frac{\|Lu\|_{\mathcal{U}}}{\|u\|_{\mathcal{U}}} < 1 \qquad \text{also for time-varying systems}$$

and also for infinite dimensional systems

Stability is guaranteed if L is contraction

$$\sup_u \frac{\|Lu_1 - Lu_2\|_{\mathcal{U}}}{\|u_1 - u_2\|_{\mathcal{U}}} < 1 \qquad \text{also for nonlinear systems}$$

The structured singular value



$$\Delta = \begin{bmatrix} \tilde{\Delta}_1 & 0 & \cdots & 0 \\ 0 & \tilde{\Delta}_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \tilde{\Delta}_K \end{bmatrix}$$

The structured singular value

with

- $\tilde{\Delta}_i = \delta_i I$, $\delta_i \in \mathbb{R}$
- $\tilde{\Delta}_i = \delta_i I$, with δ_i a stable transfer function
- $\tilde{\Delta}_i$ stable transfer matrix

And assume (mild assumption) the uncertainties are scaled so that

- $\delta_i \in [-1, 1]$
- $|\delta_i(\mathrm{j}\omega)| \leq 1$
- $\|\tilde{\Delta}_i\|_\infty \leq 1$

The structured singular value

the scaling simply means that

$$\|\Delta\|_\infty \leq 1, \quad \Delta(\mathrm{j}\omega) \in \mathcal{D} := \begin{bmatrix} \mathbb{R}I & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C}I & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C}^{\cdot\times\cdot} & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix}$$

structure

Definition 16.
BPM is robustly stable (RS) if stable for all $\|\Delta\|_\infty \leq 1, \Delta(\mathrm{j}\omega) \in \mathcal{D}$

The structured singular value

Small gain theorem says:

$$\|H\|_\infty = \sup_\omega \overline{\sigma}(H(\mathrm{j}\omega)) < 1 \quad \Longrightarrow \quad \text{robustly stable}$$

This is rather onservative
We need another, less conservative, criterion

The structured singular value

As always assume $H, \tilde{\Delta}$ stable.

Recall that for unstructered $\Delta(\mathrm{j}\omega) \in \mathbb{C}^{m \times m}$:

$$\text{RS} \iff \|H\|_\infty = \sup_\omega \overline{\sigma}(H(\mathrm{j}\omega)) < 1$$

We would like to have an alternative μ to the largest singular value $\overline{\sigma}$ such that

$$\text{RS} \iff \sup_\omega \mu(H(\mathrm{j}\omega)) < 1$$

The structured singular value

Definition 17. Let $M \in \mathbb{C}^{n\times m}$ and $\mathcal{D} \subseteq \mathbb{C}^{m\times n}$.

The structured singular value $\mu(M)$ of M (w.r.t. $\mathcal{D}$) is defined as

$$\frac{1}{\mu(M)} = \inf_{\Delta\in\mathcal{D},\ \det(I-M\Delta)=0} \overline{\sigma}(\Delta)$$

If $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathcal{D}$ then $\mu(M) = 0$.

“ $\frac{1}{\mu} =$ the smallest Δ that singularizes”

The structured singular value

So (sketch):

$$\text{RS} \iff \det(I - \Delta H(\mathrm{j}\omega)) \neq 0 \text{ for all } \omega,\ \Delta \in \mathcal{D},\ \overline{\sigma}(\Delta) \leq 1$$

$$\iff \text{"smallest } \Delta \text{ that singularizes } > 1\text{"}$$

$$\iff \mu(H(\mathrm{j}\omega)) < 1 \text{ for all } \omega$$

$$\iff \sup_\omega \mu(H(\mathrm{j}\omega)) < 1$$

compare with singular value

Summary:

$$\text{RS} \iff \sup_{\omega} \mu(H(\mathrm{j}\omega)) < 1$$

Aint that great, or is it:

- compute μ and the problem is solved!
- unfortunately computation is μ is hard
- Okay, then only test if $\mu < 1$ holds
- The test that $\mu < 1$ is hard [NP-hard]

The structured singular value

Example 18.

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} \delta & 0 \\ 0 & \epsilon \end{bmatrix} \in \mathcal{D} = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix}$$

To compute $\mu(M)$ we look at

$$\begin{aligned} \det(I - M\Delta) &= \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \delta & 2\epsilon \\ 2\delta & 4\epsilon \end{bmatrix}\right) \\ &= (1-\delta)(1-4\epsilon) - 4\delta\epsilon \\ &= 1 - \delta - 4\epsilon \end{aligned}$$

Then

$$\det(I - M\Delta) = 0 \text{ for } \Delta = \begin{bmatrix} 1 - 4\epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$$

The smallest singularizing Δ is when

$$\delta = \epsilon = \frac{1}{5}$$

so:

$$\frac{1}{\mu(M)} := \inf_{\Delta \in \mathcal{D},\ \det(I - M\Delta) = 0} \overline{\sigma}(\Delta) = \frac{1}{5}$$

implying $\mu(M) = 5$.

Unfortunately computation of μ usually is practically impossible: Need upper bounds

Upper bound of the structured singular value

Some properties

- $\mu(aM) = |a|\mu(M)$ for all $a \in \mathbb{R}$
- $\mu(M) \leq \overline{\sigma}(M)$
- $\mu(M) = \mu(DMD^{-1})$ for any

$$D \in \begin{bmatrix} \mathbb{C}^{\cdot\times\cdot} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C}^{\cdot\times\cdot} & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{C}I & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix}$$

Upper bound of the structured singular value

$$\mu(M) \leq \underbrace{\inf_D \overline{\sigma}(DMD^{-1})}_{\nu(M)}$$

$\nu(M)$ is easy to compute (convex)

Upper bound of the structured singular value

Lemma 19 (Remarkable)**.** *If no real blocks then in practice*

$$0.8\nu(M) \leq \mu(M) \leq \nu(M)$$

Lemma 20 (Incredible)**.** *If no real blocks then*

$$\text{RS against time varying } \Delta \quad \Longleftrightarrow \quad \inf_{D\ldots} \|DHD^{-1}\|_\infty < 1$$

$$\text{RS against “almost” LTI } \Delta \quad \Longleftrightarrow \quad \sup_{\omega} \inf_{D_\omega\ldots} \overline{\sigma}(D_\omega H(\mathrm{j}\omega) D_\omega^{-1}) < 1$$

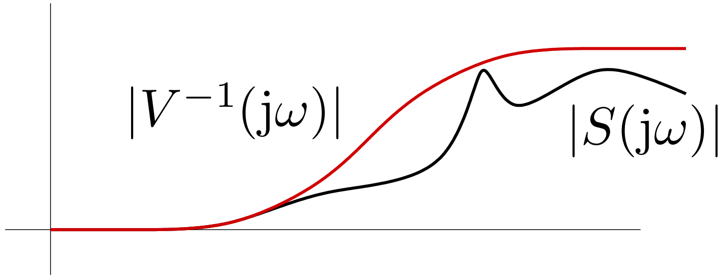
Robust performance

Idea: model 'adequate performance' as ∞-norm constraint

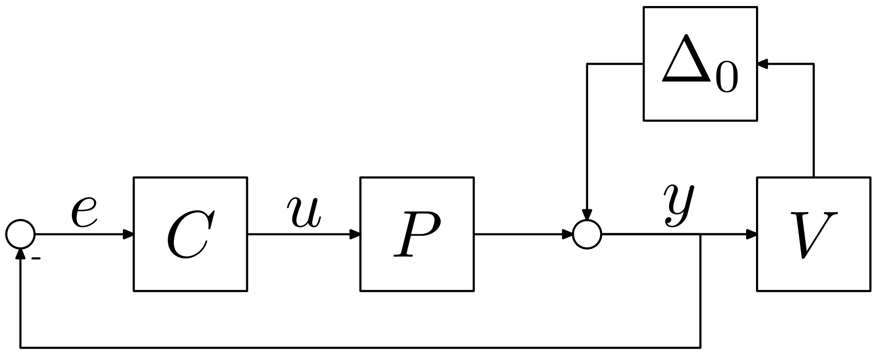
Robust performance

Example 21. Say performance is ‘adequate’ if S below given V^{-1}. The following 3 conditions are equivalent:

- $|S| \leq |V^{-1}|$



- $\|VS\|_\infty \leq 1$
- system below is robustly stable:



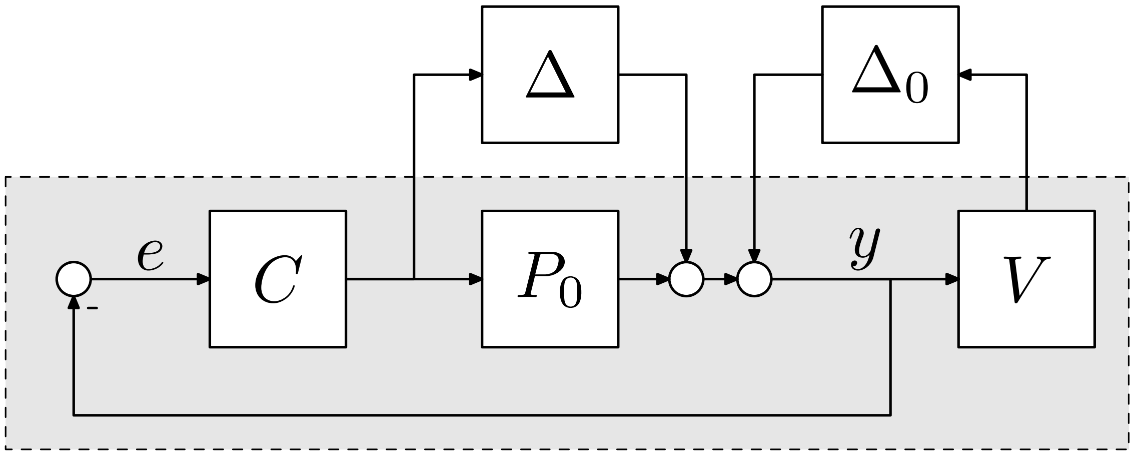
Robust performance

Definition 22. A system has robust performance if performance is adequate for every possible perturbation

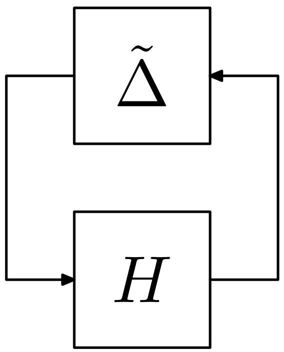
Robust performance is a μ-test

Robust performance

Assume Δ, Δ_0, H are stable. The system with plant $P = P_0 + \Delta$



has robust performance (& is robustly stable) iff



$$\tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix}$$ is robustly stable

(i.e. $\sup_\omega \mu(H(j\omega)) < 1$)

$\mathcal{H}_\infty$-Optimization and μ-Synthesis (Chapter 6)

How should I choose my controller
to achieve maximal
stability robustness & performance robustness

$\mathcal{H}_\infty$-Optimization and μ-Synthesis (Chapter 6)

Overview

- The mixed sensitivity problem
- The standard $\mathcal{H}_\infty$ problem
- Optimal & suboptimal solutions
- Integral control
- μ synthesis
- Glover-McFarlane's loop shaping

The mixed sensitivity problem

Idea:

Shape the loop gain L by choice of controller C through minimization of weighted sum of S and T.

Using $\mathcal{H}_\infty$ theory:

$$\min_{\text{stabilizing controller}} \left\| \begin{bmatrix} W_1 S V \\ W_2 T V \end{bmatrix} \right\|_\infty$$

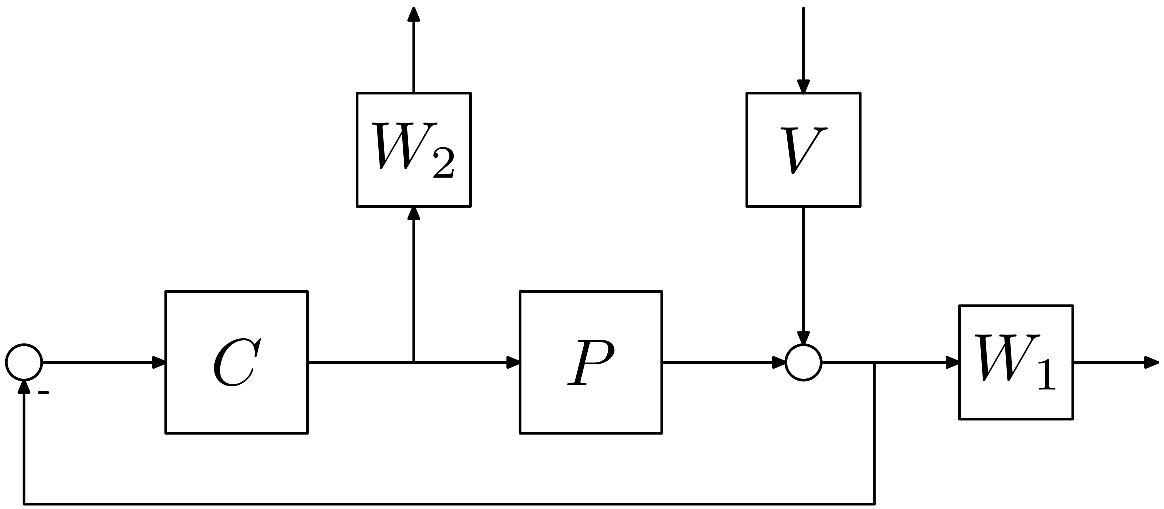
'Controller design = choice of W_1, W_2, V'

The mixed sensitivity problem

Essentially the same (for design, not for ‘math’) is

$$\min_{\text{stabilizing controller}} \left\| \begin{bmatrix} W_1 S V \\ W_2 U V \end{bmatrix} \right\|_\infty$$

where $U := C(I + PC)^{-1}$ is the input sensitivity matrix



Choice of W_1, W_2, V in mixed sensitivity problem

SISO:

$$\left\| \begin{bmatrix} W_1 S V \\ -W_2 U V \end{bmatrix} \right\|_\infty^2 \;<\; \gamma^2$$

$$\sup_{\omega\in\mathbb{R}} \left(|W_1(\mathrm{j}\omega)S(\mathrm{j}\omega)V(\mathrm{j}\omega)|^2 + |W_2(\mathrm{j}\omega)U(\mathrm{j}\omega)V(\mathrm{j}\omega)|^2 \right) \;<\; \gamma^2$$

Hence

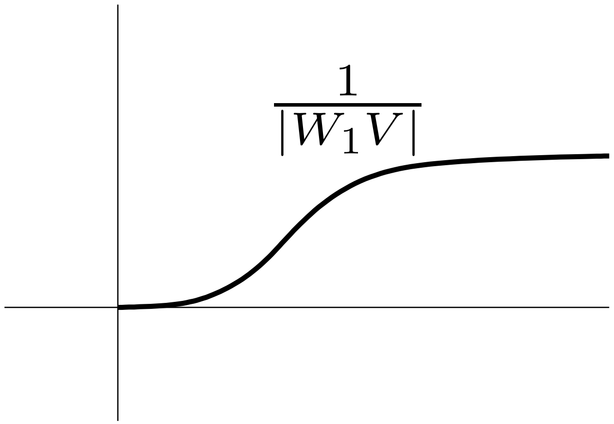
$$|S(\mathrm{j}\omega)| \;\le\; \frac{\gamma}{|W_1(\mathrm{j}\omega)V(\mathrm{j}\omega)|}, \qquad \omega \in \mathbb{R}$$

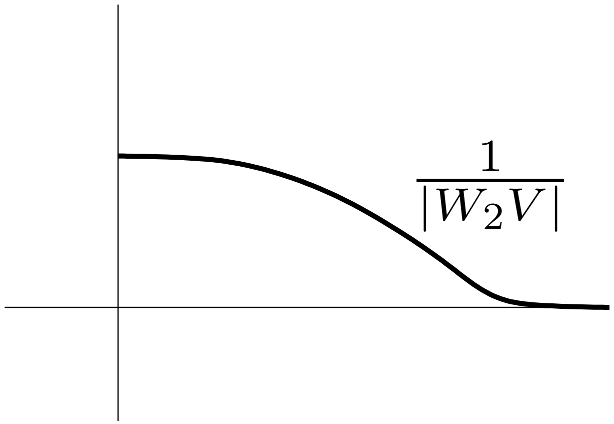
$$|U(\mathrm{j}\omega)| \;\le\; \frac{\gamma}{|W_2(\mathrm{j}\omega)V(\mathrm{j}\omega)|}, \qquad \omega \in \mathbb{R}$$

Choice of W_1, W_2, V in mixed sensitivity problem

$$
\begin{aligned}
|S(\mathrm{j}\omega)| &\leq \frac{\gamma}{|W_1(\mathrm{j}\omega)V(\mathrm{j}\omega)|}, \qquad \omega \in \mathbb{R} \\
|U(\mathrm{j}\omega)| &\leq \frac{\gamma}{|W_2(\mathrm{j}\omega)V(\mathrm{j}\omega)|}, \qquad \omega \in \mathbb{R}
\end{aligned}
$$

So want to choose W_1, W_2, V such that





Kwakernaak's pole placement technique

Let (SISO):

$$\gamma = \min_{\text{stabilizing } C} \|H\|_\infty$$

Lemma 23.
If optimal controller is unique then $|H_{opt}(j\omega)| = \gamma$ *for all freq.* ω*!*

Lemma 24.
If optimal controller is not unique then there always exists an optimal controller such that $|H_{opt}(j\omega)| = \gamma$ *for all freq.* ω*!*

Kwakernaak's pole placement technique

$$
\begin{aligned}
|H(\mathrm{j}\omega)| = \gamma \quad &\Longleftrightarrow \quad |H(\mathrm{j}\omega)|^2 = \gamma^2 \\
&\Longleftrightarrow \quad \overline{H(\mathrm{j}\omega)} H(\mathrm{j}\omega) = \gamma^2 \\
&\Longleftrightarrow \quad H(-\mathrm{j}\omega) H(\mathrm{j}\omega) = \gamma^2 \\
&\Longleftrightarrow \quad H(-s) H(s) = \gamma^2 \qquad \forall s \in \mathbb{C} \\
&\Longleftrightarrow \quad H^{\sim} H = \gamma^2
\end{aligned}
$$

Definition 25. $H^{\sim}(s) := H(-s)$

Kwakernaak's pole placement technique

$$H := \begin{bmatrix} W_1 S V \\ -W_2 U V \end{bmatrix}$$

introduce polynomial fractions

$$P = \frac{N}{D}, \quad W_1 = \frac{A_1}{B_1}, \quad W_2 = \frac{A_2}{B_2}, \quad V = \frac{M}{E}, \quad C = \frac{Y}{X}$$

Then

$$
\begin{aligned}
H^{\sim}H &= \gamma^2 \\
W_1^{\sim}W_1S^{\sim}SV^{\sim}V + W_2^{\sim}W_2U^{\sim}UV^{\sim}V &= \gamma^2 \\
\frac{D^{\sim}D \cdot M^{\sim}M \cdot (A_1^{\sim}A_1B_2^{\sim}B_2X^{\sim}X + A_2^{\sim}A_2B_1^{\sim}B_1Y^{\sim}Y)}{E^{\sim}E \cdot B_1^{\sim}B_1 \cdot B_2^{\sim}B_2 \cdot \chi_{\mathsf{cl}}^{\sim}\chi_{\mathsf{cl}}} &= \gamma^2
\end{aligned}
$$

Kwakernaak's pole placement technique

$$\frac{D^{\sim}D \cdot M^{\sim}M \cdot (A_1^{\sim}A_1B_2^{\sim}B_2X^{\sim}X + A_2^{\sim}A_2B_1^{\sim}B_1Y^{\sim}Y)}{E^{\sim}E \cdot B_1^{\sim}B_1 \cdot B_2^{\sim}B_2 \cdot \chi_{\text{cl}}^{\sim}\chi_{\text{cl}}} = \gamma^2$$

All zeros of numerator must cancel against zeros of denominator.

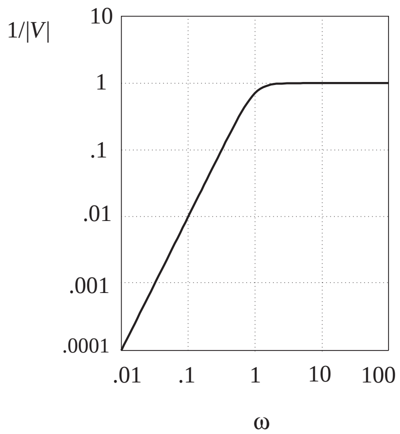
- If $E^{\sim}E \cdot B_1^{\sim}B_1 \cdot B_2^{\sim}B_2$ no zeros in common with $D^{\sim}D$ then: closed loop poles are open loop poles (possibly mirrored in im-axis)
- Simple remedy: choose $E = D$
- Also: choose M to have your favorite closed loop poles (and with same degree as D)

Kwakernaak's pole placement technique

Example 26. Suppose

$$P_0(s) = \frac{1}{s^2}$$

- We aim for closed loop bandwidth of 1
- Hence $V = \frac{M}{E} := \frac{M}{D} := \frac{s^2 + s\sqrt{2} + 1}{s^2}$
- Then $\frac{1}{|V|} =$



- Since $|S| \leq \dfrac{\gamma}{|W_1 V|}$ so $W_1 = 1$ seems okay
- Since $|U| \leq \dfrac{\gamma}{|W_2 V|}$ we take

$$W_2(s) = c(1 + rs)$$

with

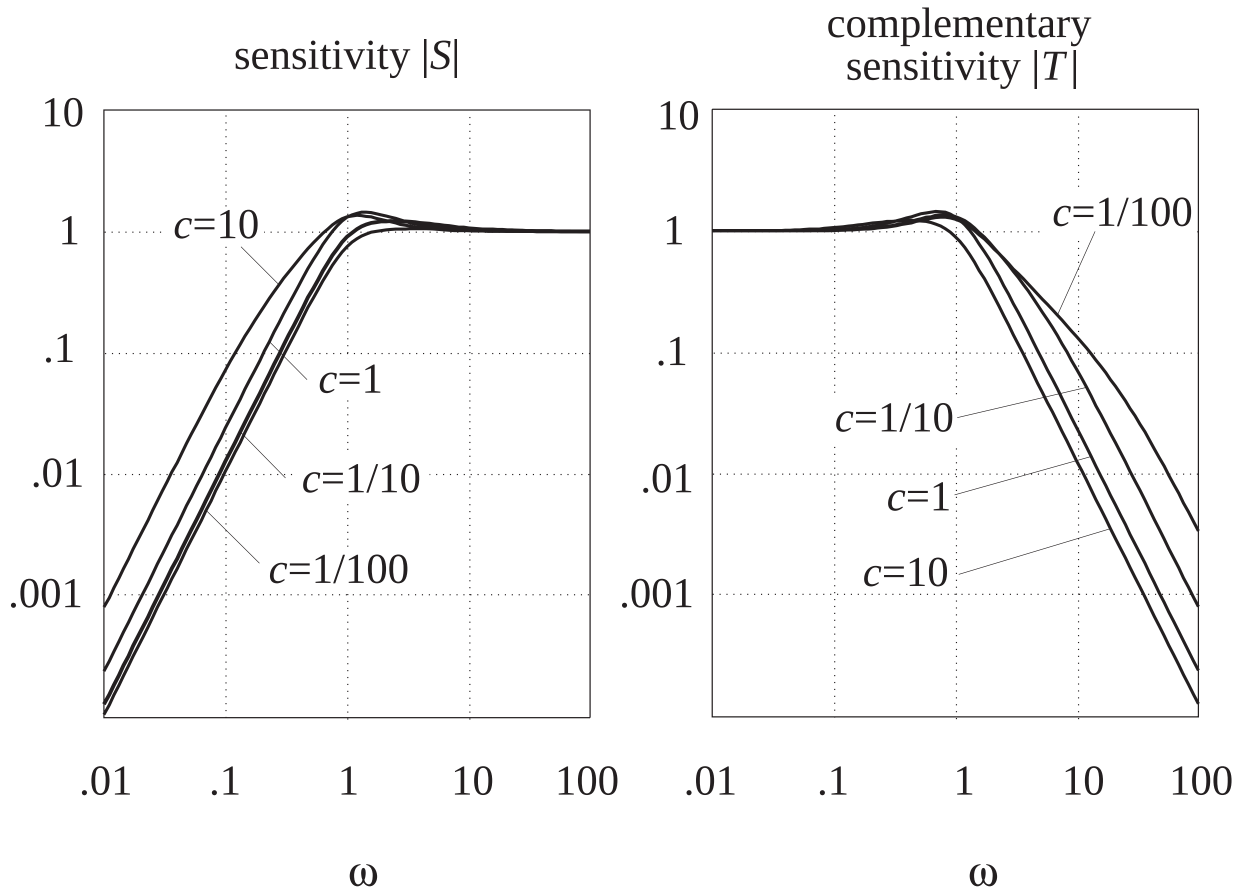
$$r \in [0, 1]$$

and

$$c \in \mathbb{R}$$

which weights relative importance of small $|S|$ versus small $|U|$

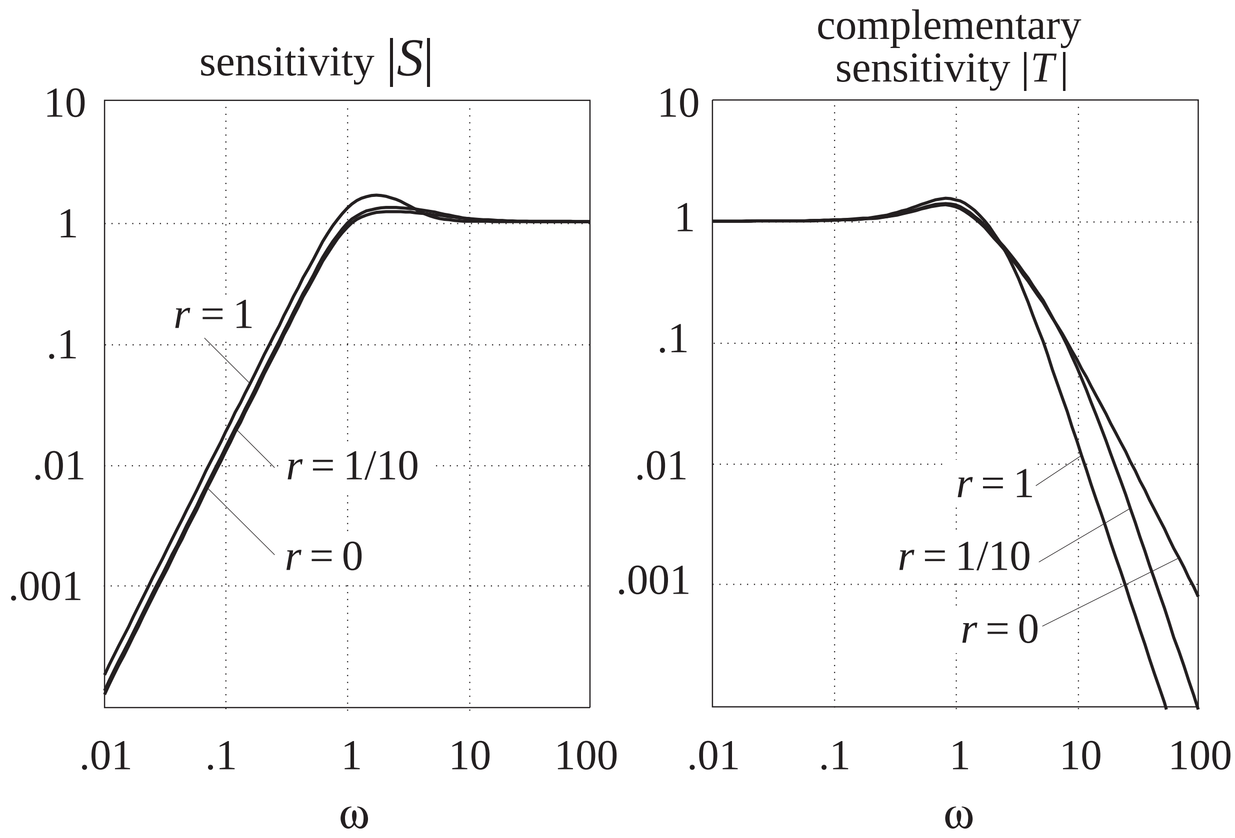
- For $r = 0$:



$$c = 1/10 \quad \Longrightarrow \quad C(s) = 1.2586\ \frac{s + 0.61967}{1 + 0.15563s}$$

CL poles: $\frac{1}{2}\sqrt{2}(-1 \pm \mathrm{j})$ & -5.0114

- For $c = 1/10$:



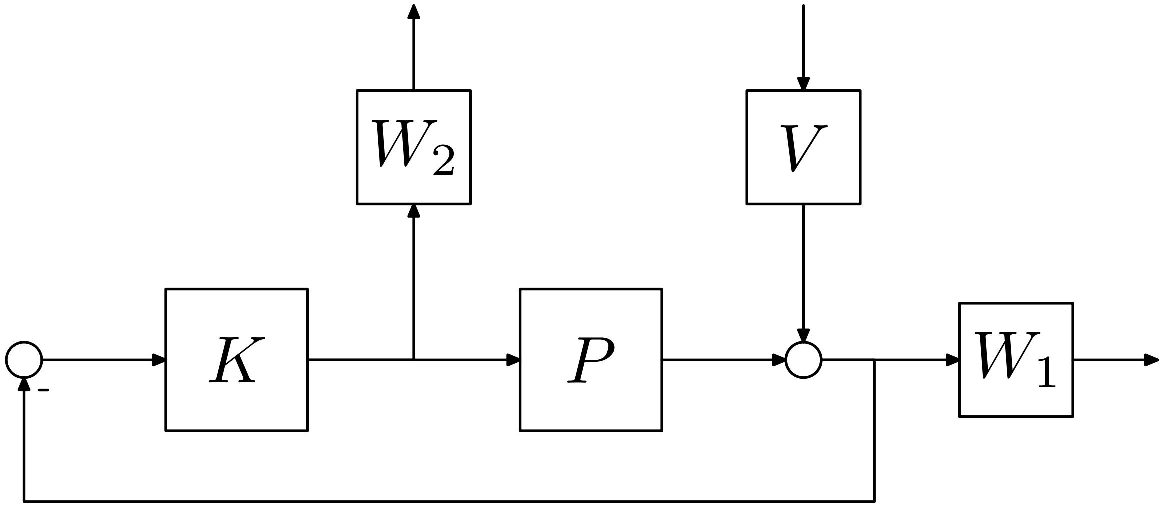
$$r = 1/10 \quad \Longrightarrow \quad C(s) = 1.2107\,\frac{s + 0.5987}{1 + 0.20355s + 0.01267s^2}$$

CL poles: $\frac{1}{2}\sqrt{2}(-1 \pm \mathrm{j})$ & $-7.3281 \pm \mathrm{j}1.8765$

The standard $\mathcal{H}_\infty$ problem

The mixed sensitivity problem is:

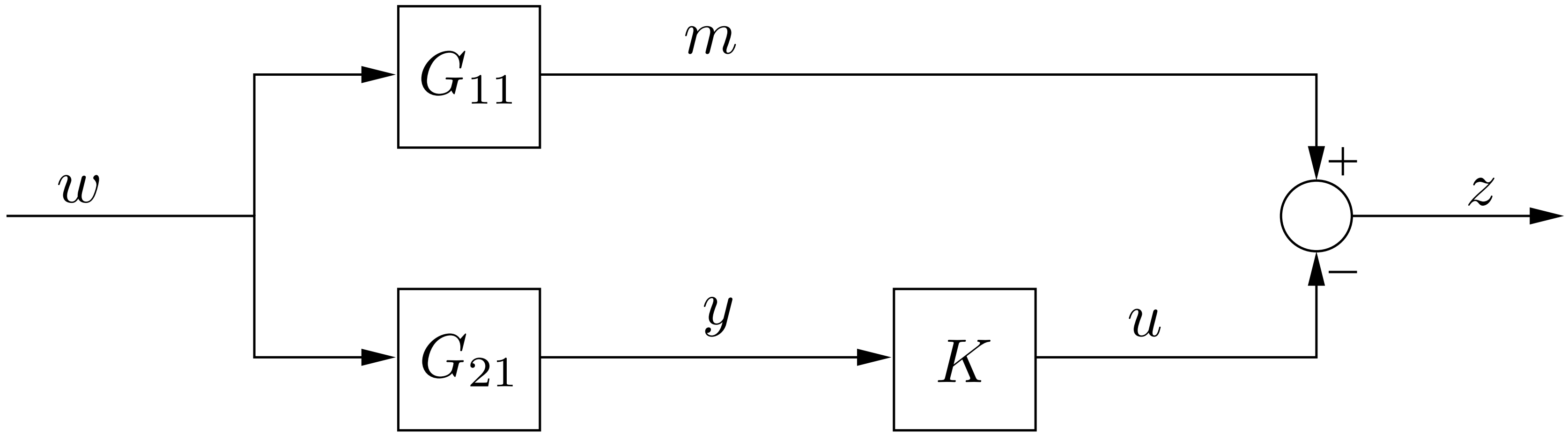
$$\min_{\text{stabilizing controller } K} \left\| \begin{bmatrix} W_1 S V \\ W_2 U V \end{bmatrix} \right\|_\infty$$



The standard $\mathcal{H}_\infty$ problem

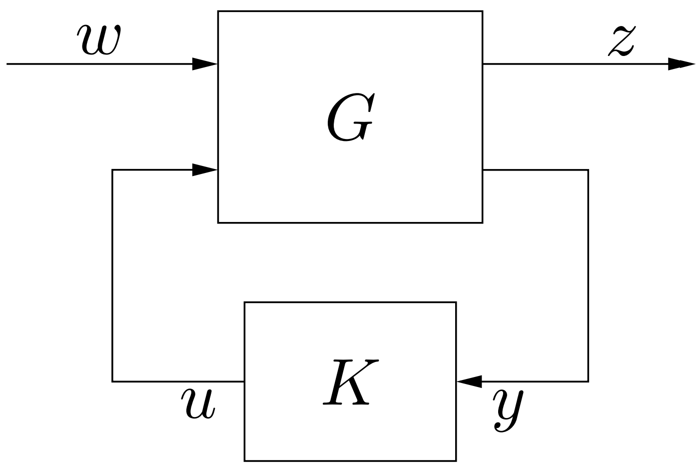
The filtering problem is:

$$\min_{\text{stable filter } K} \|G_{11} - KG_{21}\|_\infty$$



The standard $\mathcal{H}_\infty$ problem

Pull out the controller

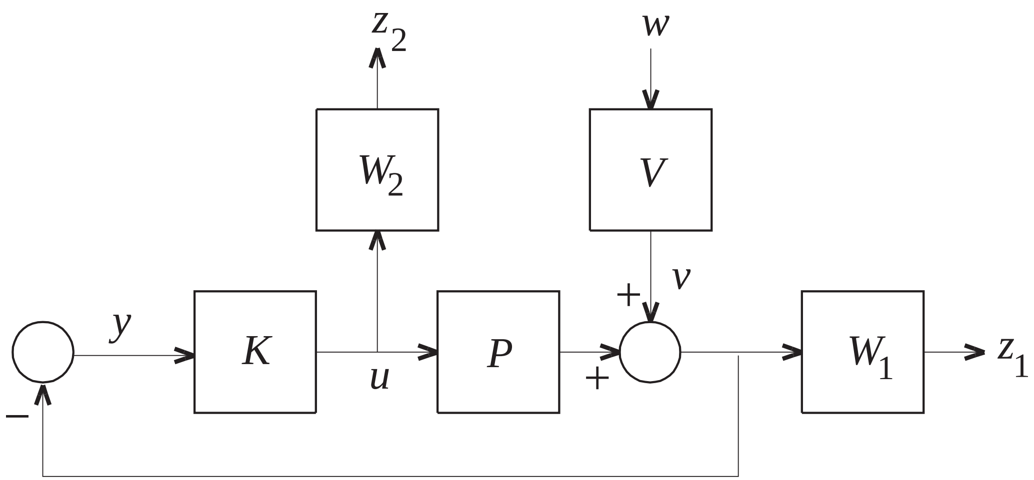


generalized plant: $\begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}$

The standard $\mathcal{H}_\infty$ problem:

$$\min_{\text{internally stabilizing } K} \|H\|_\infty$$

The standard $\mathcal{H}_\infty$ problem



Example 27. Delete K to determine generalized plant G:

$$\begin{aligned} z_1 &= W_1 V w + W_1 P u \\ z_2 &= W_2 u \\ y &= -V w - P u \end{aligned} \quad \Longrightarrow \quad G = \left[\begin{array}{c|c} W_1 V & W_1 P \\ 0 & W_2 \\ \hline -V & -P \end{array}\right]$$

Some $\mathcal{H}_\infty$ intricacies

1. Finding optimal K in one shot is not easy
2. Finding suboptimal K is easy:

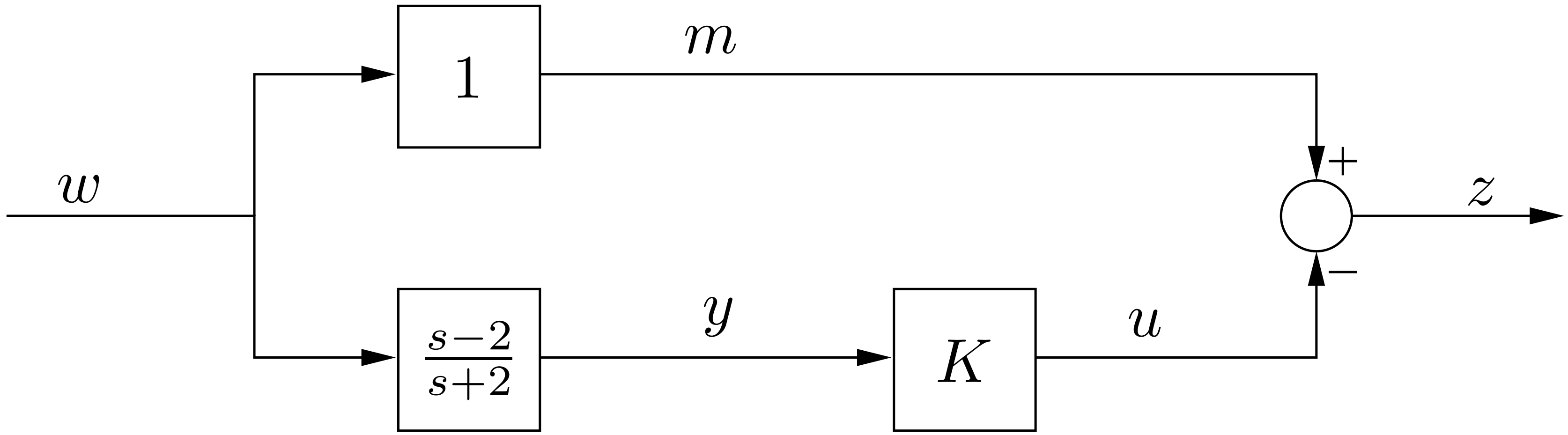
   Given $\gamma > 0$ give me a K such that $\|H\|_\infty < \gamma$
   or tell me that none such exist
3. Then K is rational if G is rational
4. Do a line search on γ to find γ_{opt} and K_{opt}

.....more intricacies:

Some $\mathcal{H}_\infty$ intricacies

Suppose

$$H(s) = 1 - K(s)\frac{s-2}{s+2}$$



Some $\mathcal{H}_\infty$ intricacies

$$
\begin{aligned}
\|H\|_\infty \le \gamma \quad &\Longleftrightarrow \quad H^\sim H \le \gamma^2 \\
&\Longleftrightarrow \quad \left(1 - \tfrac{s+2}{s-2} K^\sim\right) \left(1 - \tfrac{s-2}{s+2} K\right) \le \gamma^2 \\
&\Longleftrightarrow \quad 1 - \gamma^2 - \tfrac{s+2}{s-2} K^\sim - \tfrac{s-2}{s+2} K + K^\sim K \le 0 \\
&\Longleftrightarrow \quad \begin{bmatrix} K^\sim & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{s+2}{s-2} \\ -\frac{s-2}{s+2} & 1 - \gamma^2 \end{bmatrix} \begin{bmatrix} K \\ 1 \end{bmatrix} \le 0 \\
&\Longleftrightarrow \quad \begin{bmatrix} K^\sim & 1 \end{bmatrix} W^\sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} W \begin{bmatrix} K \\ 1 \end{bmatrix} \le 0
\end{aligned}
$$

for $\qquad W(s) = \begin{bmatrix} \frac{\gamma^2 - \frac{s-2}{s+2}}{\gamma^2 - 1} & 1 \\ \frac{\gamma}{\gamma^2 - 1}\left(1 - \frac{s-2}{s+2}\right) & \gamma \end{bmatrix}$

Some $\mathcal{H}_\infty$ intricacies

$$\|H\|_\infty \leq \gamma \quad \Longleftrightarrow \quad \begin{bmatrix} K^\sim & 1 \end{bmatrix} W^\sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} W \begin{bmatrix} K \\ 1 \end{bmatrix} \leq 0$$

RHS holds for

$$K = -\frac{W_{12}}{W_{11}} = \frac{(\gamma^2 - 1)(s+2)}{(\gamma^2-1)s + \gamma^2 + 2}$$

This K is stable for $\gamma > 1$

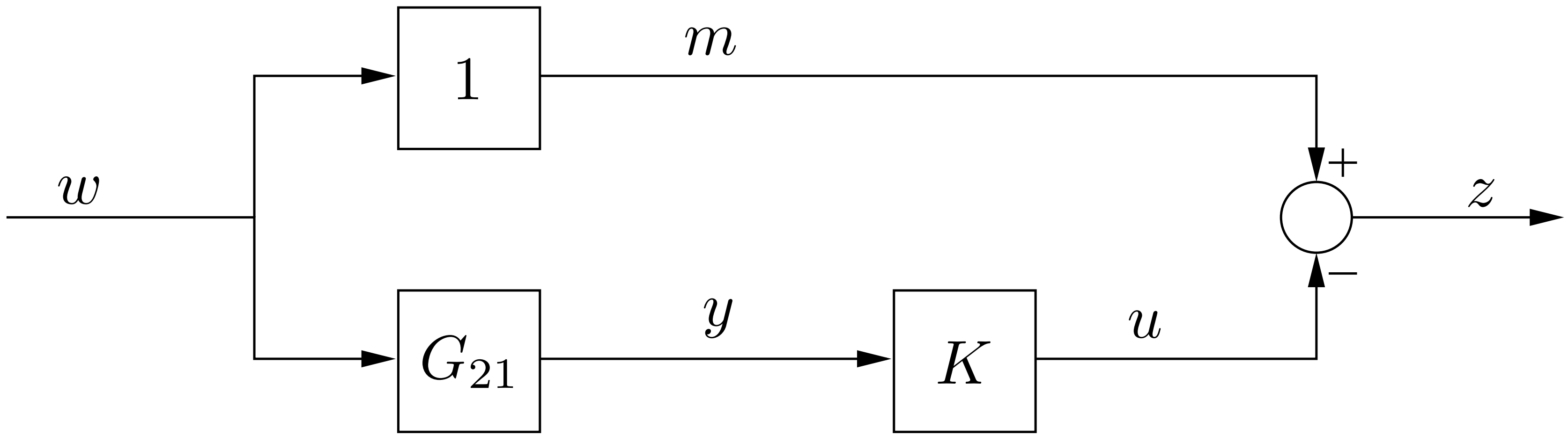
Some $\mathcal{H}_\infty$ intricacies

$$K = -\frac{W_{12}}{W_{11}} = \frac{(\gamma^2 - 1)(s+2)}{(\gamma^2-1)s + \gamma^2 + 2} \qquad \text{[has pole } s = -\frac{\gamma^2+2}{\gamma^2-1}\text{]}$$

For this example:

- $\gamma_{\mathsf{opt}} = 1$
- For all $\gamma > \gamma_{\mathsf{opt}} = 1$ the above K is a suboptimal solution
- $\lim_{\gamma \downarrow 1} K(s) = 0$ is the optimal solution
- For $\gamma = \gamma_{\mathsf{opt}} + \epsilon$ the suboptimal controller has a pole $\approx -\infty$
- The optimal K is unique (suboptimal K are not unique)

Some $\mathcal{H}_\infty$ intricacies



$$H = 1 - KG_{21}$$

Some $\mathcal{H}_\infty$ intricacies

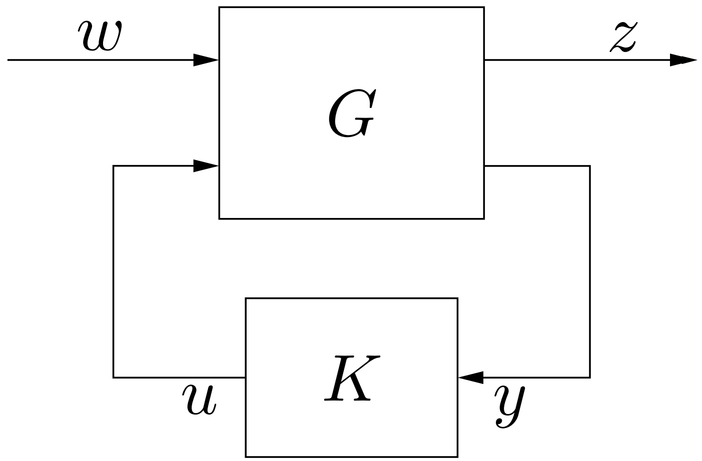
so (if SISO):

$$K = \frac{1 - H}{G_{21}}$$

Annoying discontinuity with imaginary zeros of G_{21}:

- If $G_{21} = \dfrac{s+\epsilon}{s+1}$ implies $H_{\mathrm{opt}} = 0$
- If $G_{21} = \dfrac{s-\epsilon}{s+1}$ implies $H_{\mathrm{opt}} = 1$

State space solution of the standard $\mathcal{H}_\infty$ problem



$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

State space solution of the standard $\mathcal{H}_\infty$ problem

Assumptions:

1. $(A,\ B_2)$ is stabilizable

2. $(C_2,\ A)$ is detectable

3. $\begin{bmatrix} A - \mathrm{j}\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$

4. $\begin{bmatrix} A - \mathrm{j}\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$

5. D_{12} and D_{21} have full rank

Theorem 28. *Suppose also*

$$\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}.$$

Then there exists a stabilizing controller for which $\|H\|_\infty < \gamma$ *iff the following conditions hold.*

1. $AQ + QA^T + Q(\frac{1}{\gamma^2}C_1^T C_1 - C_2^T C_2)Q + B_1 B_1^T = 0$ *has a stabilizing solution* $Q \geq 0$
2. $PA + A^T P + P(\frac{1}{\gamma^2}B_1 B_1^T - B_2 B_2^T)P + C_1^T C_1 = 0$ *has a stabilizing solution* $P \geq 0$
3. *All eigenvalues of* QP *have magnitude less than* γ^2

State space solution of the standard $\mathcal{H}_\infty$ problem

Then a suboptimal controller is:

$$\begin{aligned}\dot{\hat{x}} &= (A + [\tfrac{1}{\gamma^2}B_1B_1^\mathsf{T} - B_2B_2^\mathsf{T}]P)\hat{x} + (I - \tfrac{1}{\gamma^2}QP)^{-1}QC_2^\mathsf{T}(y - C_2\hat{x}) \\ u &= -B_2^\mathsf{T}\hat{x}\end{aligned}$$

A final note on $\mathcal{H}_\infty$ intricacies

As γ approaches γ_{opt} from above, two things can happen:

type-A:
they don't blow up but for $\gamma < \gamma_{\text{opt}}$ no such P and Q exist

type-B:
P or Q or $(I - \frac{1}{\gamma^2}QP)^{-1}$ blow up

If type-B then coefficients in controller blow up (or go to zero) which disappear at $\gamma = \gamma_{\text{opt}}$

A final note on $\mathcal{H}_\infty$ intricacies

Example 29 (Mixed sensitivity)**.**

$$G(s) = \left[\begin{array}{c|c} W_1 V & W_1 P \\ 0 & W_2 \\ \hline -V & -P \end{array}\right] = \left[\begin{array}{c|c} \frac{M(s)}{s^2} & \frac{1}{s^2} \\ 0 & c(1+rs) \\ \hline -\frac{M(s)}{s^2} & -\frac{1}{s^2} \end{array}\right]$$

with $M(s) = s^2 + \sqrt{2}s + 1,\ r = 0,\ c = 0.1.$

Then (using MU-TOOLS):

$$
\begin{aligned}
\dot{\hat{x}} &= \begin{bmatrix} -0.3422 \times 10^9 & -1.7147 \times 10^9 \\ 1 & -1.4142 \end{bmatrix} \hat{x} + \begin{bmatrix} -0.3015 \\ -0.4264 \end{bmatrix} y \\
u &= [-1.1348 \times 10^9 \quad - 5.6871 \times 10^9]\, \hat{x}
\end{aligned}
$$

A final note on $\mathcal{H}_\infty$ intricacies

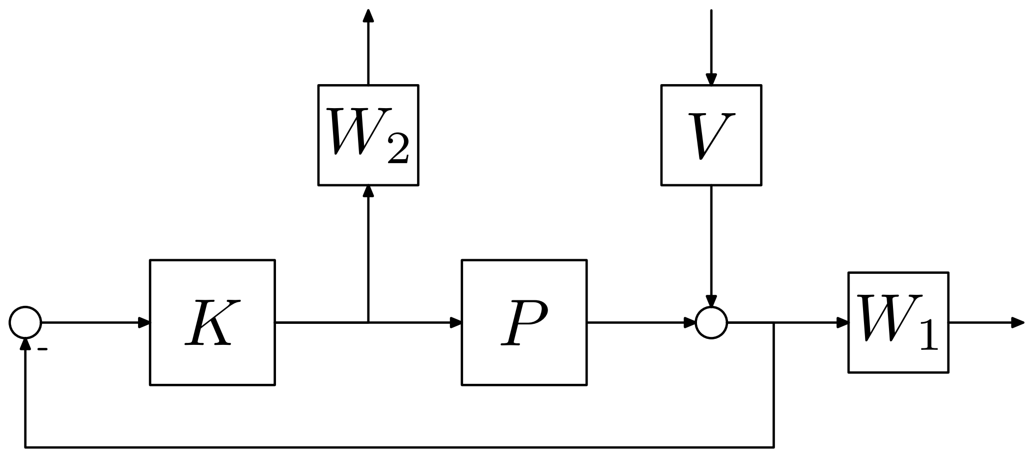
as a transfer function

$$K(s) = \frac{2.7671 \times 10^9 s + 1.7147 \times 10^9}{s^2 + 0.3422 \times 10^9 s + 2.1986 \times 10^9}$$

Optimal (after manually cancelling s^2):

$$K(s) = 1.2586 \, \frac{s + 0.6197}{1 + 0.1556 s}$$

Integral control & HF roll-off (mixed sensitivity)



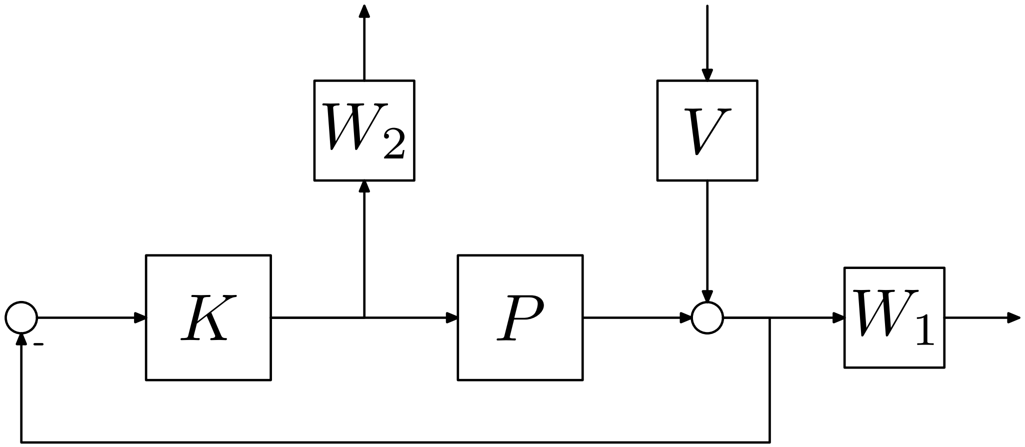
$$\min_{\text{stab.}K} \left\| \begin{bmatrix} W_1 S V \\ W_2 U V \end{bmatrix} \right\|_\infty$$

Integrator in weight W_1: $W_1(s) = W_{10}(s)/s$

Then $\|H\|_\infty < \infty \implies S(0) = 0$ implying integral control!

Bummer: but then loop cannot be stabilized

Integral control & HF roll-off (mixed sensitivity)



$$\min_{\text{stab.}K} \left\| \begin{bmatrix} W_1 S V \\ W_2 U V \end{bmatrix} \right\|_\infty$$

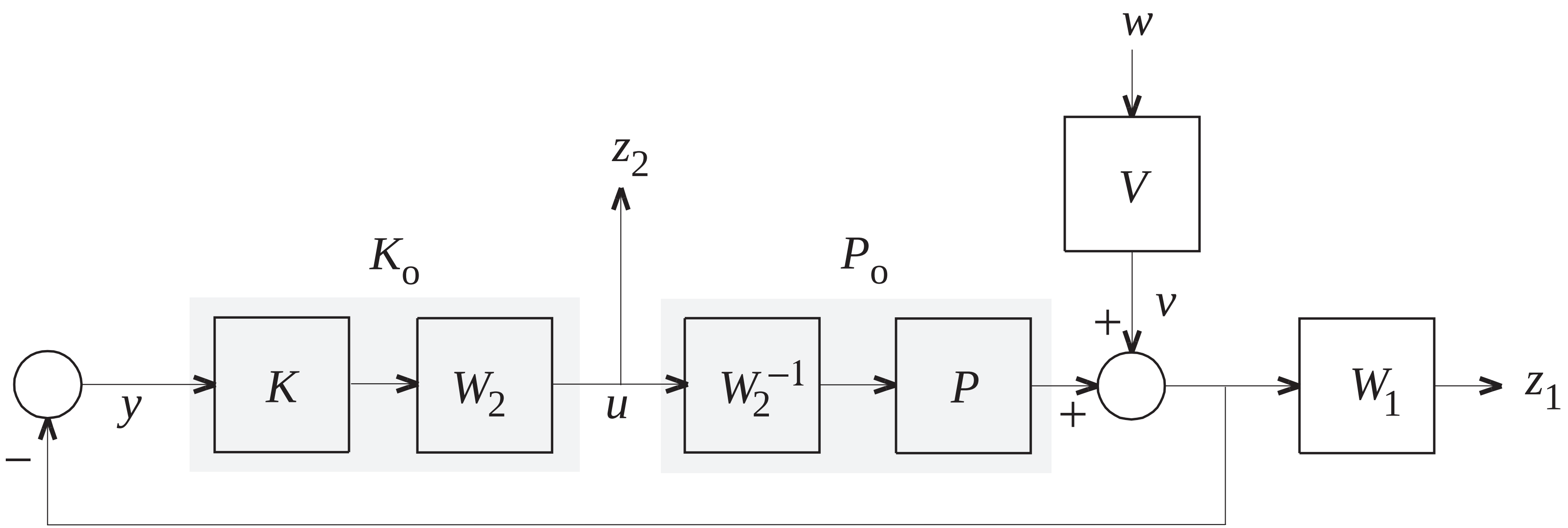
Take W_2 to be polynomial: $W_2(s)$ polynomial

Then $\|H\|_\infty < \infty \implies U(\infty) = 0$ implying HF roll-off

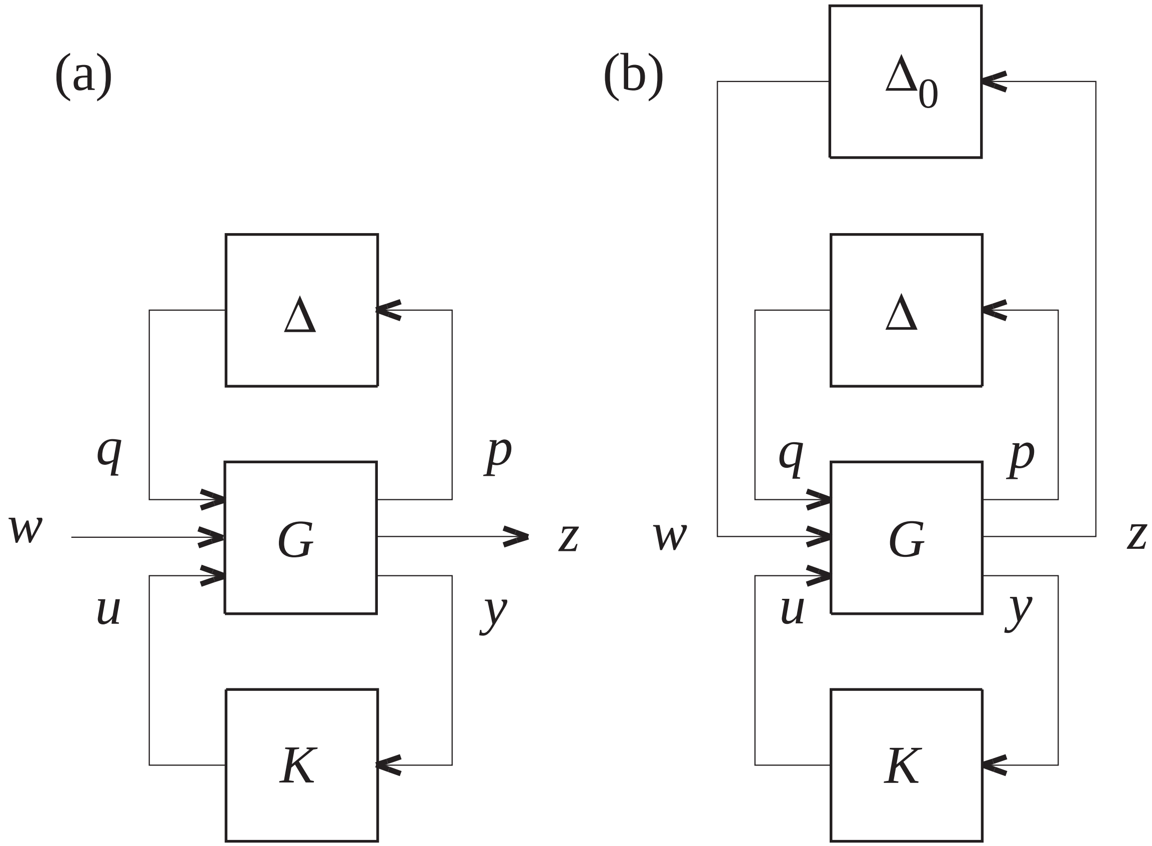
Bummer: then G is not proper: standard state space not applicable

Integral control & HF roll-off (mixed sensitivity)

Solution: absorb weights inside loop



μ-synthesis



μ-synthesis (2)

μ synthesis:

$$\min_{\text{stabilizing } K} \sup_{\omega} \mu(H(\mathrm{j}\omega))$$

This is extremely incredibly hard

Have to settle for approximate solution

μ-synthesis

D-K-iteration:

1. Find whatever stabilizing controller K
2. Choose frequency grid $\omega_m, \;\; m = 1, 2, 3, \ldots$
3. Compute for all grid points

$$\nu(H(\mathrm{j}\omega_m)) := \min_{D_m} \overline{\sigma}(D_m H(\mathrm{j}\omega_m) D_m^{-1})$$

4. Find rational fit $D(s)$ such that

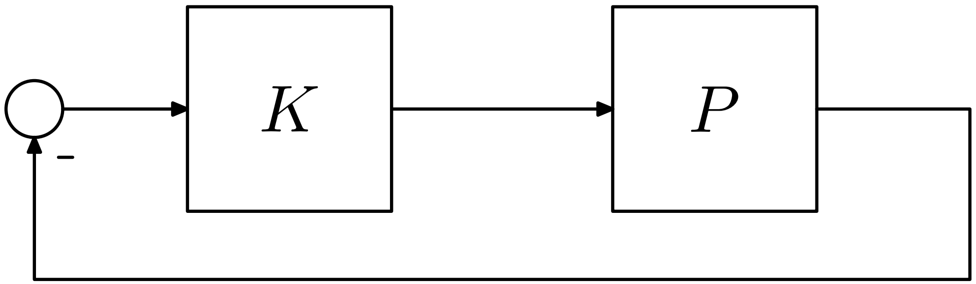
$$D(\mathrm{j}\omega_m) \approx D_m \qquad \text{(in magnitude)}$$

Hence we have

$$
\begin{aligned}
\mu(H(\mathrm{j}\omega)) &\approx \mu(H(\mathrm{j}\omega_m)) \\
&\approx \nu(H(\mathrm{j}\omega_m)) \\
&\approx \overline{\sigma}(D(\mathrm{j}\omega)H(\mathrm{j}\omega_m)D^{-1}(\mathrm{j}\omega)) \\
&\leq \|DHD^{-1}\|_\infty
\end{aligned}
$$

5. Find new controller K that (approximately) minimizes $\|DHD^{-1}\|_\infty$ [this a standard $\mathcal{H}_\infty$ problem]
6. Return to 3 (or stop)

μ-synthesis



Example 30.

$$P(s) = \frac{g}{s^2(1+\theta s)}, \quad \text{nominally } g = 1, \quad \theta = 0$$

Controller designed for nominal plant:

$$K(s) = \frac{k + T_d s}{1 + T_0 s}, \qquad k = 1, \quad T_d = \sqrt{2}, \quad T_0 = \tfrac{1}{10}$$

Performance deemed adequate if

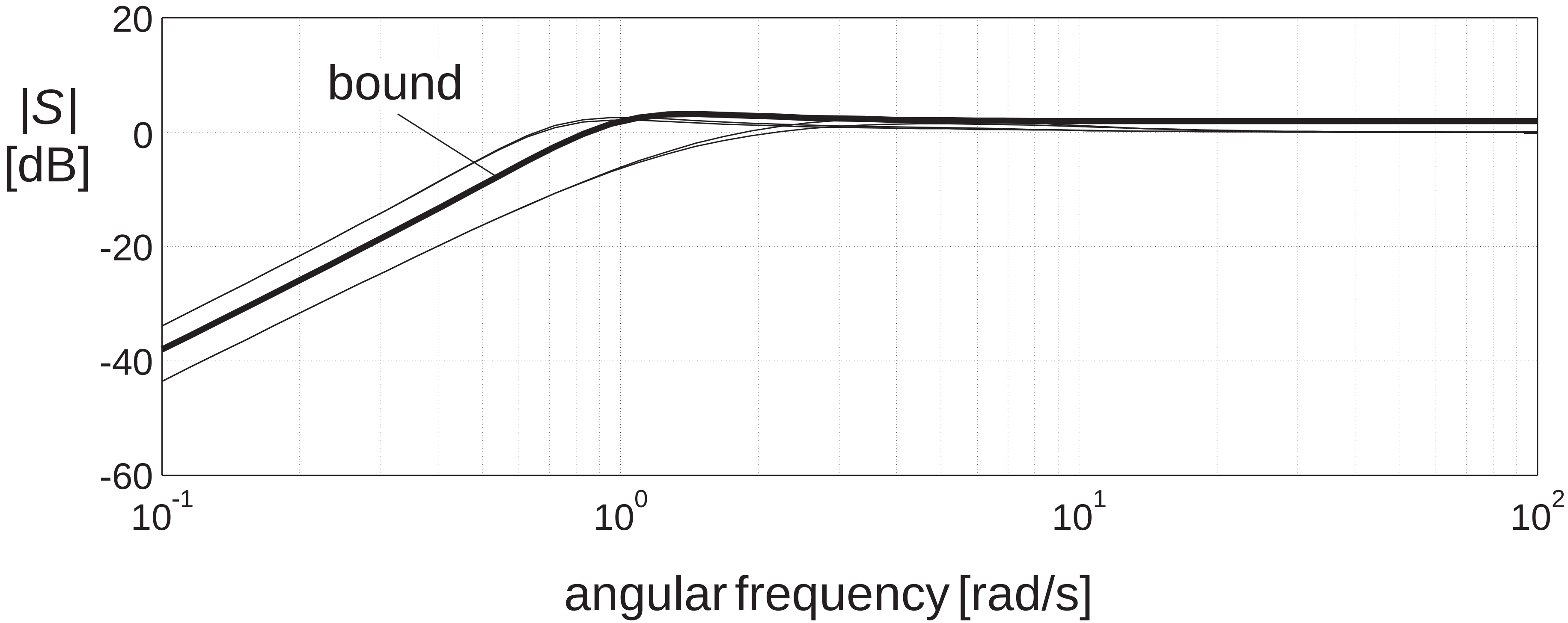
$$|S(\mathrm{j}\omega)| \le (1+\epsilon)\left|\frac{s^2}{s^2+2\zeta\omega_0 s+\omega_0^2}\right| \qquad \epsilon = 0.25,\ \omega_0 = 1,\ \zeta = \frac{1}{2}$$

the perturbed one

Equivalently

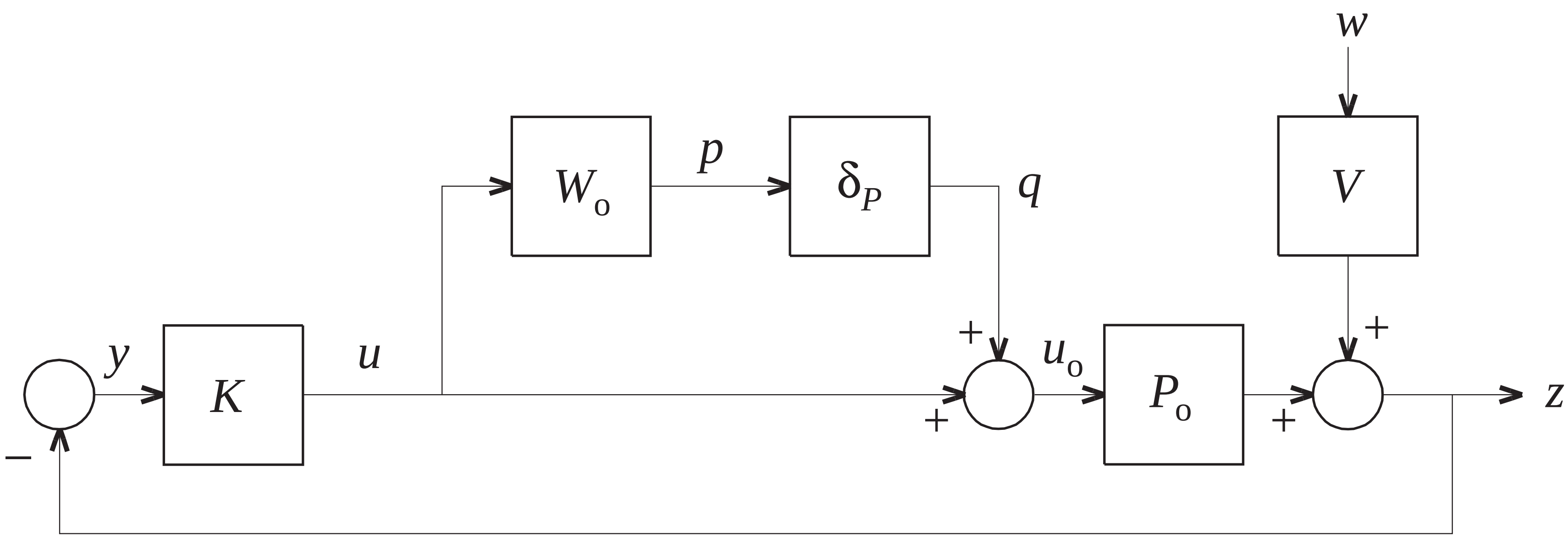
$$\|SV\|_\infty \le 1, \qquad \frac{1}{V(s)} = \frac{1}{(1+\epsilon)}\,\frac{s^2+2\zeta\omega_0 s+\omega_0^2}{s^2}$$

Plot perturbed S for $g \in [0.5, 1.5]$ and $\theta \in [0, .05]$:



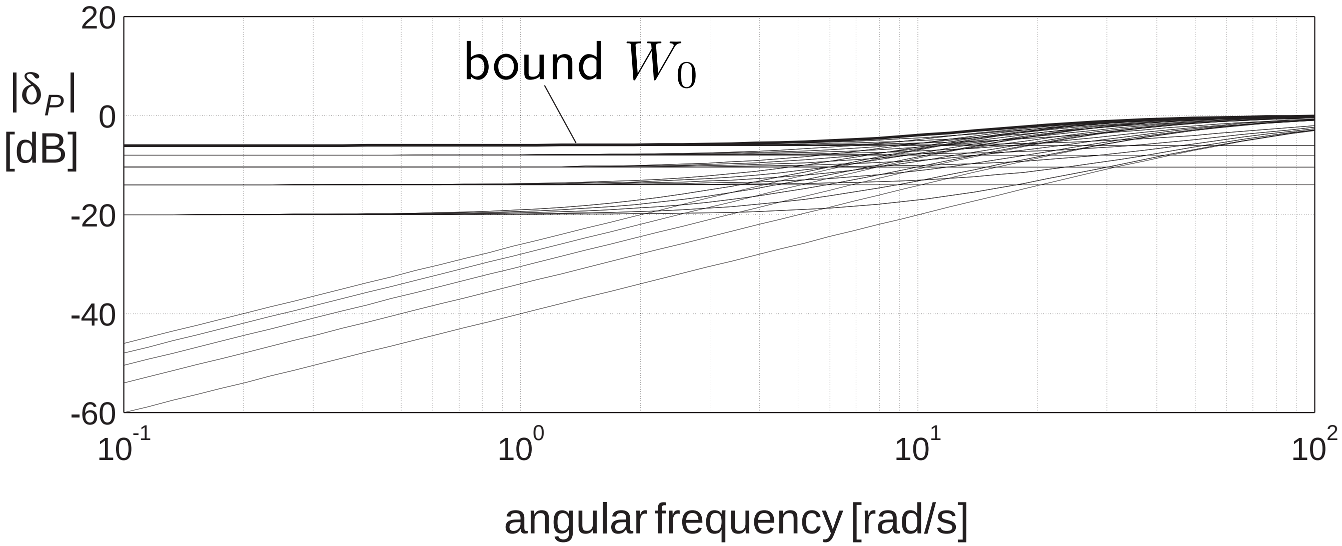
Bound is not satisfied: need to modify controller

Setting up μ problem:

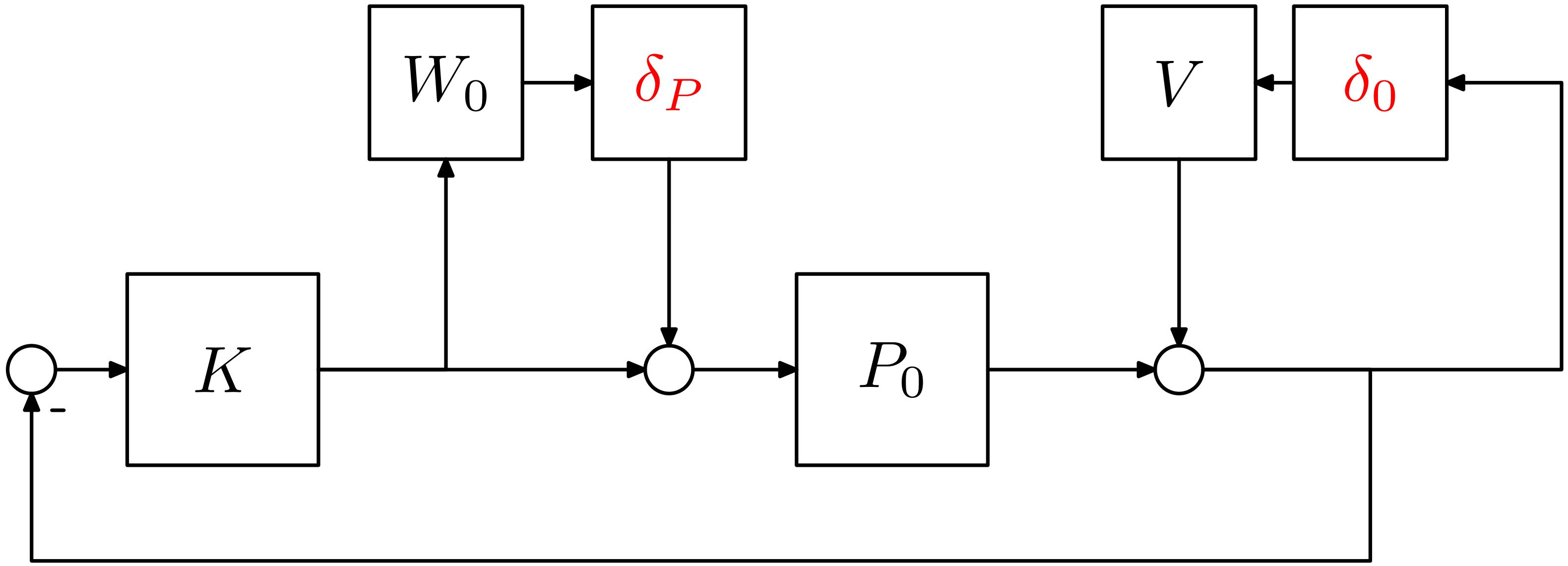


$$\Delta_P(s) = \frac{P(s) - P_0(s)}{P_0(s)} = \frac{\frac{g-g_0}{g_0} - s\theta}{1 + s\theta}$$

Plot magnitude of Δ_P for $g \in [0.5, 1.5]$ and $\theta \in [0, .05]$:



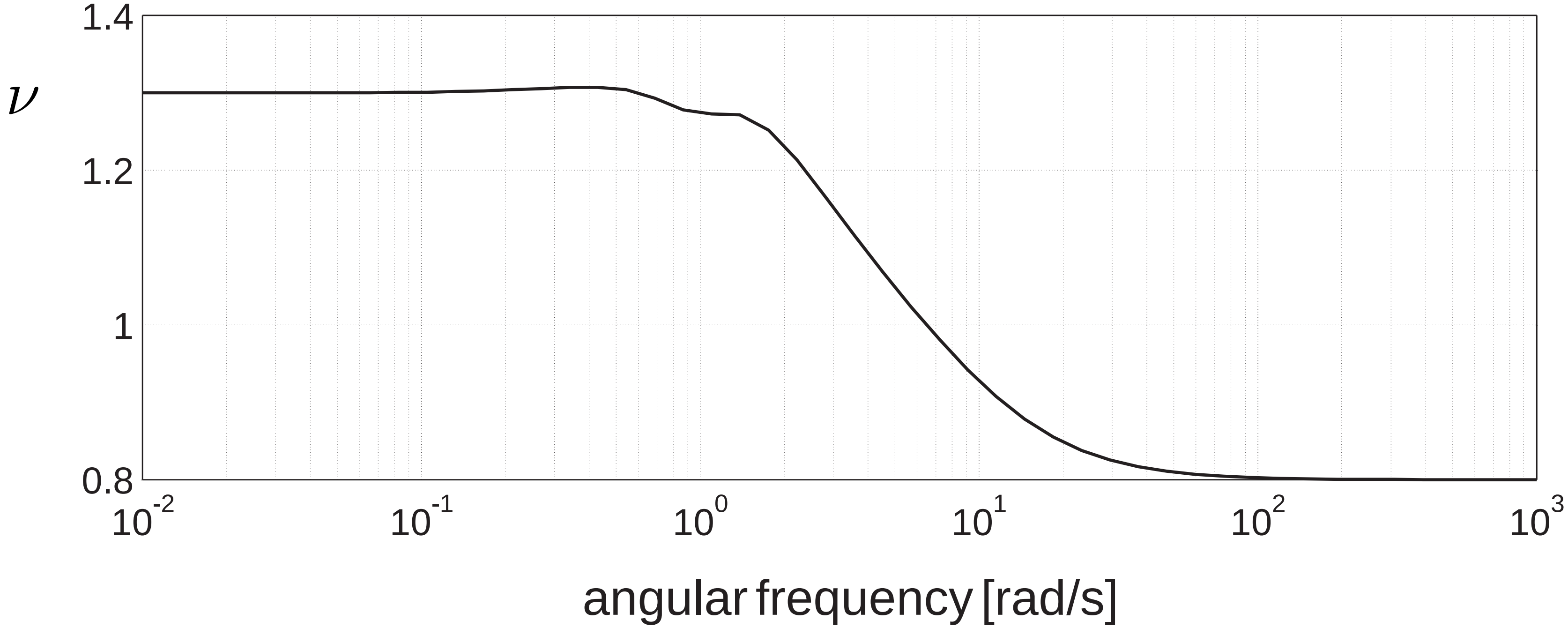
All these are below bound $W_0(s) = \frac{0.5+s/20}{1+s/20}$



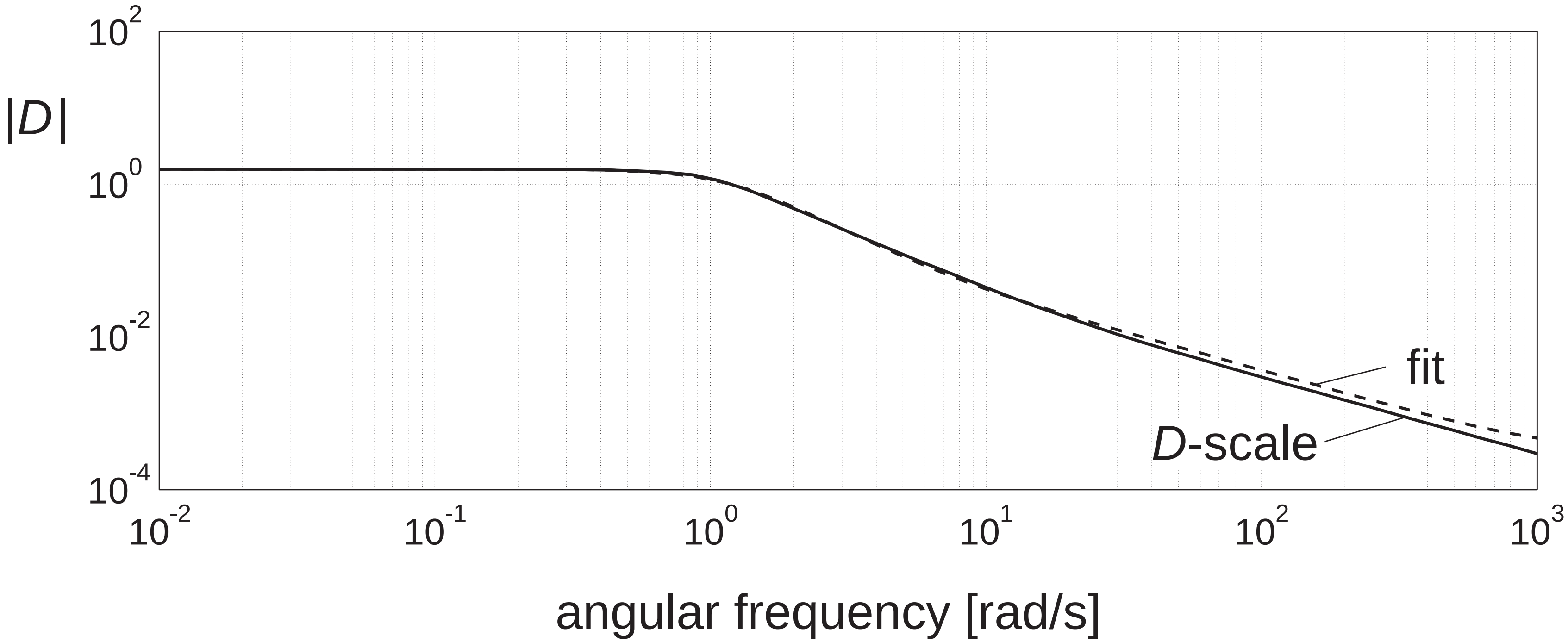
Structured perturbation

$$\Delta = \begin{bmatrix} \delta_0 & 0 \\ 0 & \delta_P \end{bmatrix}$$

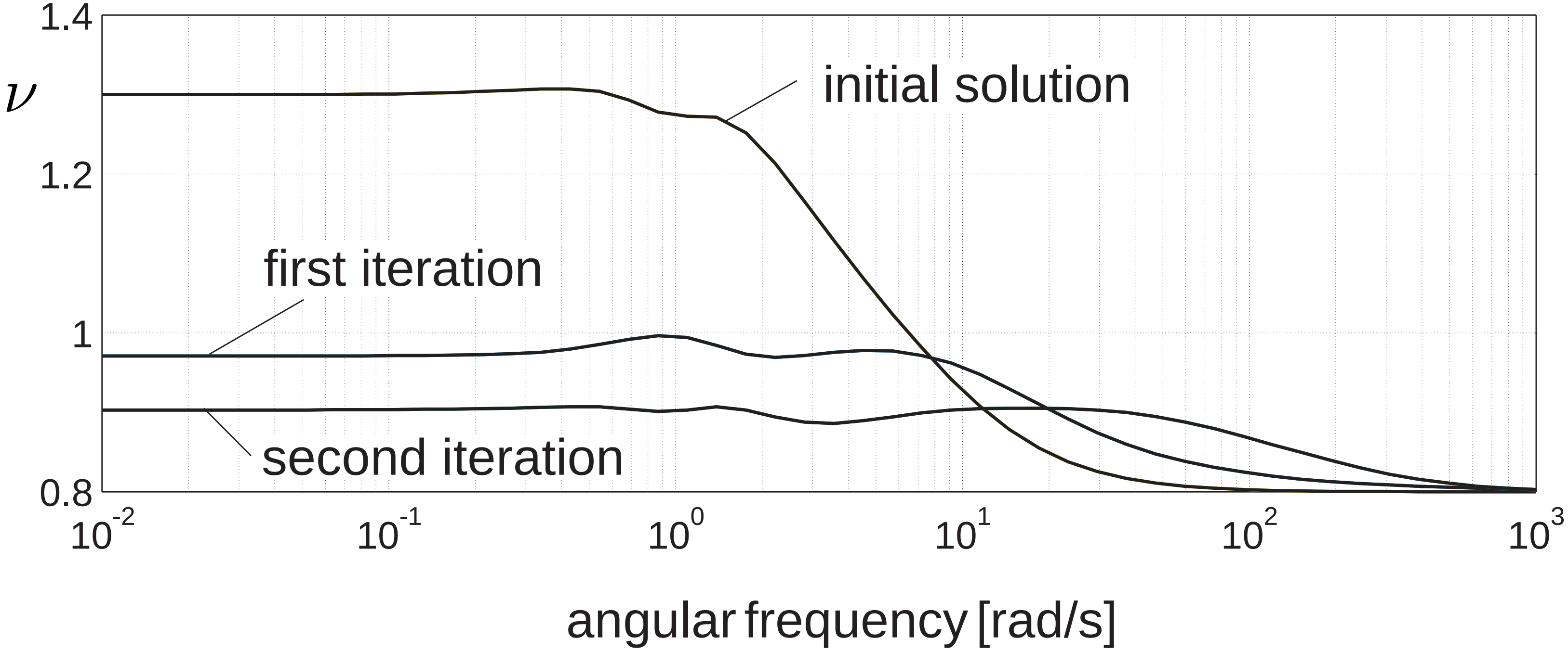
Structured singular value (well, its upper bound ν):



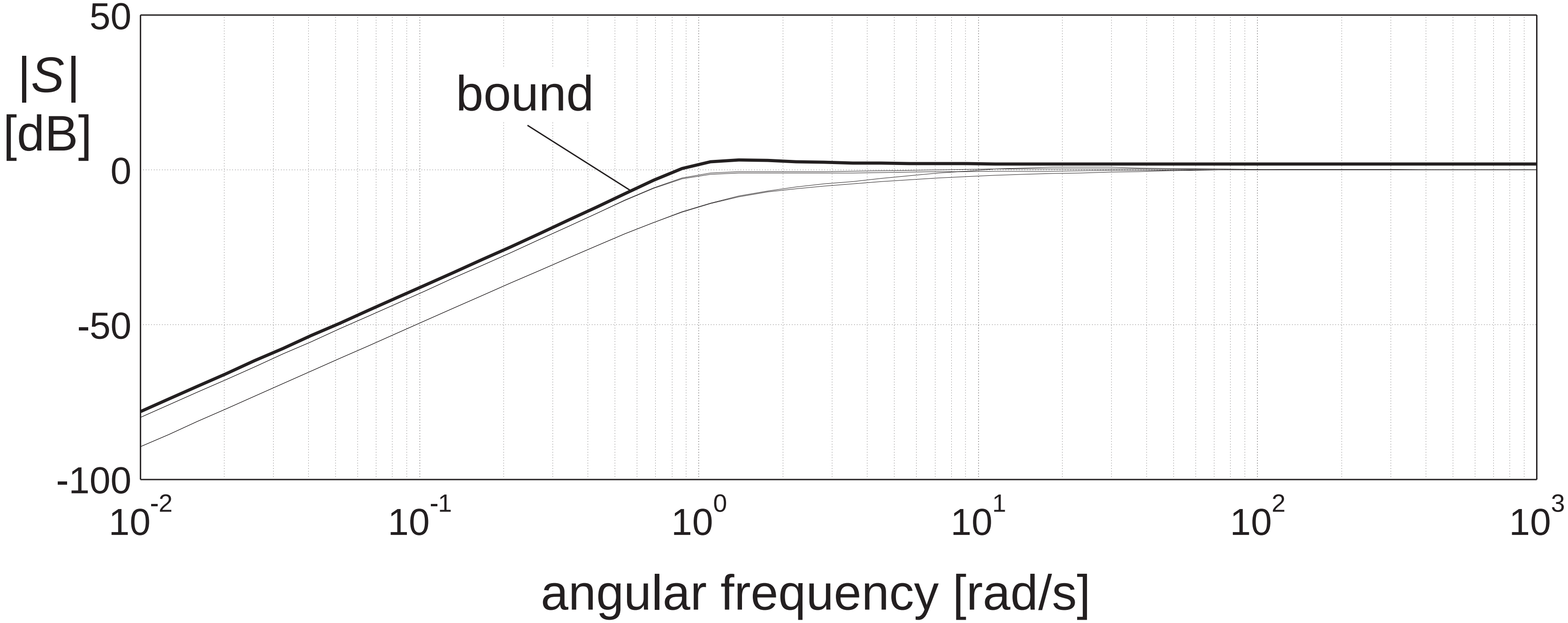
D-scale and fit



Structured singular value subsequent iterations



Perturbed S after two $D - K$ iterations



Further iterations make $\sup_\omega \mu$ smaller and smaller

Reason:

- No bandwidth limitations imposed (controller $\rightarrow$ high gain)
- Plant dynamics imply no inherent bandwidth limitations [Freudenberg–Looze]

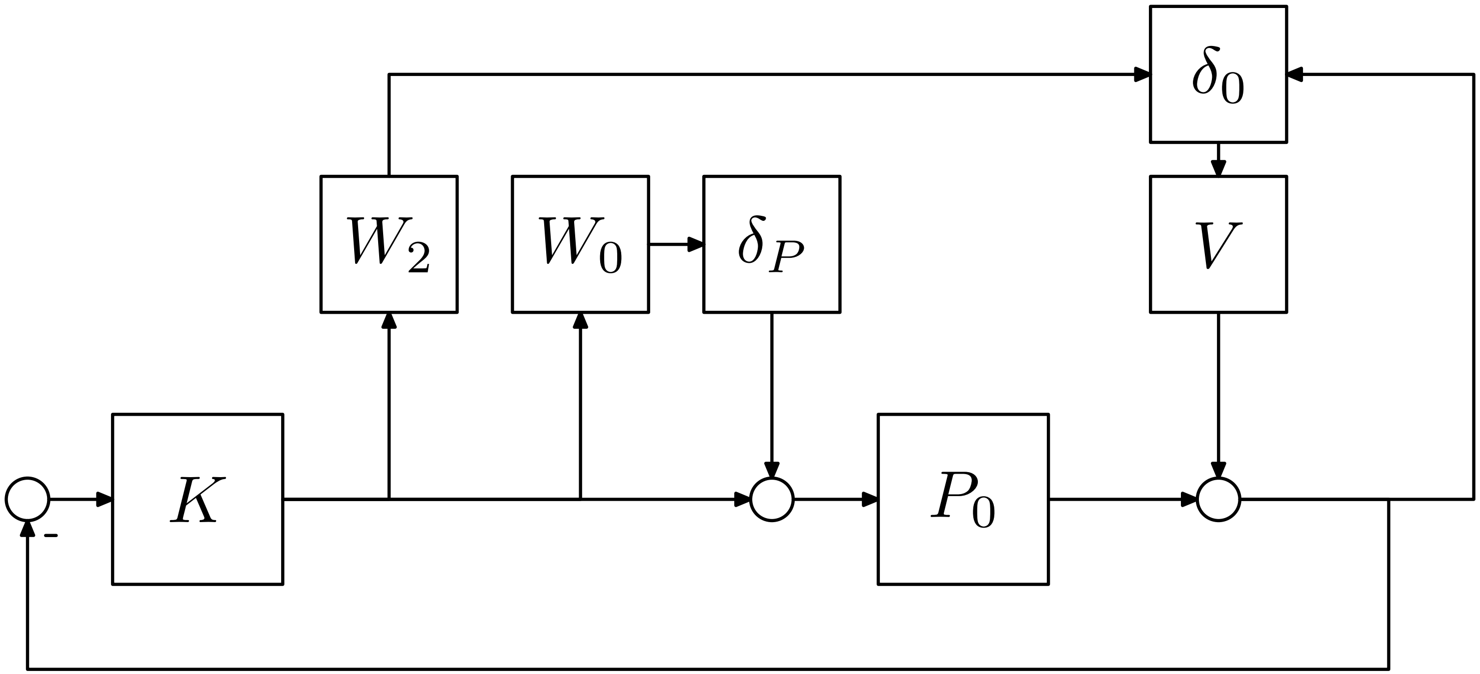
Remedy: adjust 'adequate performance' to mean, say

$$\left\| \begin{bmatrix} W_1 SV \\ W_2 UV \end{bmatrix} \right\|_\infty \leq 1, \qquad W_1(s) = 1, \quad W_2(s) = \frac{s + \omega_1}{\beta}$$

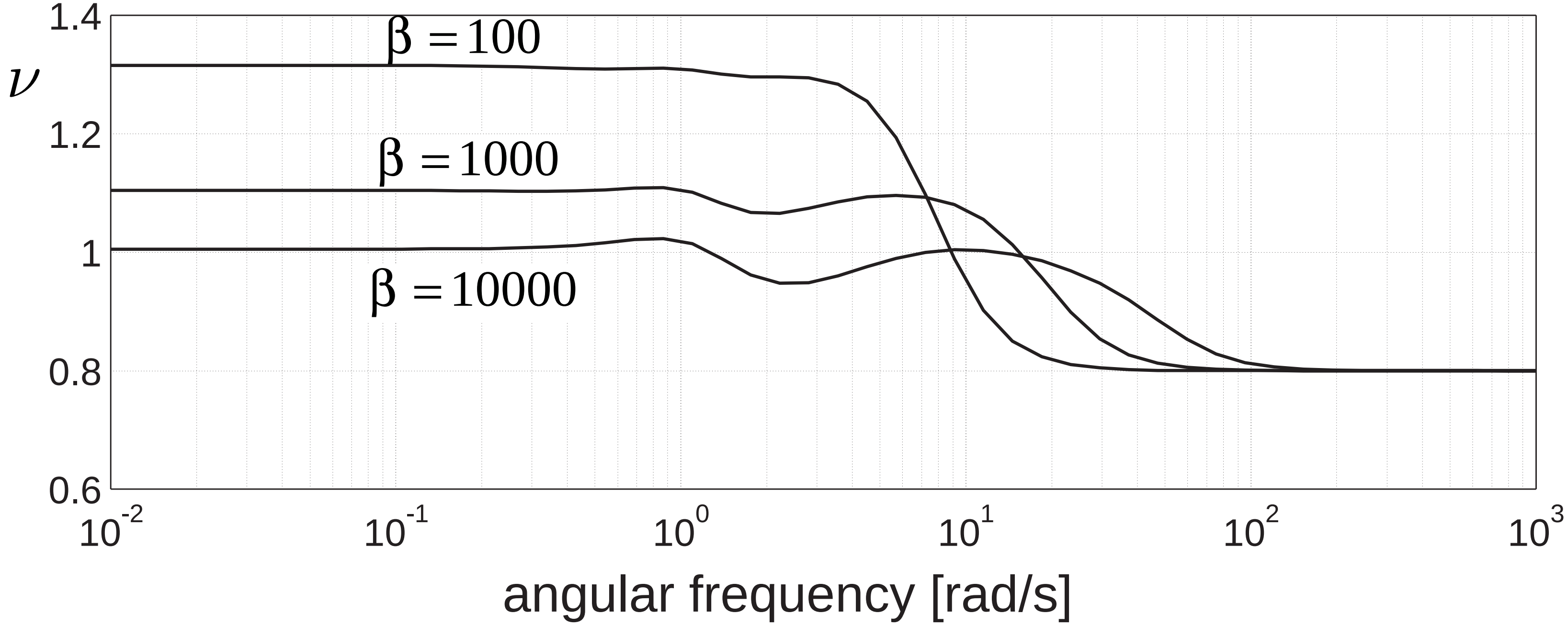
with, say, $\beta = 100$ and $\omega_1 = 1$.

This enforces HF roll-off

Again setting up the problem:



Structured singular value plots for three successive designs



$\beta > 10000$ is needed to get robust performance $(\sup_\omega \mu < 1)$

Important final step: order reduction of final controller:

Originally final controller has order 9. After reduction order 2:

$$K(s) = 6420 \frac{s + 0.6234}{(s + 22.43)^2 + 45.31^2}$$

$\mathcal{H}_\infty$ loop shaping (MacFarlane & Glover)

Algorithm (simplified version)

1. Choose weights W_1 and W_0 to make $P_a := W_0 P W_1$ your favorite loop gain

2. Solve standard H_∞ problem

$$\gamma_{\text{opt}} := \min_{\text{stabilizing } K_a} \left\| \begin{bmatrix} S & SP_a \\ K_a S & K_a S P_a \end{bmatrix} \right\|_\infty$$

3. Take $K = W_1 K_a W_0$ if stability margin $\epsilon := 1/\gamma_{\text{opt}}$ is large enough otherwise modify W_0, W_1

Why useful?

1. Optimal K_a can be solved in one shot
(it's a Nehari extension problem; no γ-iteration required)

2. Penalizes all closed-loop transfer functions
(hence controller doesn't cancel slow modes in plant [if any])

3. Loop shaping "independent" from stability consideration
($\mathcal{H}_\infty$ optimization takes care of c.l. stability)

4. $\epsilon \in [0, 1]$ is an indicator how well favorite loop gain P_a can
be achieved by stabilizing controller.
(fantastic if $\epsilon = 1$, terrible if $\epsilon \approx 0$)

5. Explicit bounds on achieved singular values exist, for instance:

$$\overline{\sigma}(S) \leq \frac{1}{\epsilon}\sqrt{\frac{1}{1+\underline{\sigma}^2(P_a)}}\, \mathrm{cond}(W_0).$$

6. Works for high order and MIMO plants