MIMO control design (Chapter 4)

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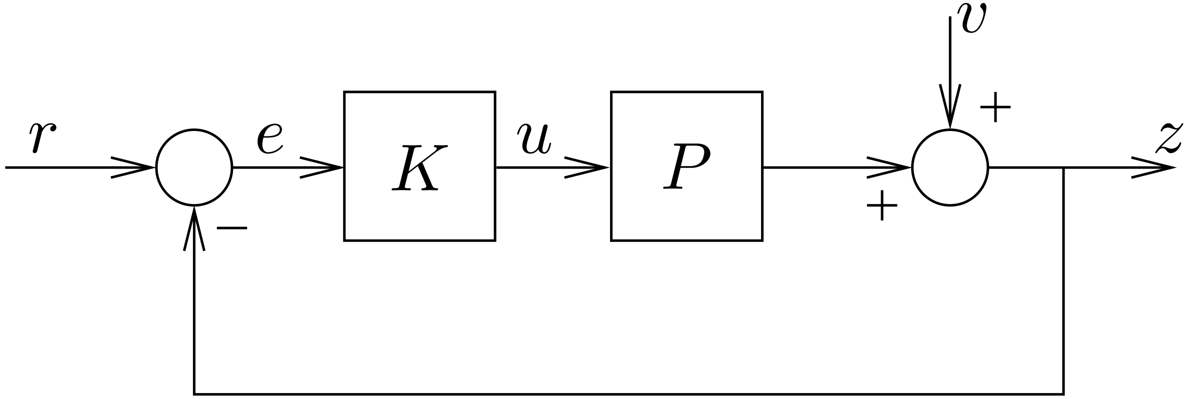
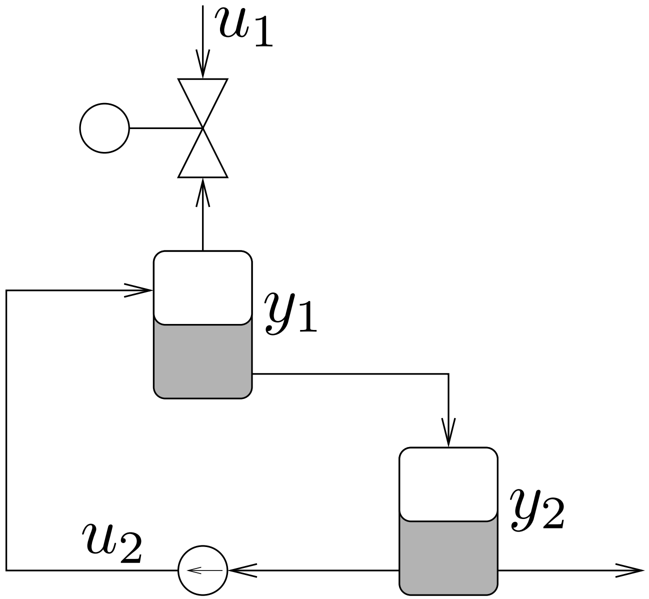
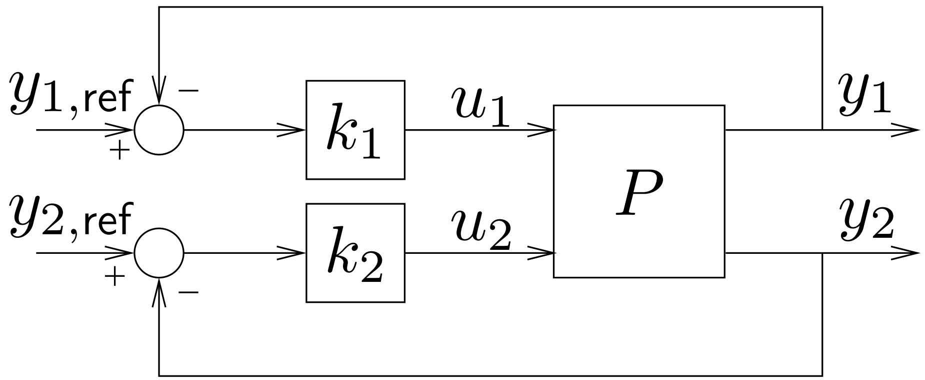
Intro



- input $u(t) \in \mathbb{R}^{n_u}$ and output $y(t) \in \mathbb{R}^{n_y}$ are vector valued

Intro





Example 1 (Decentralized & Interaction). A natural i/o-pairing:

$$u_1 = k_1\ (y_{1,\text{ref}} - y_1), \qquad u_2 = k_2\ (y_{2,\text{ref}} - y_2).$$

This is decentralized control.

Intro

Suppose

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s(s+1)} & -\frac{1}{s+1} \end{bmatrix}.$$

When lower loop closed with

$$u_2 = k_2 \left(y_{2,\text{ref}} - y_2\right)$$

then

$$\underbrace{y_1 = P_{11}\big|_{\text{llc}} u_1}_{\text{upper loop}}, \qquad P_{11}(s)\big|_{\text{llc}} = \frac{1}{s+1}\left(1 - \frac{k_2}{s(s+1-k_2)}\right).$$

Intro

For instance

$$k_2 = 0: \qquad P_{11}(s)\Big|_{\text{llc}} = \frac{1}{s+1}$$

$$k_2 = -1: \qquad P_{11}(s)\Big|_{\text{llc}} = \frac{s+1}{s(s+2)}$$

$$k_2 = -\infty: \qquad P_{11}(s)\Big|_{\text{llc}} = \frac{1}{s}$$

There is interaction.

Dynamics changes with k_2.

Cannot design k_1 independent of k_2.

Intro

Example 2 (Zeros & decoupling). Suppose

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}.$$

Then

$$\underbrace{u_2 = \infty y_2}_{\text{perfect tracking lower loop}} \qquad \Longrightarrow \qquad y_1 = -\frac{s-1}{(s+1)(s+3)} u_1$$

Bummer: non-minimumphase upper loop.

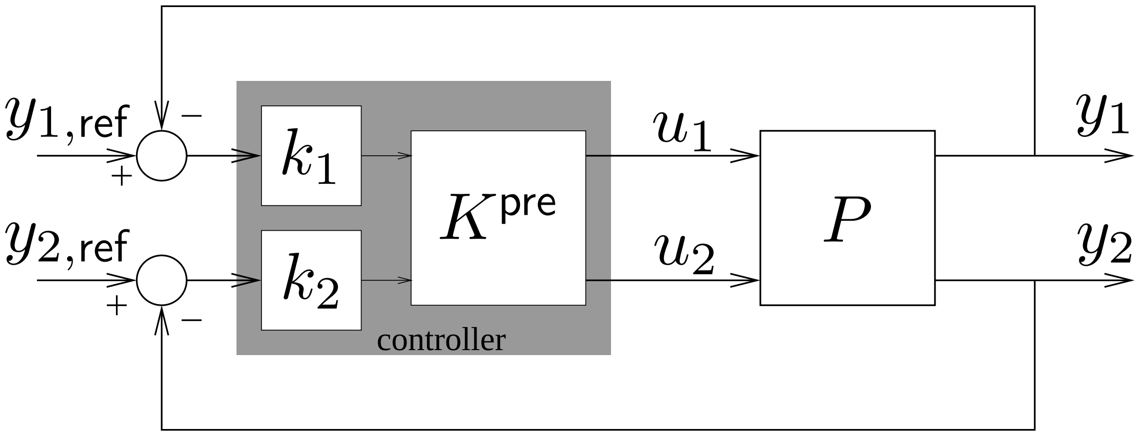
Likewise

$$\underbrace{u_1 = \infty y_1}_{\text{perfect tracking upper loop}} \qquad \Longrightarrow \qquad y_2 = -\frac{s-1}{(s+1)(s+3)}u_2$$

Bummer: non-minimumphase lower loop.

Intro

Let's instead try decoupling:



- "Ideal" $K^{\text{pre}} := P^{-1}$ cancels unstable zero of P: not allowed.
- For (4.14) — see lect.notes — we get

$$P(s)K^{\text{pre}}(s) = \begin{bmatrix} -\frac{s-1}{(s+1)(s+3)} & 0 \\ 0 & -\frac{s-1}{(s+1)(s+3)} \end{bmatrix}.$$

Bummer: non-minimumphase in both channels.

Intro

Claim: There is such a thing as MIMO zeros (here $s = 1$) which we can not get rid of by choice of stabilizing controller.

Need notion of zeros & poles for MIMO systems.

Overview (chap 4)

Technical machinary

- Poles and zeros of MIMO systems
  - Smith-McMillan of rational transfer matrix
  - Transmission zero (state space)

Design methods

- Decoupling control: "take $K = P^{-1}K_{\mathsf{d}}$"
- Decentralized control: "take K diagonal"
- servocompensators (state space "integrating action")

Poles and zeros of MIMO systems

Three examples of polynomial matrices:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}}_{\text{single zero}} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix}}_{\text{double zero}} \qquad \underbrace{\begin{bmatrix} 1 & s+1 \\ 0 & 1 \end{bmatrix}}_{\text{no zero}}$$

Definition 3. $V(s)$ is unimodular if $\det V(s) = c \neq 0$.

Poles and zeros of MIMO systems

For diagonal matrices (poles and) zeros are trivial:

$$\begin{bmatrix} s & 0 \\ 0 & s(s+1) \end{bmatrix}$$

has zeros:

$$s_1 = -1, \qquad s_2 = 0 \text{ (twice)}$$

Poles and zeros of MIMO systems

Lemma 4 (Smith form of a polynomial matrix). *For every polynomial matrix* N *there exist unimodular* U *and* V *such that*

$$UNV = S := \begin{bmatrix} \epsilon_1' & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \epsilon_r' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

Its zeros are the zeros of $\prod \epsilon_i'$. *Its rank is* r.

Poles and zeros of MIMO systems

To find Smith form S of N apply elementary row and column operations:

- multiply a row/column by a nonzero constant;
- interchange two rows or two columns;
- add row/column multiplied by polynomial to another row/column.

Poles and zeros of MIMO systems

Example 5.

$$N(s) := \begin{bmatrix} s+1 & s-1 \\ s+2 & s-2 \end{bmatrix} \overset{(1)}{\Rightarrow} \begin{bmatrix} s+1 & s-1 \\ 1 & -1 \end{bmatrix} \overset{(2)}{\Rightarrow} \begin{bmatrix} s+1 & 2s \\ 1 & 0 \end{bmatrix}$$

$$\overset{(3)}{\Rightarrow} \begin{bmatrix} 0 & 2s \\ 1 & 0 \end{bmatrix} \overset{(4)}{\Rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} =: S(s).$$

Zeros and poles of rational matrices

Let P be a rational matrix $P = \frac{1}{d}N$

Lemma 6 (Smith-McMillan form of a rational matrix). *For every rational matrix P there exist unimodular polynomial matrices U and V such that*

$$UPV = M := \begin{bmatrix} \frac{\epsilon_1}{\psi_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon_2}{\psi_2} & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon_r}{\psi_r} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

with ϵ_i and ψ_i coprime. The zeros of P are the zeros of $\prod \epsilon_i$ and its poles are the zeros of $\prod \psi_i$.

Zeros and poles of rational matrices

- s is a pole of P iff it is a pole of some of its entries P_{ij}.

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- s can be a pole and a zero at the same time

$$\begin{bmatrix} 1/s & 0 \\ 0 & s/(s+1) \end{bmatrix}$$

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- Zeros are often difficult to spot: E.g. $s = 0$ is a zero of

$$P(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \qquad \text{(see next statement).}$$

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- If P is invertible, then s is a zero of P iff it is a pole of its inverse:

$$P(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad P^{-1}(s) = \begin{bmatrix} 1 & -1/s \\ 0 & 1 \end{bmatrix}.$$

CL Stability $\implies$ no unstable pole-zero cancellation

Theorem 7. *A controller K destabilizes if there are unstable pole-zero cancellations in forming the product $L = PK$.*

That is, closed-loop stability requires

$$\mathcal{Z}(K) \cup \mathcal{Z}(P) = \mathcal{Z}(PK).$$

($\mathcal{Z}$ denotes the set of CRHP zeros).

State-space realizations

State space description are useful for MIMO systems:

$$P(s) = C(sI - A)^{-1}B + D$$

with $A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times n_u}, \quad C \in \mathbb{R}^{n_y \times n}, \quad D \in \mathbb{R}^{n_y \times n_u}$.

$$y = Pu \qquad \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array} \right.$$

Transmission zeros

Definition 8. The transmission zeros are the zeros of the polynomial matrix

$$\begin{bmatrix} A - s_0 I & B \\ C & D \end{bmatrix},$$

i.e., the values where this polynomial matrix loses rank.

Transmission zeros

Lemma 9. *If realization* $P(s) = C(sI - A)^{-1}B + D$ *is minimal, then* $s \in \mathbb{C}$ *is zero of* P *iff it is a transmission zero.*

Lemma 10. s_0 *is a transmission zero* $\iff \exists u(t) = u_0 e^{s_0 t},\ Pu = 0$

Lemma 11. *If realization* $P(s) = C(sI - A)^{-1}B + D$ *is minimal, then* $s_0 \in \mathbb{C}$ *is pole of* P *iff it is an eigenvalue of* A*.*

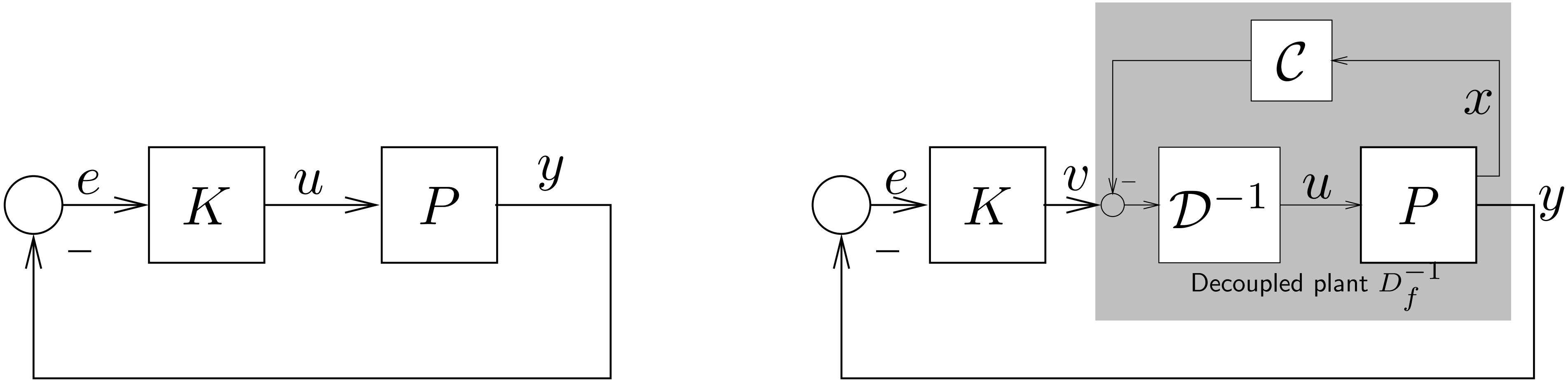
Computation of transmission zeros

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} = \begin{bmatrix} (A - BD^{-1}C) - sI & B \\ 0 & D \end{bmatrix}$$

So transmission zeros are the eigenvalues of $A - BD^{-1}C$

This assumes D invertible.
There are variations for non-invertible D

Decoupling Control



(a) (b)

Given $P(s) = C(sI - A)^{-1}B + D$, find $u = \mathcal{D}^{-1}(-\mathcal{C}x + v)$ such that resulting system $y = D_f^{-1}v$ is diagonal.

Decoupling Control

Try $v = y$!!!!

$$
\begin{aligned}
v &= y \\
v &= Cx + Du \\
u &= D^{-1}(-Cx + v)
\end{aligned}
$$

...too good to be true.

Decoupling Control

Indeed:

$$\underbrace{\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}}_{\text{system matrix of } P} \begin{bmatrix} I & 0 \\ -D^{-1}C & D^{-1} \end{bmatrix} = \underbrace{\begin{bmatrix} (A - BD^{-1}C) - sI & BD^{-1} \\ 0 & I \end{bmatrix}}_{\text{system matrix of } D_f^{-1}}$$

All transmission zeros end up as poles-of-the-realization of D_f and none of these are observable!

Conclusion: Not allowed if there is one or more CRHP tranmission zero.

Decoupling Control

What to do if $D := \lim_{s\to\infty} P(s)$ has less than full row rank?

Define

$$\mathcal{D} := \lim_{s\to\infty} D_f(s)P(s), \qquad D_f(s) = \text{diag}(s^{f_1}, s^{f_2}, \ldots, s^{f_m})$$

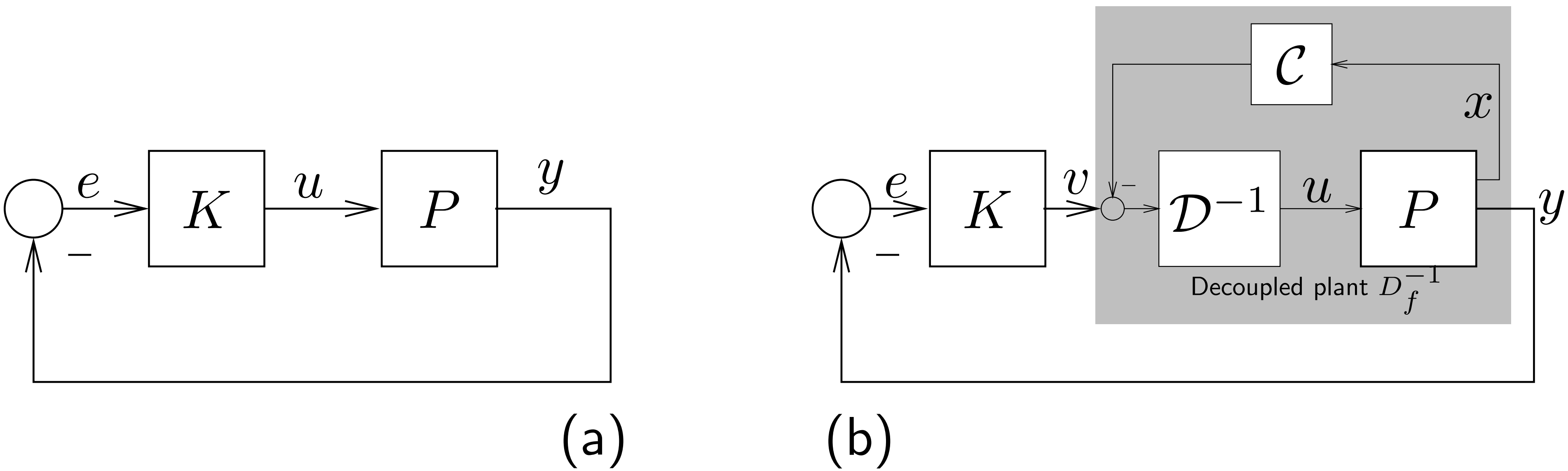
The D_f is a differentiator

$$v = D_f P u \quad \Longleftrightarrow \quad \begin{aligned} \dot{x} &= Ax + Bu, \\ v = \begin{bmatrix} \frac{d^{f_1}}{dt^{f_1}} y_1 \\ \frac{d^{f_2}}{dt^{f_2}} y_2 \\ \vdots \\ \frac{d^{f_m}}{dt^{f_m}} y_m \end{bmatrix} &= \underbrace{\begin{bmatrix} C_{1*}A^{f_1} \\ C_{2*}A^{f_2} \\ \vdots \\ C_m A^{f_m} \end{bmatrix}}_{\mathcal{C}} x + \mathcal{D}u \end{aligned}$$

and by construction get diagonal

$$v = D_f y \quad \Longleftrightarrow \quad y = D_f^{-1} v.$$

Decoupling Control

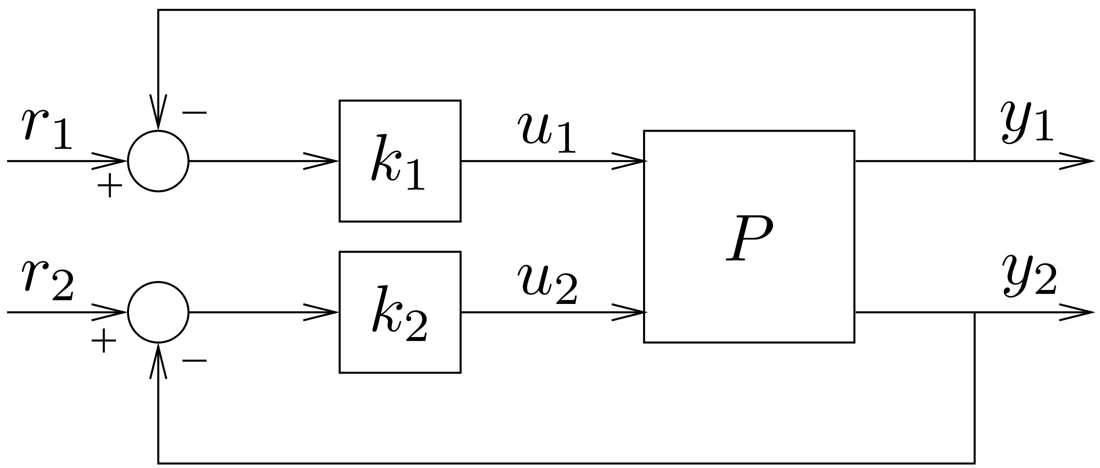


transmission zeros of P = transmission zeros of D_f^{-1}

(All transmission zeros are unobservable since D_f^{-1} has no zeros.)

Decentralized control

The relative gain array (RGA)



Find “good” pairings $u_i \leftrightarrow y_j$

(and then design $y_j = k_{ji} u_i$ without regard for other loops)

Decentralized control

Tuning of second loop independent of first loop if

$$\lambda := \frac{\text{‘gain’ from } u_2 \text{ to } y_2 \text{ if upper loop is open}}{\text{‘gain’ from } u_2 \text{ to } y_2 \text{ if upper loop is closed}}$$

is close to 1.

‘If $\lambda = 1$ then k_2 can be designed independently from k_1’

Suppose all signals are constant (in steady state). Then

$$\lambda = \frac{dy_2/du_2 \text{ if upper loop is open}}{dy_2/du_2 \text{ if upper loop is closed}}$$

Decentralized control

Assume also perfect tracking:

$$\lambda = \frac{dy_2/du_2 \text{ if upper loop is open}}{dy_2/du_2 \text{ if upper loop is closed}}$$

$$= dy_2/du_2\Big|_{u_1} \times du_2/dy_2\Big|_{y_1}$$

$$= P_{22}(0) \times (P^{-1})_{22}(0)$$

Decentralized control

Suppose

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ P_{m1} & \cdots & \cdots & P_{mm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

define

$$\Lambda = P \circ (P^{-1})^T \qquad \text{(an } m \times m \text{ matrix)}$$

where $\circ$ denotes the Hadamard product (elementwise product).

Then

$$\Lambda_{ij}(0)$$

is a measure of interconnection:

If $\Lambda_{ij}(0) \approx 1$ then pair y_i with u_j

Decentralized control

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \;\Rightarrow\; \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix} \;\Rightarrow\; \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

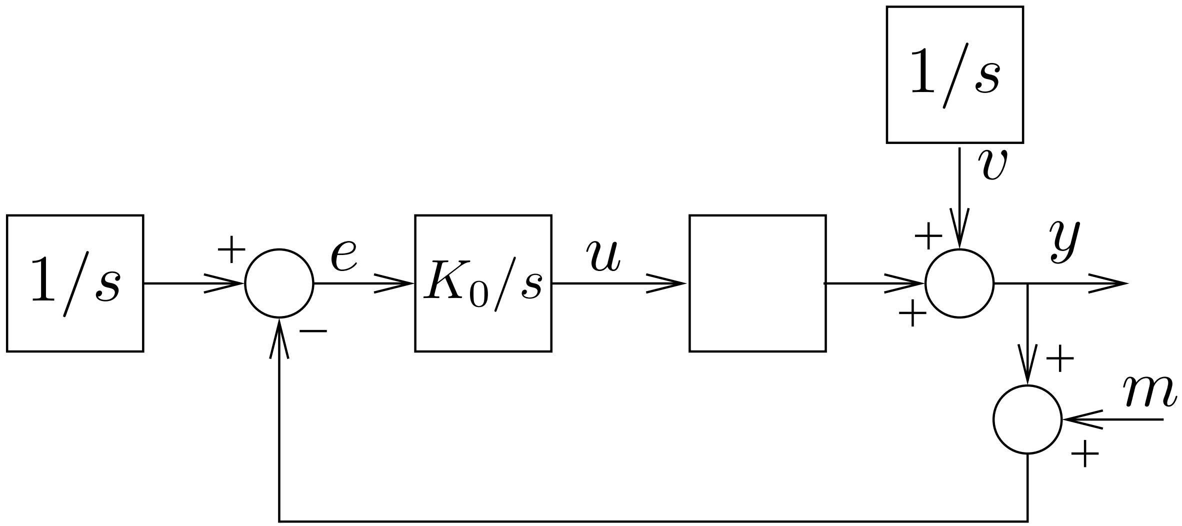
$$P = \begin{bmatrix} 0 & P_{12} \\ P_{21} & 0 \end{bmatrix} \;\Rightarrow\; \Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} p & p \\ -p & p \end{bmatrix} \;\Rightarrow\; \Lambda = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$P = \begin{bmatrix} a & a \\ a & a(1+\delta) \end{bmatrix} \;\Rightarrow\; \Lambda = \begin{bmatrix} \frac{1+\delta}{\delta} & \frac{-1}{\delta} \\ \frac{-1}{\delta} & \frac{1+\delta}{\delta} \end{bmatrix}$$

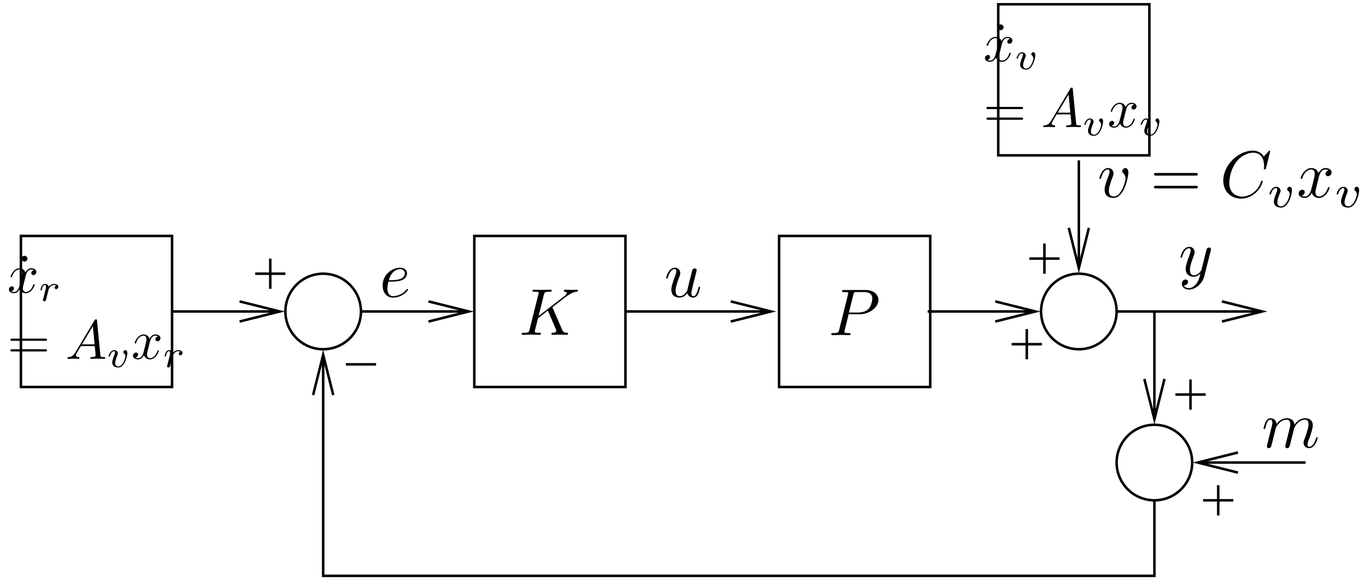
Servomechanism Problem

.. is to design a controller — called servocompensator — that renders y insensitive to disturbances v and asymptotically track certain reference inputs r.



Example: Integral action: tracks constant r and rejects constant v.
But what if r is not constant & v is not constant & MIMO?

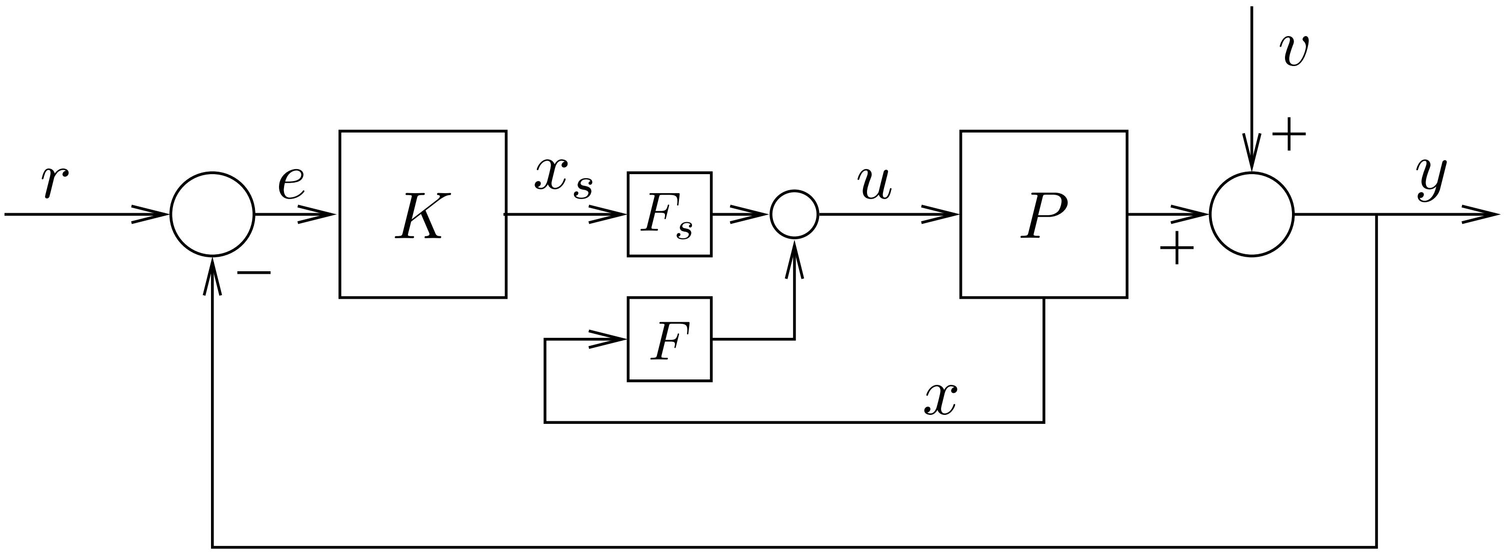
Servomechanism Problem



controller K: $\quad \dot{x}_s(t) = \operatorname{diag}\underbrace{(A_v, A_v, \ldots, A_v)}_{n_v \text{ times}} x_s(t) + B_s e(t)$

Stability not yet guaranteed.

Decentralized control



$$u = Fx + F_s x_s$$

Where F, F_s are chosen to stabilize the system.

May use observer for x (does not affect stability).

Decentralized control

Lemma 12. *Let* $P(s) = C(sI - A)^{-1}B + D$ *and* $\lambda_i(A_v) \in j\mathbb{R}$. *Suppose*

1. (A, B) *is stabilizable and* (A, C) *is detectable*
2. $n_u \geq n_y$
3. $\operatorname{rank}\left[\begin{smallmatrix} A-\lambda_i I & B \\ C & D \end{smallmatrix}\right] = n + n_y$ *for every eigenvalue* λ_i *of* A_v

Then F, F_s *can be selected such that the closed loop system matrix*

$$\begin{bmatrix} A + BF & BF_s \\ -B_s C & A_s \end{bmatrix}$$

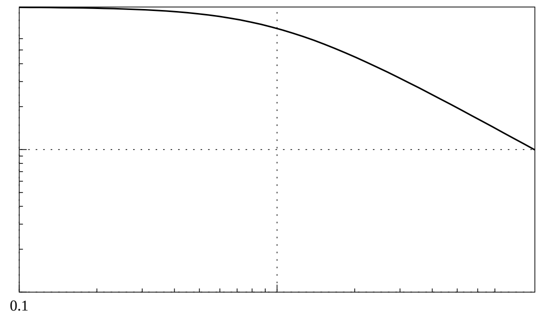
has all its eigenvalues in the open left-half plane

In that case $e(t) \to 0$ *as* $t \to \infty$ *for any of the* v *and* r *defined earlier*

Gains and norms

Bode plot for SISO is well defined

$$P(s) = \frac{1}{s} \quad \Longrightarrow \quad T(s) = \frac{1}{s+1} =$$



but Bode plot for MIMO ??

$$T(s) = \begin{bmatrix} 1 & \frac{1}{s} \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad ??$$

Plots of eigenvalues of $T(\mathrm{j}\omega)$ are no good: Norms to the rescue

Gains and norms

The Euclidean norm of vectors:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \|x\|_2 := \sqrt{\sum_i |x_i|^2} = \sqrt{x^T x}.$$

Then

$$\underline{\sigma}(M) \quad \leq \quad \frac{\|Mx\|_2}{\|x\|_2} \quad \leq \quad \overline{\sigma}(M)$$

$\sqrt{\lambda_{\text{min}}(M^*M)}$

$\sqrt{\lambda_{\text{max}}(M^*M)}$, the spectral norm

Gains and norms

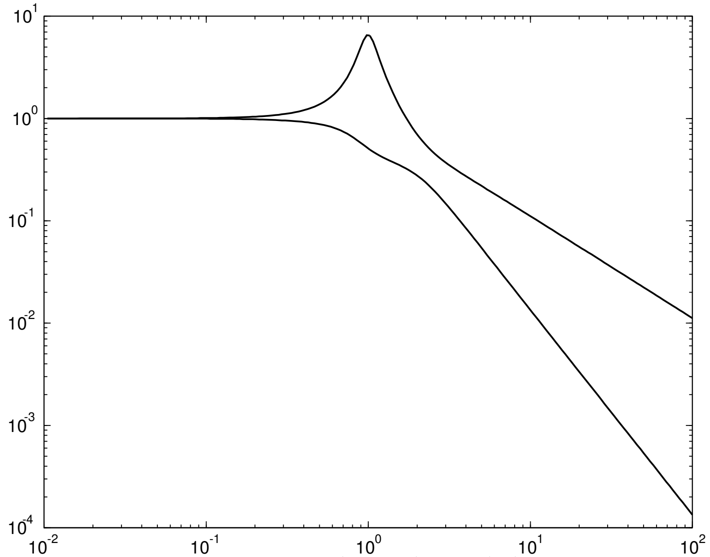
$$y = Tr$$

$$\underline{\sigma}(T(\mathrm{j}\omega)) \;\leq\; \frac{\|y(\mathrm{j}\omega)\|_2}{\|r(\mathrm{j}\omega)\|_2} \;\leq\; \overline{\sigma}(T(\mathrm{j}\omega))$$

minimal gain

maximal gain

Gains and norms



Bounds on gains: $\underline{\sigma}(T(\mathrm{j}\omega))$ and $\overline{\sigma}(T(\mathrm{j}\omega))$

Design goals:

- $\overline{\sigma}(S(\mathrm{j}\omega)) \ll 1$ low frequency
- $\overline{\sigma}(T(\mathrm{j}\omega)) \ll 1$ high frequency

The $\mathcal{H}_\infty$-norm

$$\sup_x \frac{\|Mx\|_2}{\|x\|_2} = \overline{\sigma}(M)$$

$$\sup_x \frac{\|Mx\|_2^2}{\|x\|_2^2} = \overline{\sigma}(M)^2$$

consider stable system $y = Hu$

power gain at freq. ω

$$\sup_{u(\mathrm{j}\omega)} \frac{\|y(\mathrm{j}\omega)\|_2^2}{\|u(\mathrm{j}\omega)\|_2^2} = \overline{\sigma}(H(\mathrm{j}\omega))^2$$

output power at freq. ω

input power at freq. ω

The $\mathcal{H}_\infty$-norm

$$\begin{aligned} \sup_\omega \text{power gain at freq. } \omega &= \text{maximal power gain} \\ &= \text{maximal energy gain} \end{aligned}$$

Precise: if $y = Hu$ stable then

$$\sup_u \frac{\sqrt{\int_{-\infty}^{\infty} \|y(t)\|^2 \, dt}}{\sqrt{\int_{-\infty}^{\infty} \|u(t)\|^2 \, dt}} = \sup_\omega \overline{\sigma}(H(\mathrm{j}\omega)) \quad \text{(when stable)}$$

$$=: \|H\|_{\mathcal{H}_\infty} =: \|H\|_\infty \quad \text{(when stable)}$$