MIMO HOWTO

version 1.0

Jeroen de Best, Matthijs Boerlage, Maarten Steinbuch, Marc van de Wal, Ton van der Weijden

I. Gains and Directions

The direction of a vector d is defined as;

$$d_{dir} = \frac{1}{\|d\|_2} d \tag{1}$$

In order to express the amplification of a system as a scalar value, norms can be used. The maximum and minimum amplification of a matrix are the maximum and minimum singular value respectively;

$$\overline{\sigma}(A) = \max_{d \neq 0} \frac{\|Aw\|_2}{\|w\|_2}, \quad \underline{\sigma}(A) = \min_{w \neq 0} \frac{\|Aw\|_2}{\|w\|_2} \tag{2}$$

The *condition number* of a matrix A is the ratio between the maximum singular value and the minimum singular value,

$$\gamma(A) = \overline{\sigma}(A)/\underline{\sigma}(A) \tag{3}$$

and therefore gives an indication of the ”change” of amplification when a small change in direction is present. A matrix with a large condition number is said to be ill-conditioned. The condition number is sensitive for diagonal scaling, the minimal condition number is then; $\gamma^*(A) = \min_{D_1,D_2} \gamma(D_1 A D_2)$ for arbitrary diagonal scaling D_1, D_2.

Matrix Norms

Consider the following equation $z = Aw$. Where w is the input vector and z is the output vector. We consider ”amplification” of the matrix A as defined by the ratio $\|z\| / \|w\|$. The maximum gain for all possible input directions is of particular interest. This is given by the induced norm which is defined as

$$\|A\|_{ip} \equiv \max_{w \neq 0} \frac{\|Aw\|_p}{\|w\|_p} \tag{4}$$

where $\|w\|_p = \left(\sum_{i=1} |w_i|^p\right)^{1/p}$ denotes the vector p-norm. In other words we are looking for a direction of the vector w such that the ratio $\|z\|_p / \|w\|_p$ is maximized. Thus, the induced norm gives the largest possible ”amplification power” of the matrix.

For the induced 1-, 2-, ∞-norms the following identities hold:

$\|A\|_{i1} = \max_j \left(\sum_i |a_{ij}|\right)$ ”maximum column sum”

$\|A\|_{i\infty} = \max_i \left(\sum_j |a_{ij}|\right)$ ”maximum row sum”

$\|A\|_{i2} = \overline{\sigma}$ ”singular value norm”

MATLAB: `norm.m`

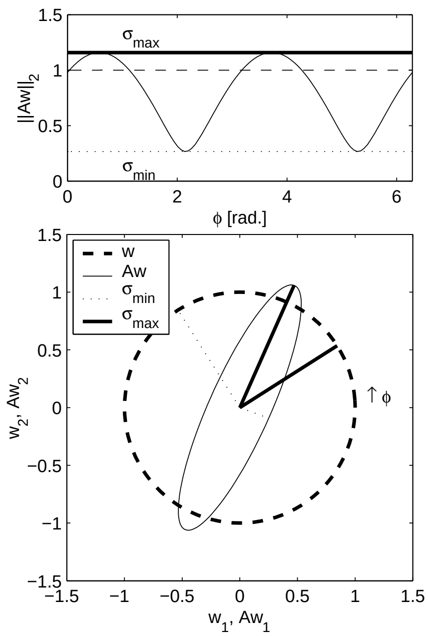

Fig. 1. w and Aw for $\phi = [0...2\pi]$, when the direction of w equals the columns of V^T, amplification is given by the maximum or minimum singular value of A. Here $A = [0.26310.4638; 0.94330.4894]$

MATLAB: `eigshow.m`

II. Eigenvalue Decomposition

For a square matrix A we have the eigenvalue problem;

$$At_i = \lambda_i t_i \tag{5}$$

where λ_i is the eigenvalue and t_i is the eigenvector. We collect the eigenvectors in a matrix $T = \{t_1, t_2, ...\}$ and the eigenvalues in a matrix $\Lambda = diag\{\lambda_1, \lambda_2, ...\lambda_n\}$ When A is 2×2 the eigenvalues of A can be found as;

$$\lambda_\pm = \frac{1}{2}[(a_{11} + a_{22}) \pm \sqrt{4a_{12}a_{21} + (a_{11} - a_{22})^2})] \tag{6}$$

Eindhoven University of Technology Faculty of Mechanical Engineering, Control Systems Technology Group P.O. Box 513, 5600 MB Eindhoven, The Netherlands Email: m.l.g.boerlage@tue.nl, m.steinbuch@tue.nl

Some tricks

$$
\begin{aligned}
tr(A) &= \sum_i \lambda_i && (7)\\
det(A) &= \prod_i \lambda_i && (8)\\
\lambda(A) &= \lambda(A^T) && (9)\\
\lambda(A+cI) &= \lambda(A)+c && (10)\\
\lambda((I+L)^{-1}) &= \frac{1}{1+\lambda_i(L)} && (11)
\end{aligned}
$$

Remarks

- When A is triangular, the eigenvalues of A equal the diagonal elements of A.
- When A is $l \times m$ and B is $m \times l$, BA is $m \times m$ and AB is $l \times l$. The nonzero eigenvalues of AB and BA are the same. So if $l > m$ the matrix AB has the same m eigenvalues of BA plus $l - m$ eigenvalues which are identically equal to zero.
- The eigenvalues of the openloop transfer function matrix $L(j\omega)$ are called the *characteristic loci*.
- If A is $n \times n$, the eigenvalues of A lie in the union of n circles in the complex plane, each with center a_{ii} and radius $r_i = \sum_{j \neq i} |a_{ij}|$ (sum of off-diagonal elements in row i). The same can be done studying the columns. This is called the *Gershgorin* theorema [10].
- When the eigenvalues of a $n \times n$ matrix are not repeated, the corresponding eigenvectors are linear independent and can be used to diagonalize the matrix as; $\Lambda = T^{-1}AT$.

MATLAB: `eig.m`

III. Singular Value Decomposition

If the matrix of eigenvectors P of a given matrix A is not a square matrix, then P cannot have a matrix inverse, and hence A does not have an eigen decomposition. However, if A is an $m \times n$ real matrix with $m > n$, then A can be written using a so-called singular value decomposition of the form

$$A = UDV^T \tag{12}$$

Here, U is an $m \times n$ matrix and V is an $n \times n$ square matrix, both of which have orthogonal columns so that

$$UU^T = VV^T = I \tag{13}$$

and D is an diagonal matrix. For a complex matrix A, the singular value decomposition is a decomposition into the form

$$A = U^H DV \tag{14}$$

where U and V are unitary matrices (A square matrix U is unitary if $U^H = U^{-1}$), U^H is the conjugate transpose of U , and D is a diagonal matrix whose elements are the singular values of the original matrix. If A is a complex matrix, then there always exists such a decomposition with positive singular values.

Some tricks:

$$
\begin{aligned}
\sigma_i(A) &= \sqrt{\lambda_i(A^H A)} && (15)\\
\overline{\sigma}(A^{-1}) &= 1/\underline{\sigma}(A) && (16)\\
A &= U\Sigma V^H = \sum_{i=1}^{r} \sigma_i u_i v_i^H && (17)\\
\overline{\sigma}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} &= \max\{\overline{\sigma}(A), \overline{\sigma}(B)\} && (18)\\
\overline{\sigma}\begin{bmatrix} A \\ B \end{bmatrix} &\leq \overline{\sigma}(A) + \overline{\sigma}(B) && (19)\\
\overline{\sigma}(AB) &\leq \overline{\sigma}(A)\overline{\sigma}(B) && (20)\\
\underline{\sigma}(A)\underline{\sigma}(B) &\leq \underline{\sigma}(AB) && (21)
\end{aligned}
$$

Remarks

- A matrix U which is orthogonal matrices is invertible and $U^{-1} = U^T$
- The columns of U, V are orthogonal.
- The pseudo inverse of a matrix is defined as
- $\underline{\sigma}(A) \leq |\lambda_i(A)| \leq \overline{\sigma}(A)$
- For a non singular $m \times m$ matrix A holds that $A^{-1} = V\Sigma^{-1}U^H$

MATLAB: `svd.m`

IV. MIMO systems

Notation

In state space form;

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned} \tag{22}
$$

or transfer function matrix form;

$$G(s) = C(sI - A)^{-1}B + D \tag{23}$$

Poles

Poles of $G(s)$, are the eigenvalues of A in Equation 22.

Zeros

z_i is a zero of $G(s)$ is the rank of $G(z_i)$ is smaller then the normal rank of $G(s)$. Note that $G(s)$ can also loose rank when the elements of $G(z_i)$ are non-zero. When this happens, we call z_i a transmission zero of $G(s)$.

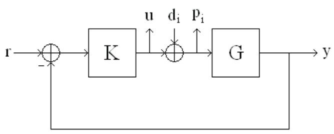

Fig. 2. Case 1

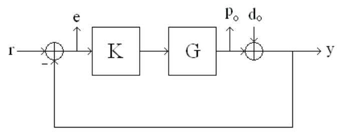

Fig. 3. Case 2

Transfer functions

$$
\begin{aligned}
u &= -KG(d_i+u) & p_o &= -GK(d_o+p_o)\\
u &= -\underbrace{(I+KG)^{-1}KG}_{T_i}\,d_i & p_o &= -\underbrace{(I+GK)^{-1}GK}_{T_o}\,d_o\\
p_i &= d_i - KGp_i & e &= -(d_o+GKe)\\
p_i &= \underbrace{(I+KG)^{-1}}_{S_i}\,d_i & e &= -\underbrace{(I+GK)^{-1}}_{S_o}\,d_o
\end{aligned}
$$

Note that in general $S_o \neq S_i, T_o \neq T_i$ as $GK \neq KG$. All transfer functions in Figure 4 are given in Equation 24.

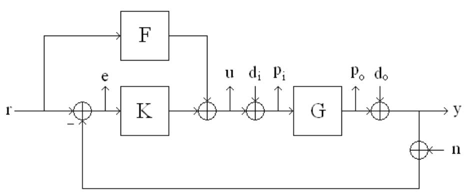

Fig. 4. Transfer functions

V. MEASURES OF INTERACTION

Interaction coefficient

The interaction index of a matrix G is defined as

$$\phi = \frac{g_{12}g_{21}}{g_{11}g_{22}} \tag{25}$$

When ϕ is close to zero this means no interaction.

Relative Gain Array

The Relative Gain Array (RGA) of a non-singular matrix G, square matrix is defined as

$$RGA(G) \triangleq G \times (G^{-1})^T \tag{26}$$

where $\times$ denote element-by-element multiplication. MATLAB: `RGA=G.*pinv(G.')` For a 2×2 matrix with elements g_{ij} the RGA equals;

$$
\begin{aligned}
\Lambda(G) &= \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix} & (27)\\
\lambda_{11} &= \frac{1}{1-\frac{g_{12}g_{21}}{g_{11}g_{22}}} = \frac{1}{1-\phi} & (28)
\end{aligned}
$$

The RGA has a number of interesting algebraic properties, see [8] for an extensive list, the most important are given below.

Remarks

1) It is independent of input and output scaling
2) Its row and columns sum to one
3) The sum-norm of the RGA, $\|\Lambda\|_{sum}$, is very close to the minimized condition number γ^*. This means that plants with Large RGA-elements are always ill-conditioned (with a large value of $\gamma(G)$), but the reverse may not hold (i.e. a plant with a large $\gamma(G)$ may have small RGA-elements
4) A relative change in an element of G equal to the negative inverse of its corresponding RGA-element yields singularity
5) The RGA is the identity matrix if G is upper or lower triangular

The RGA can be used on frequency response data (per frequency) to study two sided interaction. If the RGA is close to the identity matrix, the plant is decoupled. If not, interaction is present.

VI. STABILITY

Stability of multivariable systems is governed by the Generalized Nyquist Criterion [6]. We define the characteristic loci λ_i of the transfer function matrix $G(s)$ to be the eigenvalues of $G(j\omega_i)$ plotted per frequency ω_i. For these characteristic loci the following theorem holds: *Generalized Nyquist criterion*

> If $G(s)$ has p_0 unstable (Smith-McMillan) poles, then the closed-loop system with return ratio -$kG(s)$ is stable if and only if the characteristic loci of $kG(s)$, taken together, encircle the point -1 p_0 times anti-clockwise, assuming that there are no hidden modes.

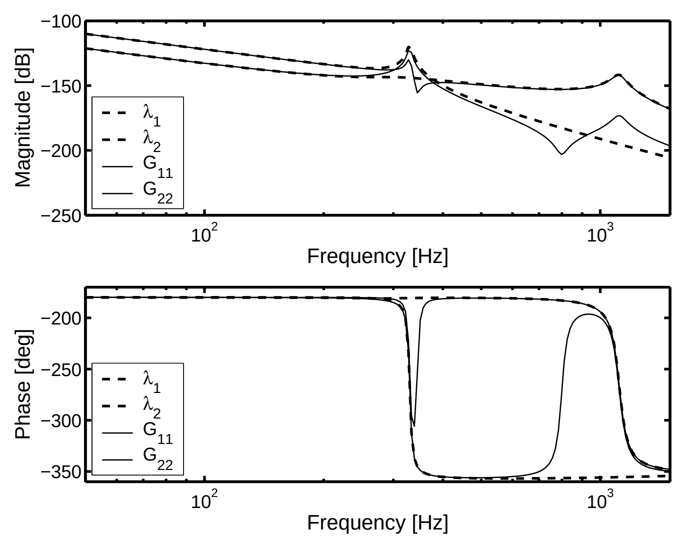

Fig. 5. The characteristic loci and frequency response function of the diagonal terms for a two by two system. Due to coupled modes, it is difficult to relate the characteristic loci with the frequency response function.

$$\begin{bmatrix} e \\ u \\ p_i \\ p_o \\ y \end{bmatrix} = \begin{bmatrix} (I+GK)^{-1}(I-GF) & -(I+GK)^{-1}G & -(I+GK)^{-1} & -(I+GK)^{-1} \\ (I+KG)^{-1}(K+F) & -(I+KG)^{-1}KG & -(I+KG)^{-1}K & -(I+KG)^{-1}K \\ (I+KG)^{-1}(K+F) & (I+KG)^{-1} & -(I+KG)^{-1} & -(I+KG)^{-1}K \\ (I+GK)^{-1}(GF+GK) & (I+GK)^{-1}G & -(I+GK)^{-1}GK & -(I+GK)^{-1}GK \\ (I+GK)^{-1}(GF+GK) & (I+GK)^{-1}G & (I+GK)^{-1} & -(I+GK)^{-1}GK \end{bmatrix} \begin{bmatrix} r \\ d_i \\ d_o \\ n \end{bmatrix} \quad (24)$$

VII. Sequential Loopclosing

The goal is to sequentially design SISO controllers (close loops) while recomputing (or measuring!) the openloop "as seen by the controller" with the latter loops closed. If every loop is stabilized, stability of the whole system is guaranteed. There are no general rules to determine the sequence in which loops should be closed. The plant with all former loops closed is called the *equivalent plant* and can be calculated for a two by two system as;

$$\begin{aligned} G_{11}^{eq} &= G_{11} - \frac{G_{12}K_{22}G_{21}}{1+G_{22}K_{22}} \\ G_{22}^{eq} &= G_{22} - \frac{G_{21}K_{11}G_{12}}{1+G_{11}K_{11}} \end{aligned} \quad (29)$$

VIII. Control limitations

Some relations for control limitations are posed, see [2] for a detailed discussion information;

A. At each frequency

For multivariable systems holds that;

$$S_o + T_o = I, \quad S_i + T_i = I \quad (30)$$

B. Frequency wise tradeoff (waterbed effect)

The multivariable sensitivity integral is stated as follows; For a closed loop stable rational multivariable system, with at least relative degree two, the output sensitivity function $S(s) = [I + G(s)K(s)]^{-1}$ must satisfy;

$$\begin{aligned} \int_{-\infty}^{\infty} log|h_i S_c(j\omega)| d\theta(\omega) &= \pi log|B_{ic}^*(z)h_{ic}| \quad (31) \\ c &\in H_i, \quad i = 1, ..., v \\ d\theta(\omega) &= \frac{x}{x^2+(y-\omega)^2} d\omega \\ B_{ic}^*(s) &= \prod_{k=1}^{c_j} \frac{s+\overline{p}_{jk}}{s-p_{jk}}, \quad z = x+jy \end{aligned}$$

and $S_c(s)$ the c^{th} column of $S(s)$. And z is a non-minimum phase zero ($h_i S(z) = h_i$) with multiplicity v associated with direction $h_i, i = 1, ..., v$. H_i is a set of integers corresponding to the non-zero columns of h_i defined as; $H_i = \{r | h_{ir} \neq 0\}$. Summarizing, the sensitivity integral for MIMO systems equals the sensitivity integral for multiloop SISO systems when the direction of the zeros is canonical (a zero is located in one loop).

IX. Mechanical systems

Here, we focus on mechanical systems which dynamics can be described with the following differential equations;

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= Fu \quad (32) \\ y &= Hq \quad (33) \end{aligned}$$

Where M, D, K denote the mass, damping and stiffness matrix respectively. F, H are the actuator and sensor matrix. By solving the eigenvalue problem $(K - \omega^2 M)t = 0$ the eigenvalues λ_i and eigenvectors t_i of this system can be determined. The matrix with eigenvectors T can be used to decouple the differential equations of Equation 32. So that using $q = T\eta$ leads to;

$$\begin{aligned} \underbrace{T^T M T}_{M_d} \ddot{\eta} + \underbrace{T^T D}_{D_d} T\dot{\eta} + \underbrace{T^T K}_{K_d} T\eta &= T^T F u \quad (34) \\ y &= HU\eta \quad (35) \end{aligned}$$

When we have proportional damping (e.g. Rayleigh damping, $D = \alpha M + \beta K$), D_d is diagonal. In this case, the differential equations Equation 34 are decoupled in a set of n independent second order differential equations. Another notation of Equation 34 is the *multiplicative modal form*:

$$Y(s) = \underbrace{HT}_{B} diag\left\{\frac{1}{m_{di}s^2 + d_{di}s + k_{di}}\right\} \underbrace{T^T F}_{A} U(s) \quad (36)$$

with $i = 1, ..n$ the number of modes. A, B are static input and output transformations respectively and n denotes the number of modes. m_{di}, d_{di} and k_{di} are the corresponding diagonal terms of M_d, D_d and K_d respectively. We can also use the *additive modal form*;

$$P(s) = \underbrace{\sum_{j=1}^{N_{rb}} \frac{u_j^T v_j}{s^2}}_{rigid\ body} + \underbrace{\sum_{i=N_{rb}+1}^{N} \frac{u_i^T v_i}{(s^2 + 2\zeta_i \omega_i s + \omega_i^2)}}_{flexible\ body} \quad (37)$$

Where N_{rb} denotes the number of rigid body modes, $N - N_{rb}$ are the number of flexible modes with resonance frequency ω_i and relative damping ζ_i. The corresponding eigenvectors are denoted by u_i, v_i.

X. MIMO Math

The inverse of a two by two matrix can be calculated algebraically:

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad (38)$$

$$det\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) = det(A_{11})det(A_{22}A_{11}^{-1}A_{12})$$
$$det(A_{22})det(A_{12}A_{2}^{-1}A_{21}) \quad (39)$$

Some tricks

$$\begin{aligned} K(I+GK)^{-1} &= (I+KG)^{-1}K \\ GK(I+GK)^{-1} &= (I+GK)^{-1}GK \\ L(I+L)^{-1} &= (I+L^{-1})^{-1} \\ A^{-1}B^{-1} &= (BA)^{-1} \\ (A^{-1}+B)^{-1} &= (I+AB)^{-1}A \end{aligned}$$

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