Design Methods for Control Systems

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Spring term 2009-2010

Schedule

March 8	MSt	
March 15	MSt	Homework # 1
March 22	MSt	
March 29	MSt	Homework # 2
April 12	SW	
April 19	SW	Homework # 3
April 26	SW	
May 3	SW	Homework # 4

Course material

http://wwwhome.math.utwente.nl/~meinsmag/dmcs/

Overview

Ch. 1. Introduction to feedback control theory

Ch. 2. Classical control system design

Ch. 3. Design of multivariable control systems

Ch. 4. LQ, LQG and H2 control system design

Ch. 5. Uncertainty models and robustness

Ch. 6. $\mathcal{H}_\infty$ optimization and μ-synthesis

Scope and features

Mature review of “classical” and
“modern” control system design techniques

- Linear time-invariant systems
- 70% SISO- 30% MIMO
- Continuous-time
- MATLAB exercises
  - Control toolbox
  - Mu-Tools and Robust Control toolboxes

Overview

Ch. 1. Introduction to feedback control theory

Introduction

- Types of control systems
- Design issues
- Configurations
- High-gain feedback

Stability

- Closed-loop characteristic polynomial
- Nyquist criterion
- Stability margins

Performance

- System functions
- Low and high frequencies

Robustness

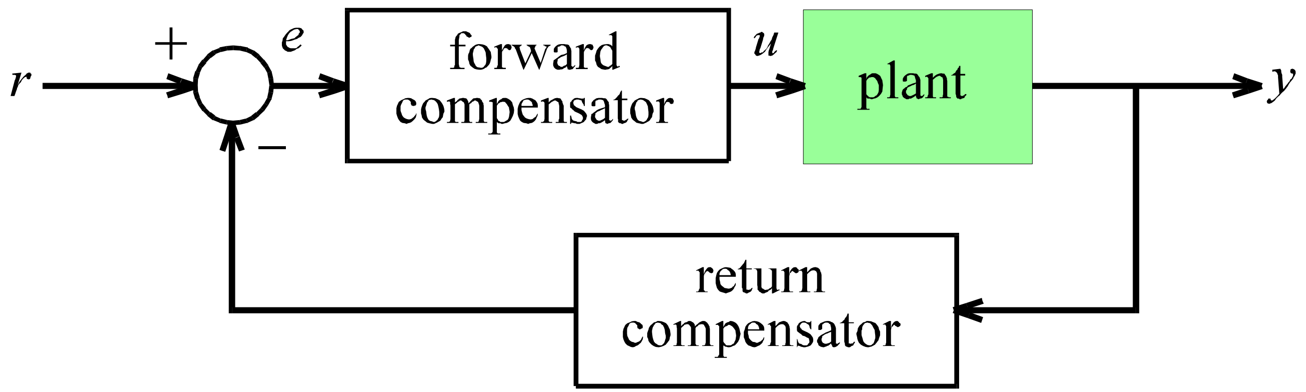
- Robustness functions

Loop shaping

Limits of performance

Two-degree-of-freedom control systems

Types of control systems



- Regulator systems
- Servo or positioning systems
- Tracking systems

Design issues

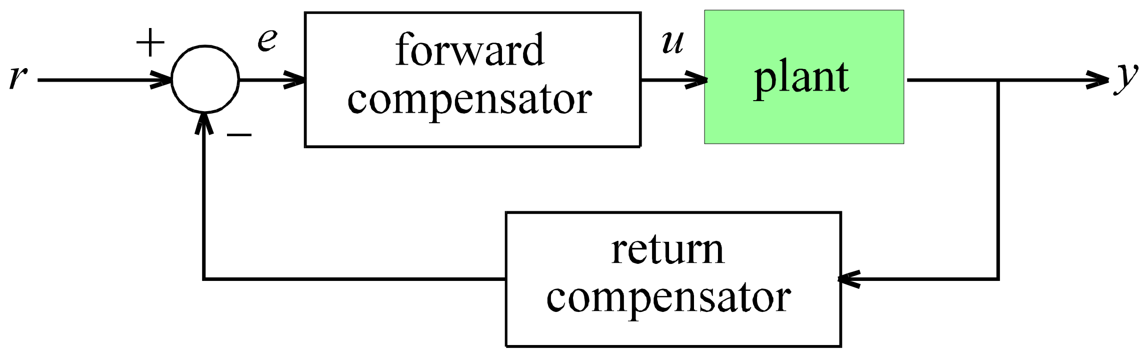
Targets

- Closed-loop stability
- Disturbance attenuation
- Good command response
- Robustness

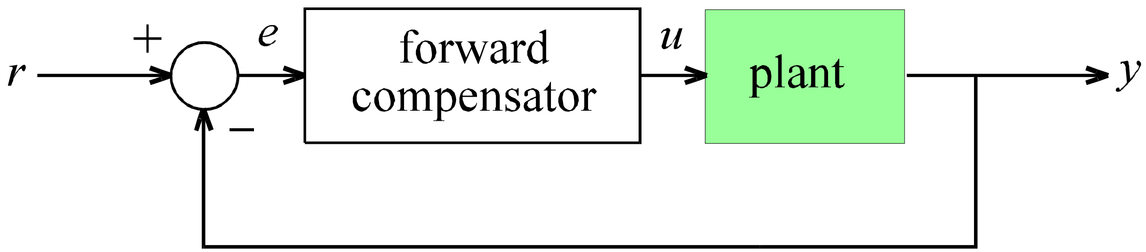
Limitations

- Plant capacity
- Measurement noise

Configurations

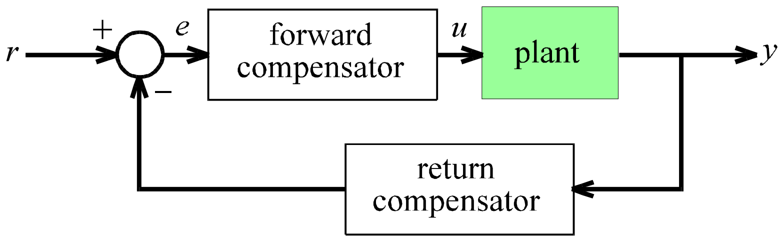


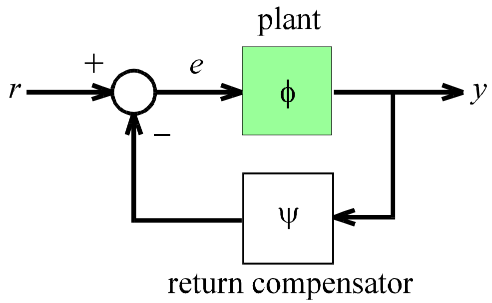
Two degrees of freedom

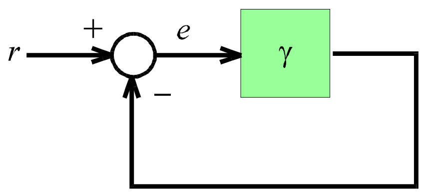


One degree of freedom

High-gain feedback-1







Loop gain

$$\gamma = \psi \circ \phi$$

Signal balance

$$e = r - \gamma(e)$$

Feedback equation

$$e + \gamma(e) = r$$

High-gain feedback-2

Feedback equation $e + \gamma(e) = r$

High gain: $|\gamma(e)| >> |e|$

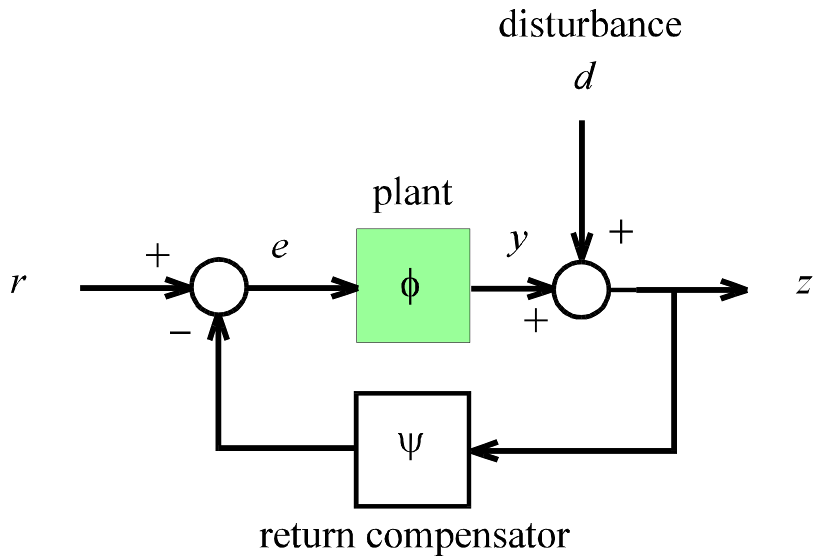
Implies: $\gamma(e) \approx r$

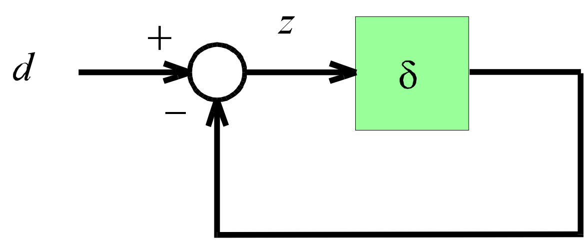
Hence: $|e| << |r| \quad \Rightarrow \quad r \approx \psi(y)$

So that $y \approx \psi^{-1}(r)$

In case of unit feedback $y \approx r$ ***Good tracking***

High-gain feedback-3





Loop gain $\delta = (-\phi) \circ (-\psi)$

Signal balance: $z = d - \delta(z)$

High gain: $|\delta(z)| >> |z|$

Hence: $|z| << |d|$

Good disturbance reduction

High-gain feedback-4

Need *closed-loop stability*

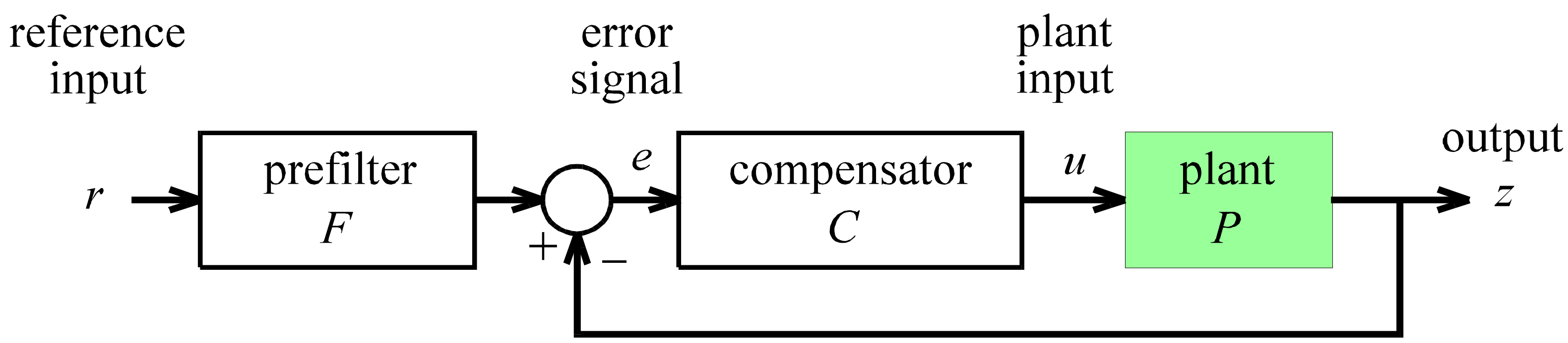
- Good tracking and disturbance attentuation are retained as long as
- the closed-loop system remains stable
- the gain remains high

Under these conditions high-gain feedback implies **robustness** with respect to loop uncertainty

Pitfalls of high-gain feedback

- High-gain feedback has pitfalls:
- Naively making the gain large easily results in an *unstable* feedback system
- Even if the feedback system is stable overly large plant inputs may occur that exceed the plant capacity
- Measurement noise causes loss of performance

Stability-1



State space representation:

$$\dot{x}(t) = Ax(t) + Br(t)$$

$$\begin{bmatrix} e(t) \\ u(t) \\ z(t) \end{bmatrix} = Cx(t) + Dr(t)$$

Stability-2

$$\dot{x}(t) = Ax(t) + Br(t)$$

$$\begin{bmatrix} e(t) \\ u(t) \\ z(t) \end{bmatrix} = Cx(t) + Dr(t)$$

The closed-loop system is *stable* if its state space representation is *asymptotically stable*

Equivalent statements:

- $x(t) \to 0 \quad \text{as} \quad t \to \infty$ for every solution of $\dot{x}(t) = Ax(t)$
- All eigenvalues of A have strictly negative real parts
- All roots of $\det(sI - A)$ have strictly negative real parts

Stability-3

The control system is *BIBO stable* if every bounded input signal r results in bounded output signals e, u and z.

BIBO = “bounded input bounded output”

Asymptotic stability $\Rightarrow$ BIBO stability

The converse is *not* true

Stability-4

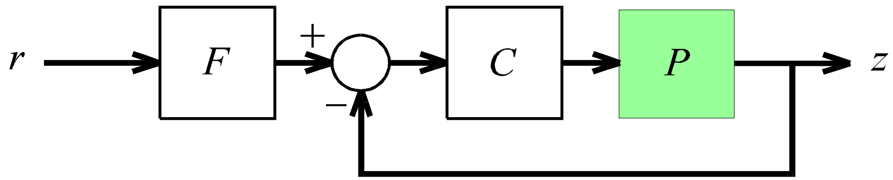
Internal stability

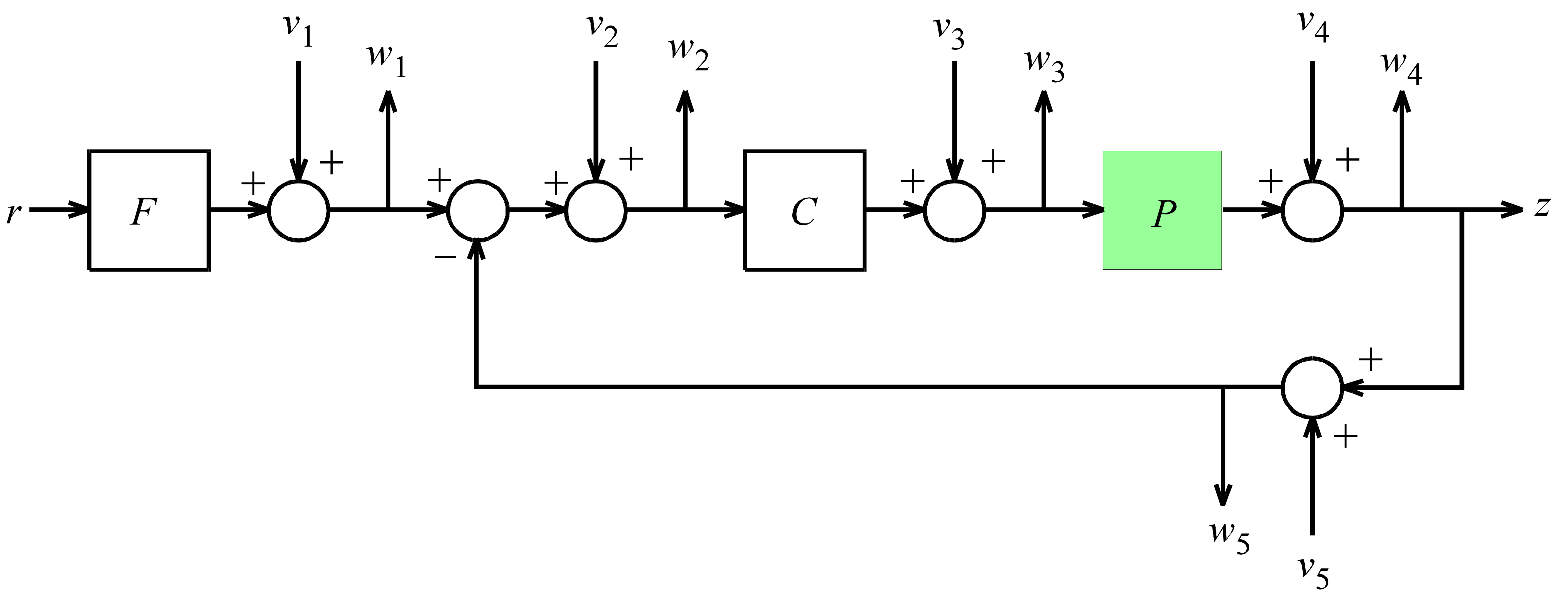
Inject "internal" signals into each "exposed interconnection" of the system, and define additional "internal" output signals after each injection point

Then the system is *internally stable* if it is BIBO stable with respect to all inputs (external and internal) and all (external and internal) outputs

Stability-5

Example





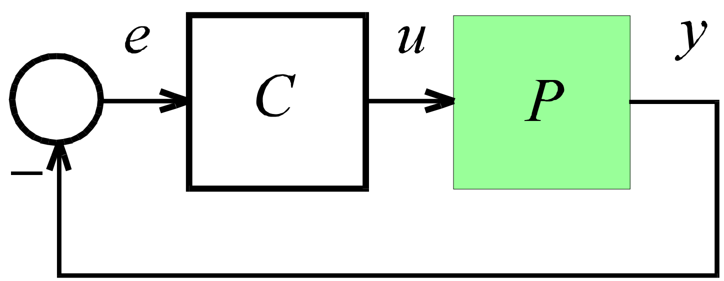
Stability-6

If each component system is stabilizable and detectable ("has no hidden unstable modes") then

Stability $\Leftrightarrow$ Internal stability

When input-output descriptions are used (such as transfer functions) internal stability is often easier to check than asymptotic stability

Closed-loop characteristic polynomial-1



State space representation of the open-loop system:

$$\dot{x}(t) = Ax(t) + Be(t)$$
$$y(t) = Cx(t) + De(t)$$

Characteristic polynomial:

$$\chi(s) = \det(sI - A)$$

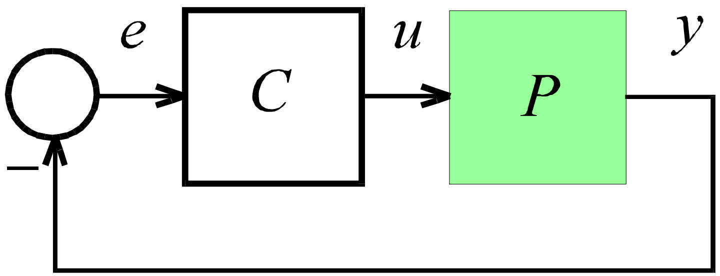
State space representation of the closed-loop system:

$$\dot{x}(t) = A_{cl}x(t),$$
$$A_{cl} = A - B(I + D)^{-1}C$$

Characteristic polynomial:

$$\chi_{cl}(s) = \det(sI - A_{cl})$$

Closed-loop characteristic polynomial-2



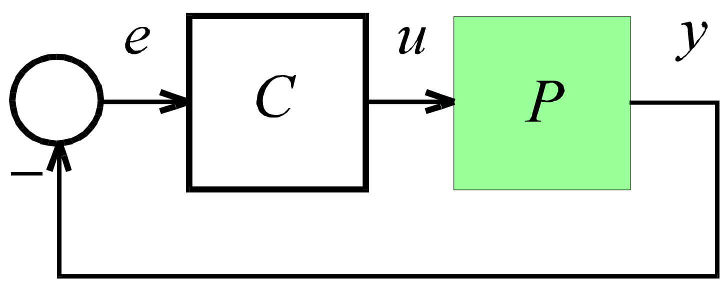
$P(s)$	plant transfer matrix
$C(s)$	compensator transfer matrix
$L(s)=P(s)C(s)$	loop gain transfer matrix

Then

$$\chi_{cl}(s) = \chi(s)\frac{\det(I + L(s))}{\det(I + L(\infty))}$$

Closed-loop characteristic polynomial-3

$$\chi_{cl}(s) = \chi(s)\frac{\det(I + L(s))}{\det(I + L(\infty))}$$



SISO case: $L(s) = P(s)C(s) = \dfrac{N(s)}{D(s)} \cdot \dfrac{Y(s)}{X(s)}$

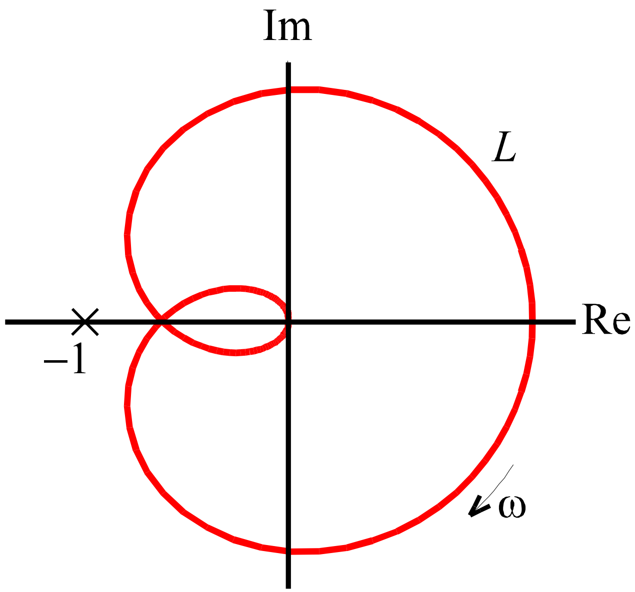
Then (within a nonzero constant factor)

$$\chi(s) = D(s)X(s) \qquad \chi_{cl}(s) = D(s)X(s) + N(s)Y(s)$$

Nyquist criterion

Consider the SISO case

The locus of $L(j\omega), \quad \omega \in \mathbb{R}$ in the complex plane is called the *Nyquist plot* of the loop gain



The number of unstable closed-loop poles

=

The number of times the Nyquist plot encircles the point −1

+

The number of unstable open-loop poles

Generalized Nyquist criterion

Consider the MIMO case

The number of unstable closed-loop poles

=

The number of times the locus of

$$\det(I + L(j\omega)), \quad \omega \in \mathrm{R}$$

encircles the origin

+

The number of unstable open-loop poles

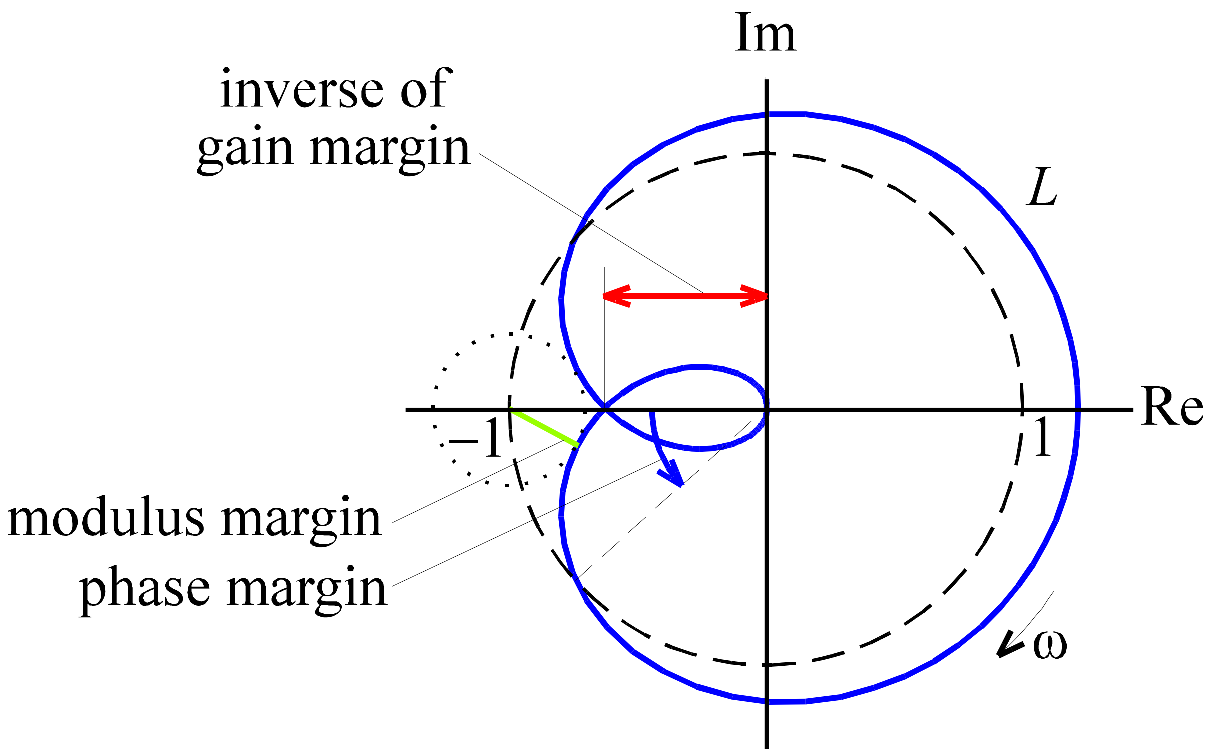
(Principle of the argument)

Stability margins-1

- In the SISO case, the point –1 is a *critical point* for the Nyquist plot of the closed-loop system. If the Nyquist plot is changed so that it crosses the point –1 then the system becomes unstable
- If the closed-loop system is stable but the Nyquist plot passes closely by –1 then
- the system is near-unstable, that is, has an oscillatory response
- the system may become unstable by small perturbations of the plant, that is, the system is not robust

Stability margins-2

There exist various *stability margins*. They measure how close the Nyquist plot gets to –1

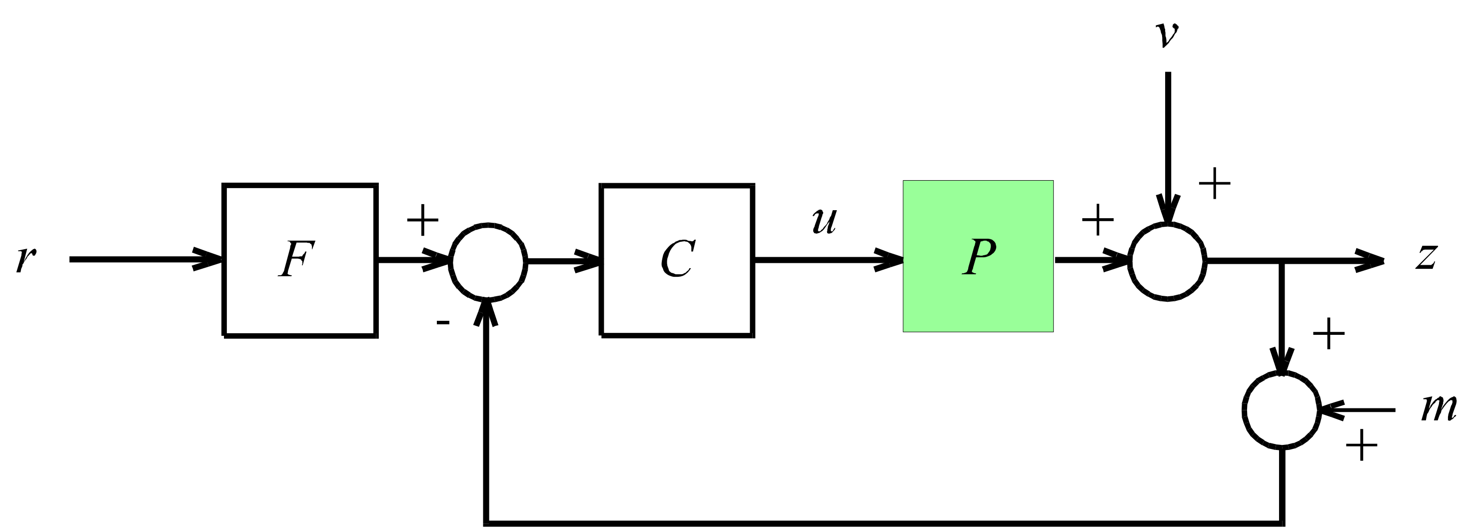


Gain margin k_m

Phase margin ϕ_m

Modulus margin τ_m (Landau)

System functions: L and S



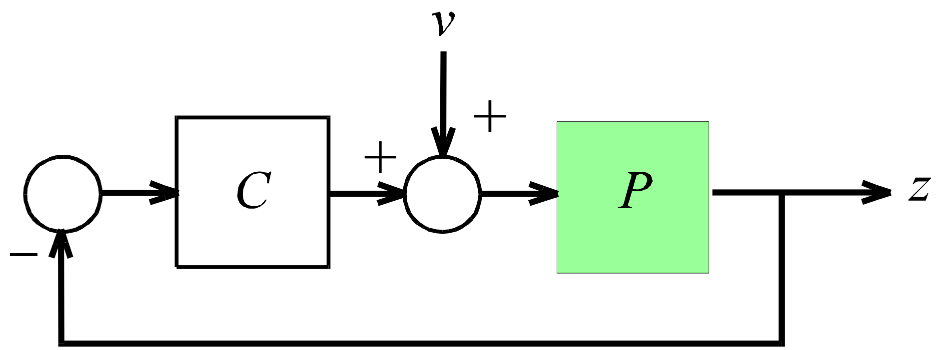
Loop gain L

$$L = PC$$

Sensitivity function S

$$z = \underbrace{\frac{1}{1+L}}_{S} v$$

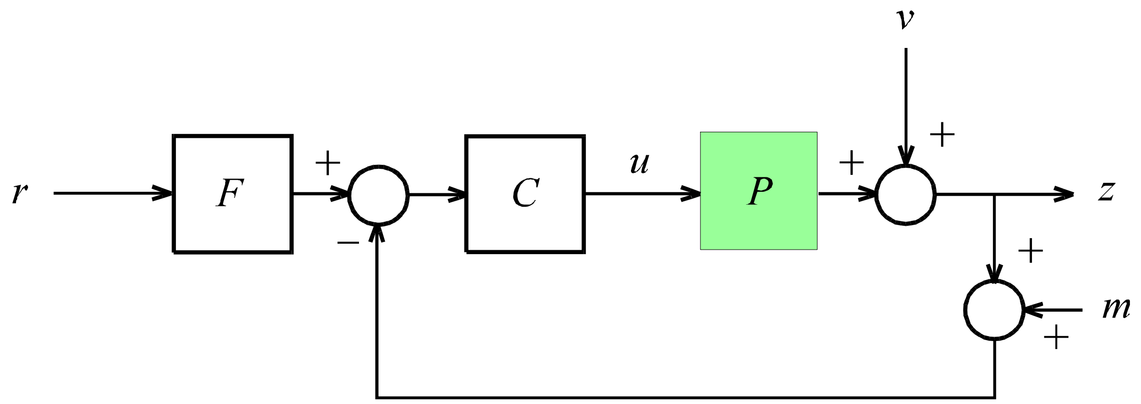
System functions: *R*



Input disturbance ('proces') sensitivity function R

$$z = \frac{1}{1+L} P v = \underbrace{SP}_{R} r$$

System functions: *H* and *T*



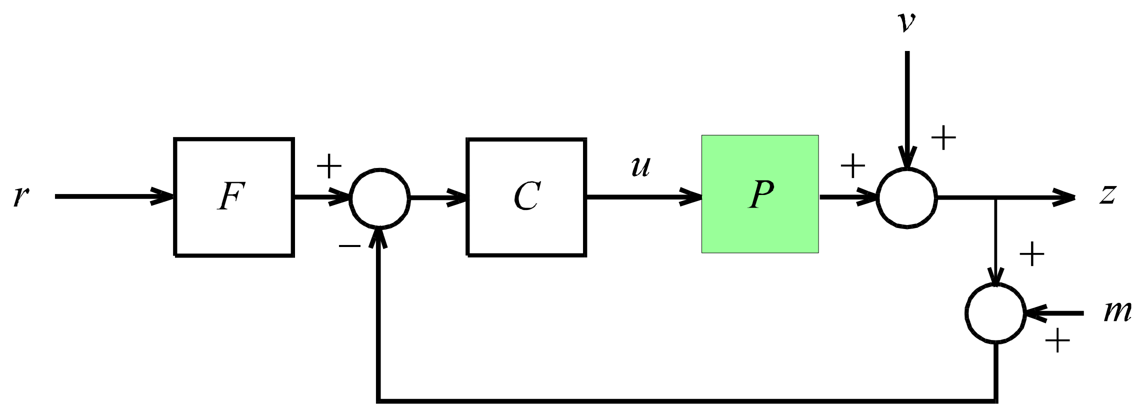
Closed-loop transfer function H

Complementary sensitivity function T

$$z = \underbrace{\frac{L}{1+L}}_{H} F\, r$$

$$H = \underbrace{\frac{L}{1+L}}_{T} F$$

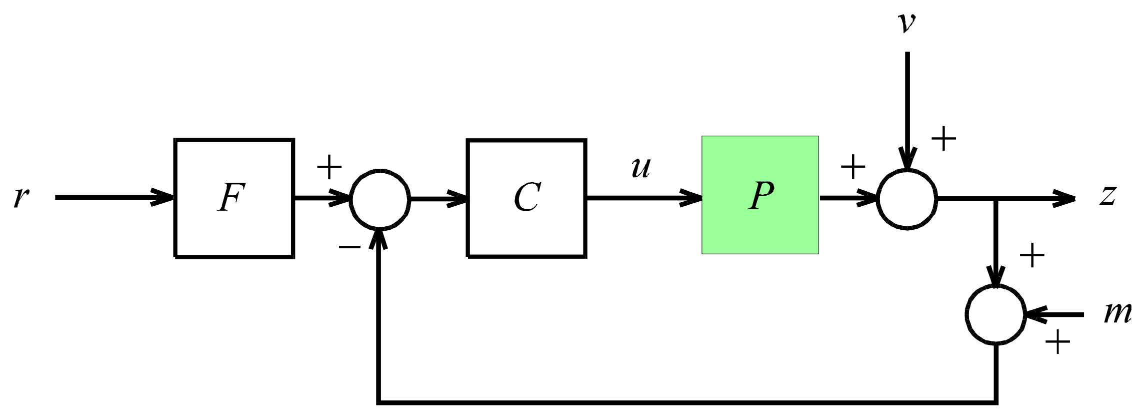
System functions: *U*



Input ('control') sensitivity function U

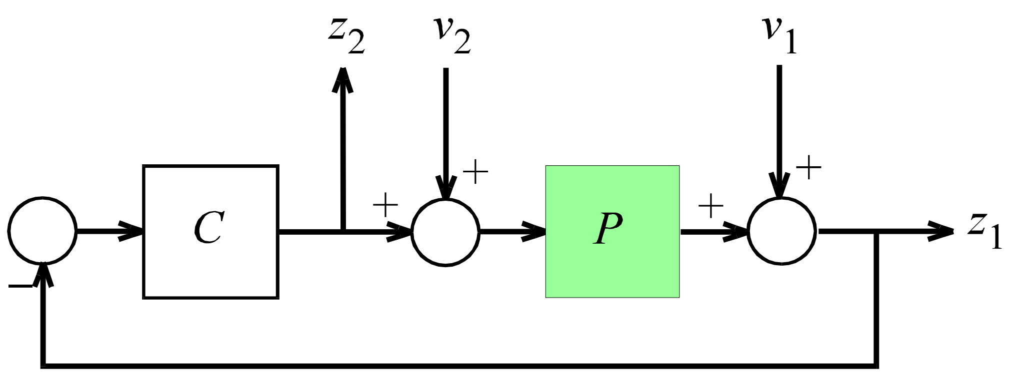
$$u = \underbrace{\frac{C}{1+CP}}_{U}(Fr - m - v)$$

Measurement noise



$$z = \underbrace{\frac{1}{1+PC}}_{S} v + \underbrace{\frac{PC}{1+PC}}_{T} Fr - \underbrace{\frac{PC}{1+PC}}_{T} m$$

S, R, U and T



$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+PC} & \dfrac{P}{1+PC} \\ -\dfrac{C}{1+PC} & -\dfrac{PC}{1+PC} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} S & R \\ -U & -T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Design interrelations

$$S = \frac{1}{1+L} \qquad U = T/P$$

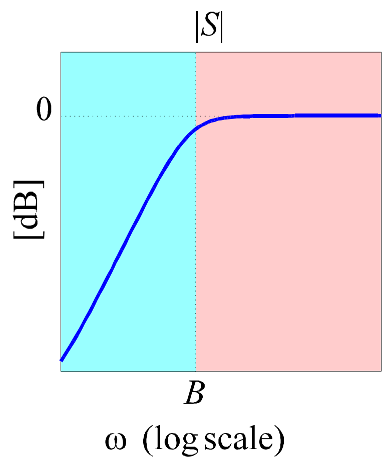
$$T = \frac{L}{1+L} \qquad R = SP$$

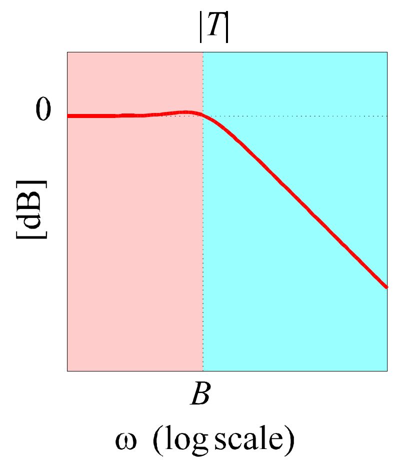
$$H = TF$$

S and T are suitable objects for manipulation

Low and high frequencies-1

Typical shapes for S and T





	S	T
low frequencies	small	$\simeq 1$
high frequencies	$\simeq 1$	small

Low and high frequencies-2

Loop gain L

- large at low frequencies: $|L(j\omega)| >> 1, \quad S \approx 1/L, \quad T \approx 1$
- small at high frequencies: $|L(j\omega)| << 1, \quad S \approx 1, \quad T \approx L$

- *Crossover* region: $|L(j\omega)| \approx 1$

Low and high frequencies-3

input sensitivity

$$U = T / P = \frac{C}{1+PC} \approx \begin{cases} 1/P & \text{for low frequencies} \\ C & \text{for high frequencies} \end{cases}$$

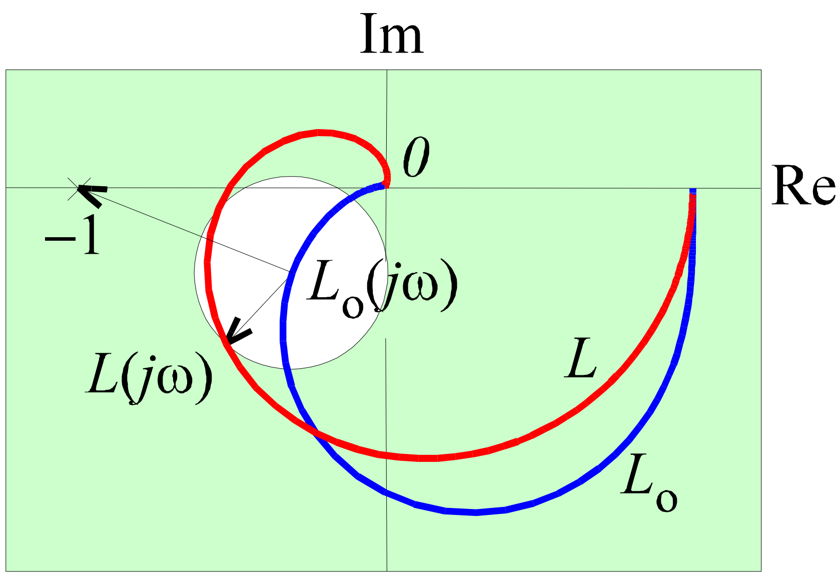
input disturbance sensitivity

$$R = SP = \frac{P}{1+PC} \approx \begin{cases} 1/C & \text{for low frequencies} \\ P & \text{for high frequencies} \end{cases}$$

closed-loop transfer function

$H = TF$ F corrects T

Robustness functions-1



Sufficient condition for stability under perturbation:

$$\left|L(j\omega) - L_o(j\omega)\right| < \left|1 + L_o(j\omega)\right|, \quad \omega \in \mathbb{R}$$

Robustness functions-2

Equivalently,

$$\left|\frac{L(j\omega)-L_o(j\omega)}{L_o(j\omega)}\right| < \left|\frac{1+L_o(j\omega)}{L_o(j\omega)}\right|, \quad \omega \in \mathbb{R}$$

or

$$\left|\frac{L(j\omega)-L_o(j\omega)}{L_o(j\omega)}\right| < \frac{1}{|T_o(j\omega)|}, \quad \omega \in \mathbb{R}$$

Robustness functions-3

Bound on the relative size of perturbations:

$$\left|\frac{L(j\omega)-L_o(j\omega)}{L_o(j\omega)}\right| \le \left|W_1(j\omega)\right|, \quad \omega \in \mathbb{R}$$

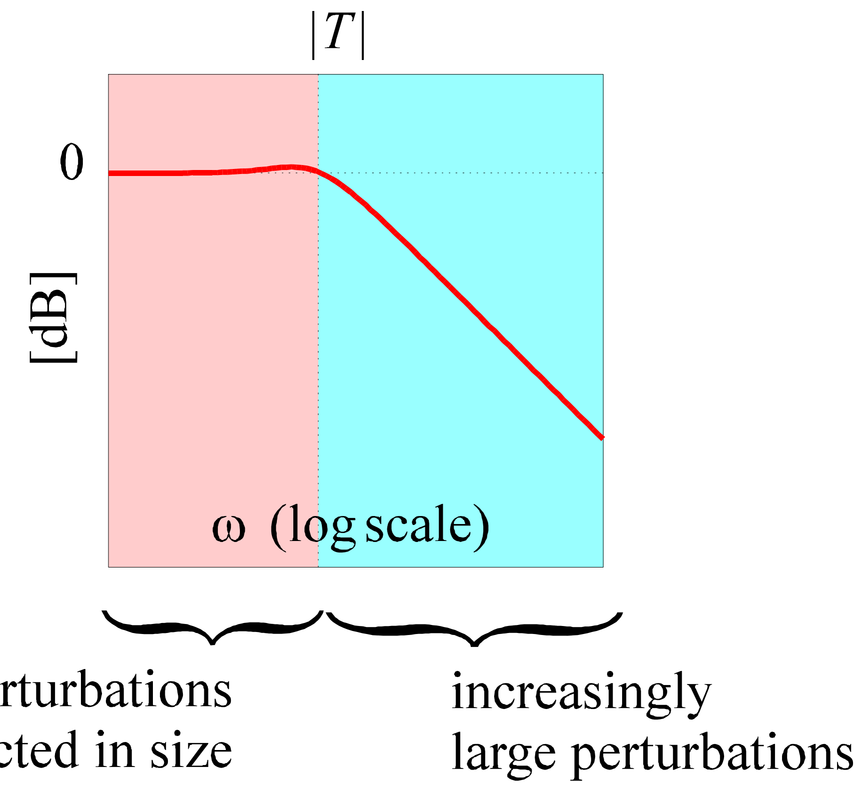
Sufficient and necessary condition for stability under all perturbations that satisfy the bound:

$$\left|W_1(j\omega)\right| < \frac{1}{\left|T_o(j\omega)\right|}, \quad \omega \in \mathbb{R}$$

Robustness functions-4

Size of the smallest perturbation that may destabilize the system:

$$|W_1(j\omega)| = \frac{1}{|T_o(j\omega)|}, \quad \omega \in \mathbb{R}$$



Robustness functions-5

The preceding discussion focuses on preventing the Nyquist plot of the loop gain L from crossing the point -1. Preventing the *inverse Nyquist plot* – that is, the Nyquist plot of $1/L$ – from crossing the point -1 also guarantees stability.

Sufficient condition:

$$\left| \frac{1}{L(j\omega)} - \frac{1}{L_o(j\omega)} \right| < \left| 1 + \frac{1}{L_o(j\omega)} \right|, \quad \omega \in \mathbb{R}$$

Robustness functions-6

Equivalently,

$$\left| \frac{\frac{1}{L(j\omega)} - \frac{1}{L_o(j\omega)}}{\frac{1}{L_o(j\omega)}} \right| < \frac{1}{|S_o(j\omega)|}, \quad \omega \in \mathbb{R}$$

Robustness functions-7

Consider perturbations such that

$$\left| \frac{\dfrac{1}{L(j\omega)} - \dfrac{1}{L_o(j\omega)}}{\dfrac{1}{L_o(j\omega)}} \right| \leq |W_2(j\omega)|, \quad \omega \in \mathbb{R}$$

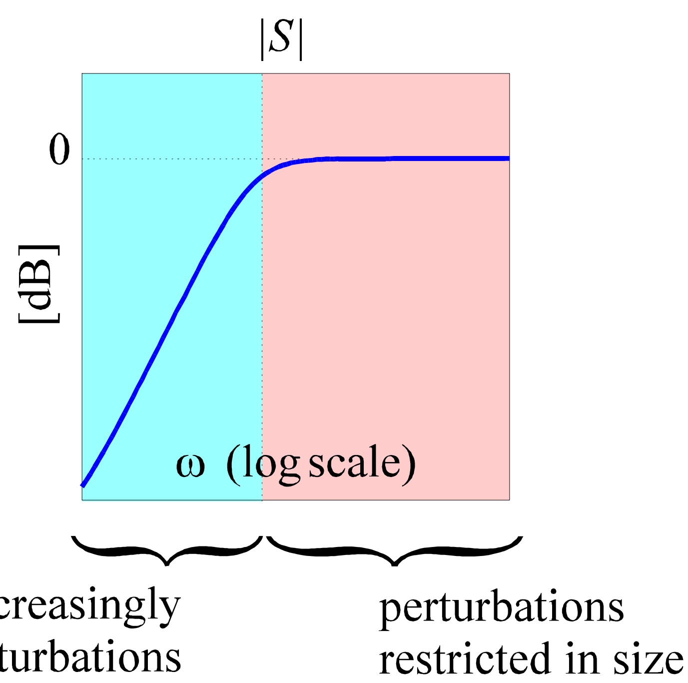
Sufficient and necessary condition for robust stability:

$$|W_2(j\omega)| < \frac{1}{|S_o(j\omega)|}, \quad \omega \in \mathbb{R}$$

Robustness functions-8

Size of the smallest perturbation that may destabilize the system:

$$|W_2(j\omega)| = \frac{1}{|S_o(j\omega)|}, \quad \omega \in \mathbb{R}$$



Combined robustness test-1

Define

$$\delta_L(j\omega) = \left| \frac{L(j\omega) - L_o(j\omega)}{L_o(j\omega)} \right|$$

$$\delta_{L^{-1}}(j\omega) = \left| \frac{\dfrac{1}{L(j\omega)} - \dfrac{1}{L_o(j\omega)}}{\dfrac{1}{L_o(j\omega)}} \right|$$

Combined robustness test-2

Then the perturbed closed-loop system is stable if

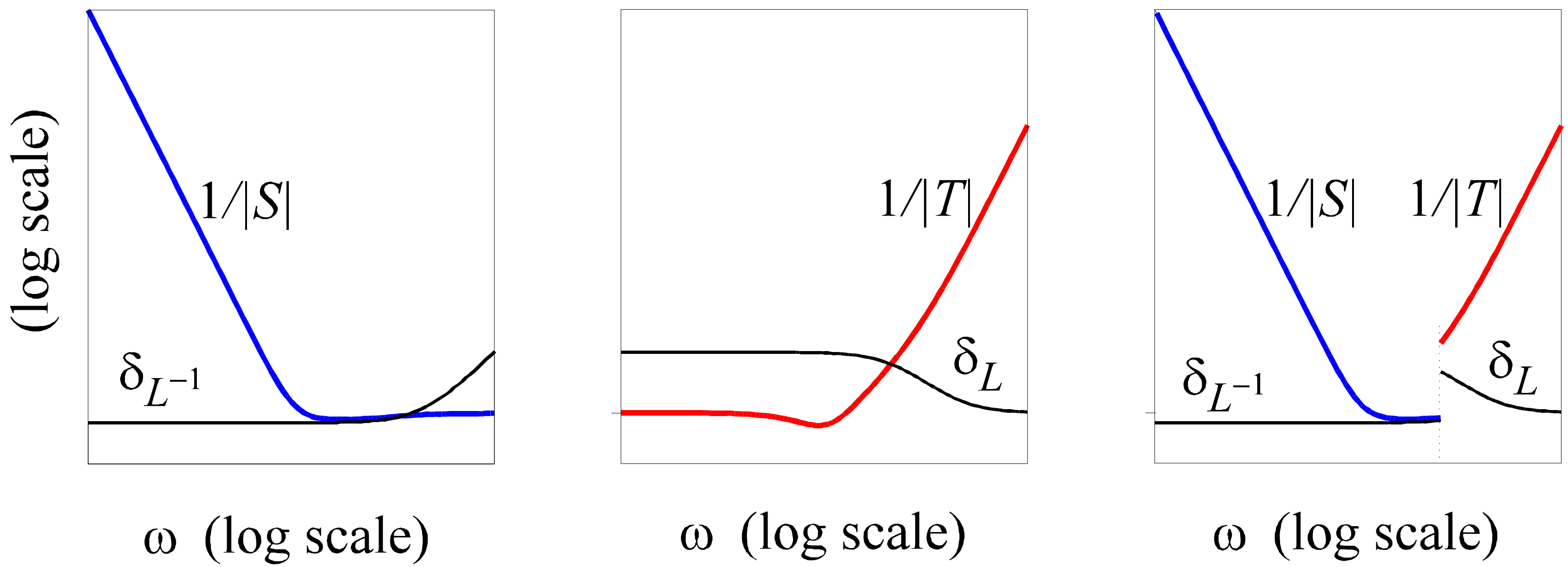
$$\forall \omega \in \mathbb{R}$$

$$\left|\delta_{L^{-1}}(j\omega)\right| < \frac{1}{\left|S(j\omega)\right|} \quad \textbf{\textit{or}} \quad \left|\delta_L(j\omega)\right| < \frac{1}{\left|T(j\omega)\right|}$$

typically satisfied at ***low*** frequencies

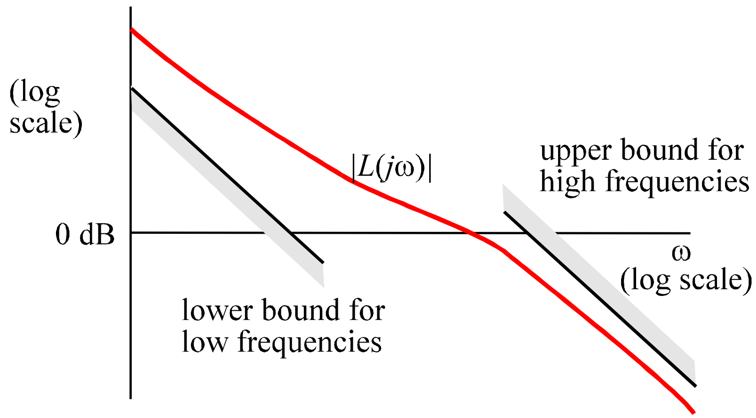
typically satisfied at ***high*** frequencies

Combined robustness test-3



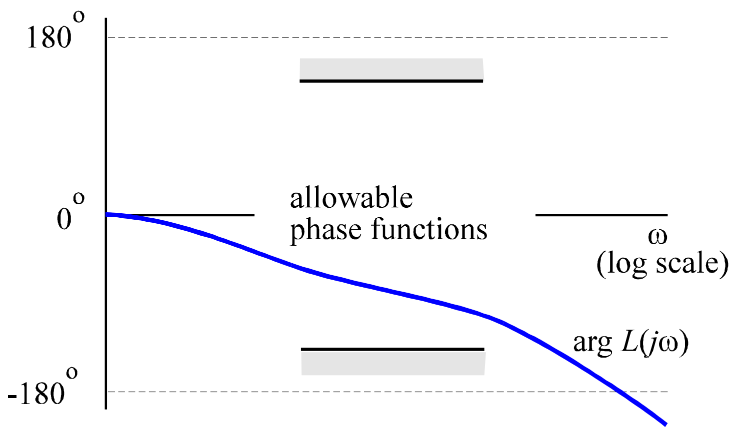
Critical frequency region: crossover area

Loop shaping



Low frequencies: large loop gain

High frequencies: small loop gain



In the crossover region the phase is constrained because of stability

Bode's gain-phase relationship-1

Between break frequencies the loop gain behaves as

$$L(j\omega) \approx c(j\omega)^n$$

Hence

$$|L(j\omega)| \approx c\omega^n$$

$$\arg L(j\omega) \approx n \times \frac{\pi}{2}$$

Phase and magnitude do not behave independently

Bode's gain-phase relationship describes the relation more accurately

Bode's gain-phase relationship-2

The limitations imposed by stability on the phase in the crossover region by Bode's gain-phase relationship limit the rate at which the loop gain decreases:

If, say,

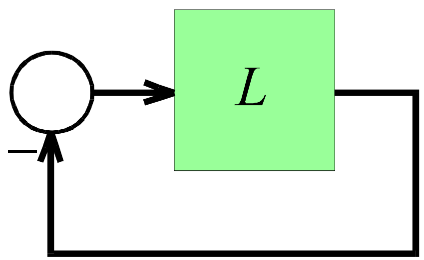
$$\arg L(j\omega) \approx -\frac{\pi}{2} \quad \text{in the crossover region}$$

then

$$| L(j\omega) | \approx c\,\omega^{-1} \quad \text{in the crossover region}$$

Limits of performance

- Bode's integral
- The Freudenberg-Looze equalities



Limitations are imposed by

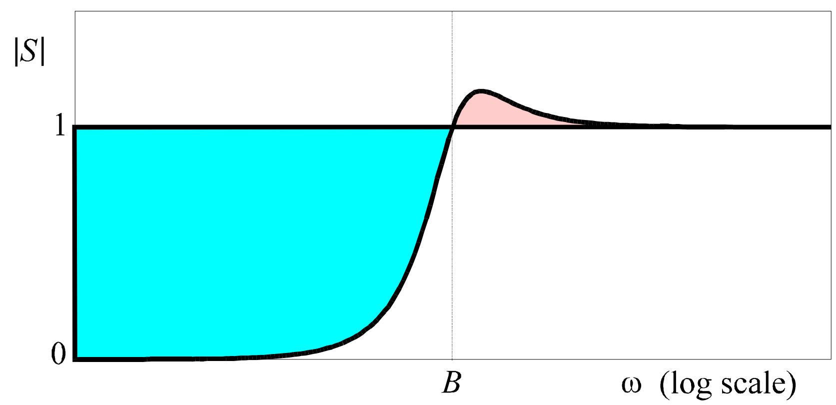
- causality
- the pole-zero configuration

Bode's integral-1

If L has at least two more poles than zeros then

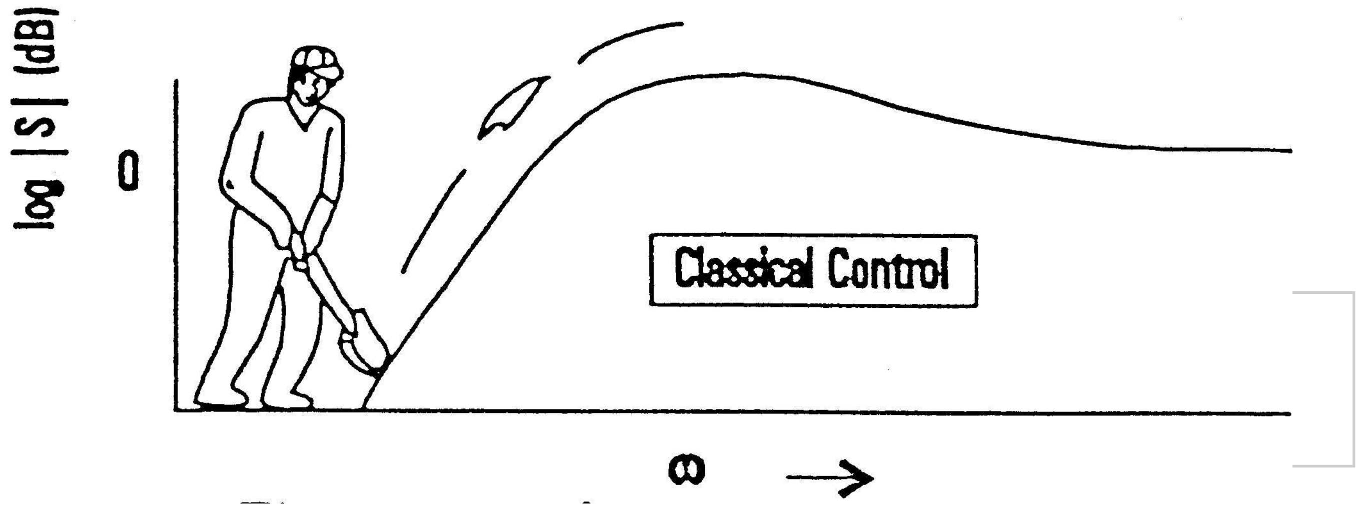
$$\int_0^\infty \log|S(j\omega)|\,d\omega = \pi\sum_i \operatorname{Re} p_i \geq 0$$

The p_i are the right-half plane poles of the loop gain.

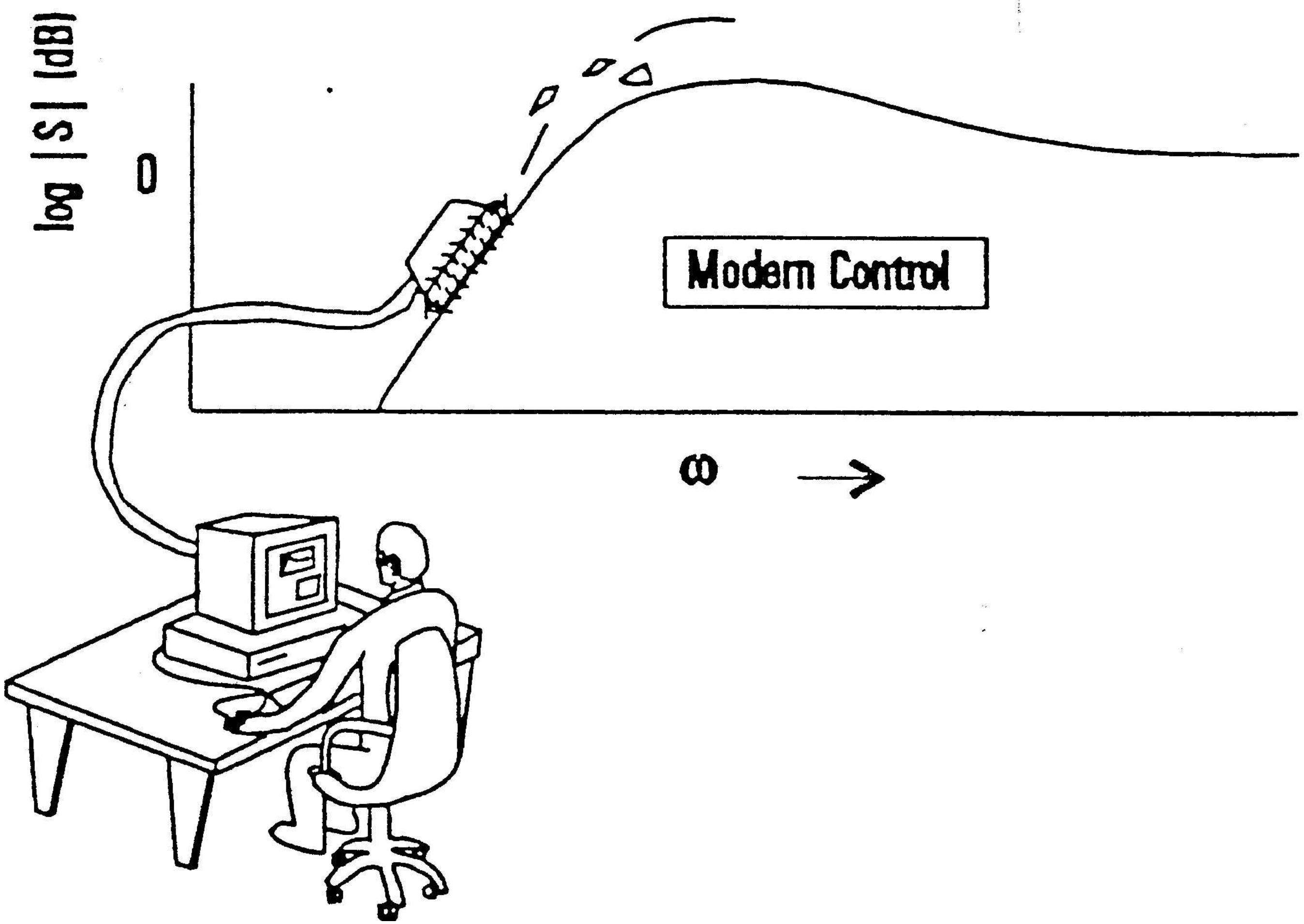


Proof: Use the Poisson integral from complex function theory

Bode's integral-1



Bode’s integral-1

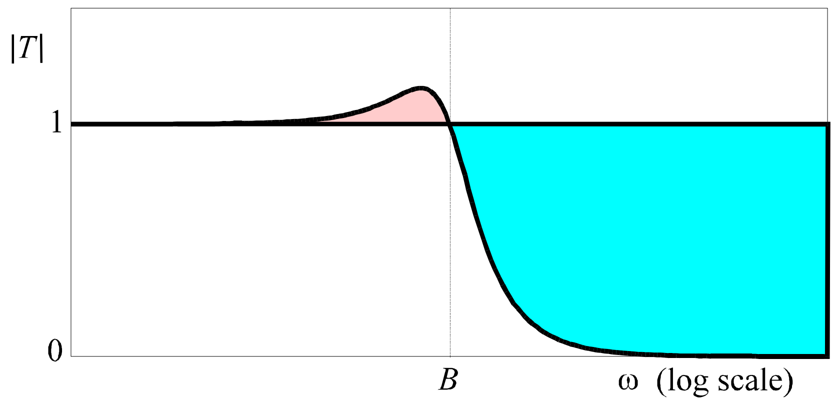


Bode's integral-2

"Dual" result: Suppose that the loop has integrating action of at least order 2. Then

$$\int_0^\infty \log\left|T(1/j\omega)\right| d\omega = \pi \sum_i \operatorname{Re}\frac{1}{z_i} \geq 0$$

The z_i are the right-half plane zeros of the loop gain.



Freudenberg-Looze equality-1

Let z be any right-half plane zero of the loop gain. Poisson's formula of complex function theory leads to the equality

$$\int_0^\infty \log(|S(j\omega)|)\, dW_z(\omega) = \log\left|B_{\text{poles}}^{-1}(z)\right| \geq 0$$

Strengthens Bode's integral

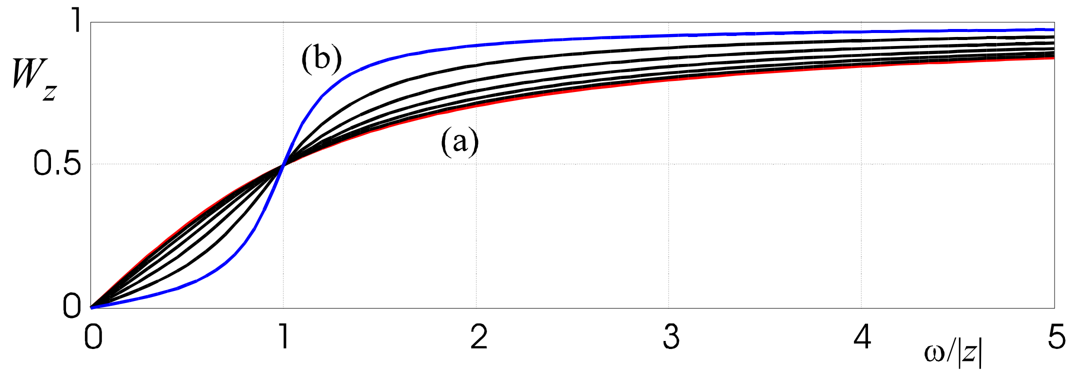
$$W_z(\omega) = \frac{1}{\pi}\arctan\frac{\omega - \operatorname{Im} z}{\operatorname{Re} z} + \frac{1}{\pi}\arctan\frac{\omega + \operatorname{Im} z}{\operatorname{Re} z}$$

Increasing function. Rises most steeply at |z|.

$$B_{\text{poles}}(s) = \prod_i \frac{p_i - s}{\overline{p}_i + s}$$

Blaschke product

Freudenberg-Looze equality-2



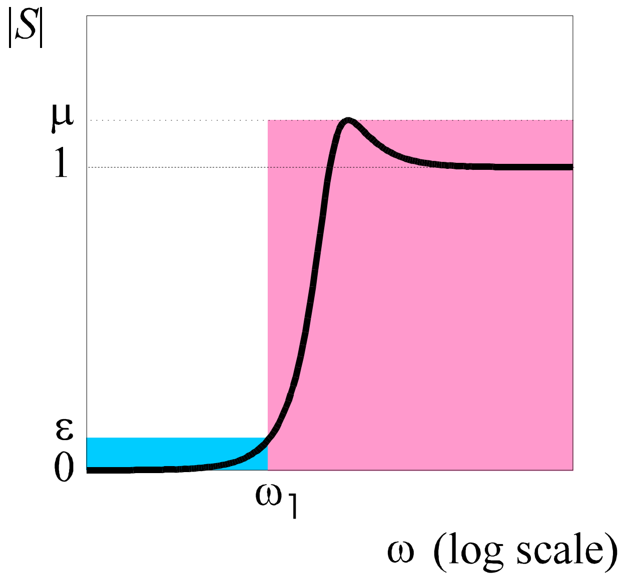
W_z for different values of $\arg z$

(a) $\arg z = 0$

(b) $\arg z$ is almost $\pi/2$

Frequencies where W_z rises most steeply contribute most to the integral

Freudenberg-Looze equality-3



The bounds for $|S|$ hold provided

$$\mu \geq \left(\frac{1}{\varepsilon}\right)^{\frac{W_z(\omega_1)}{1-W_z(\omega_1)}} \cdot \left|B^{-1}_{\text{poles}}(z)\right|^{\frac{1}{1-W_z(\omega_1)}}$$

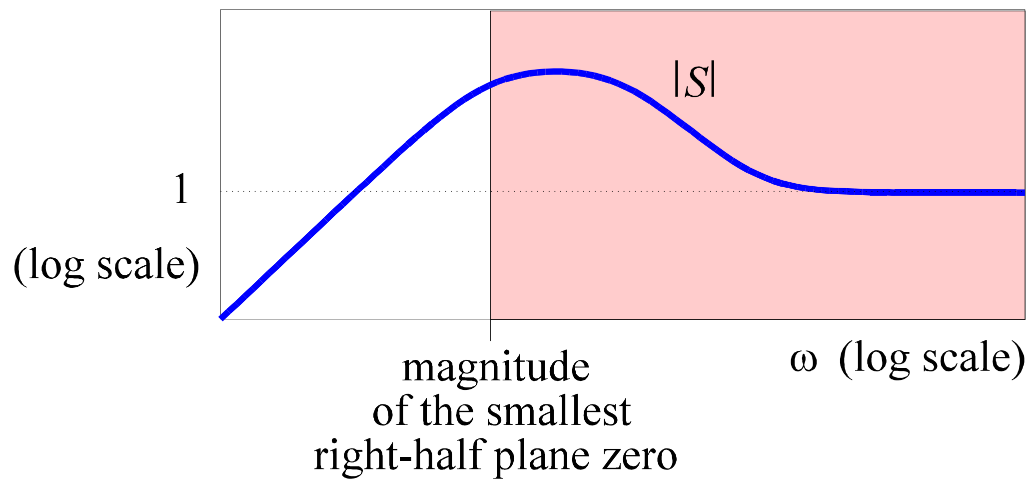
The dependence of the right-hand side on the various parameters may be analyzed

Freudenberg-Looze equality-4

Effects of right-half plane zeros on S

$|S|$ may be made small up to the frequency $\min_i |z_i|$. Attempting to make $|S|$ small beyond this frequency makes $|S|$ peak

- Right-half plane poles further impair the achievable reduction of $|S|$ (in particular, nearly-cancelling right-half plane pole-zero pairs)



Freudenberg-Looze equality-5

Rederivation of the Freudenberg-Looze equality while

- replacing L with $1/L$, so that $S = \frac{1}{1+L} \rightarrow \frac{1}{1+\frac{1}{L}} = \frac{L}{1+L} = T$
- interchanging the roles of the poles and the zeros

leads to

$$\int_0^\infty \log(|T(j\omega)|)\, dW_p(\omega) = \log\left|B_{\text{zeros}}^{-1}(p)\right| \geq 0$$

Freudenberg-Looze equality-6

$$\int_0^\infty \log(|T(j\omega)|)\, dW_p(\omega) = \log\left|B_{\text{zeros}}^{-1}(p)\right| \geq 0$$

p is any right-half plane pole of the loop gain, and

$$B_{\text{zeros}}(s) = \prod_i \frac{z_i - s}{\bar{z}_i + s}$$

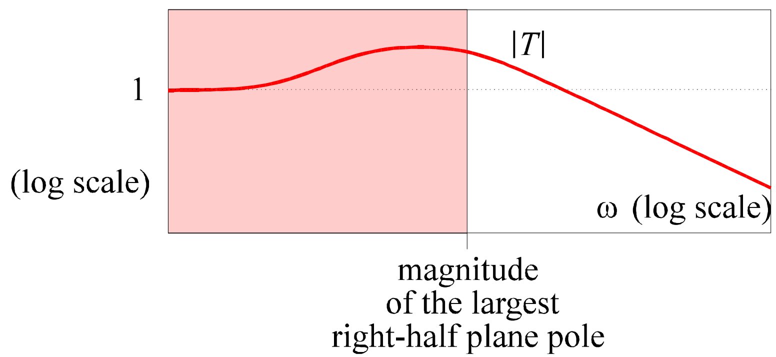
In the application of the equality, interchange the roles of *low* and *high* frequencies

Freudenberg-Looze equality-7

Effects of right-half plane poles on T

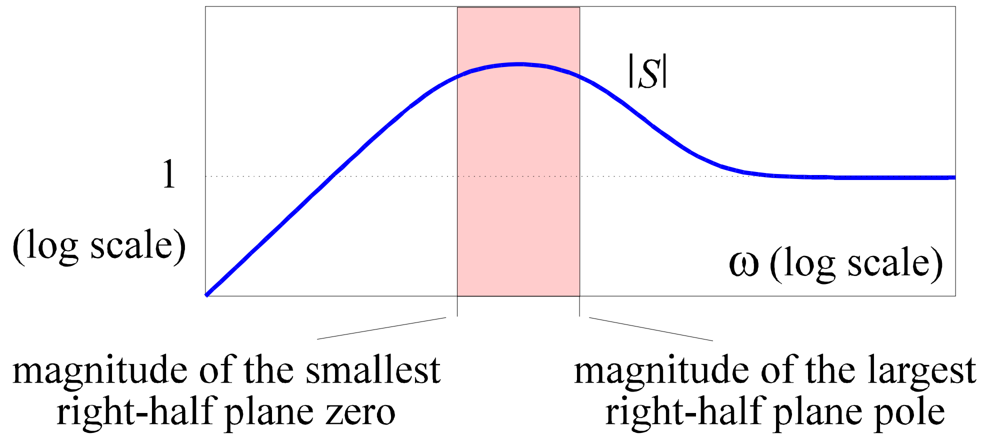
$|T|$ may be made small above the frequency $\max_i |p_i|$. Attempting to make $|T|$ small below this frequency makes $|T|$ peak

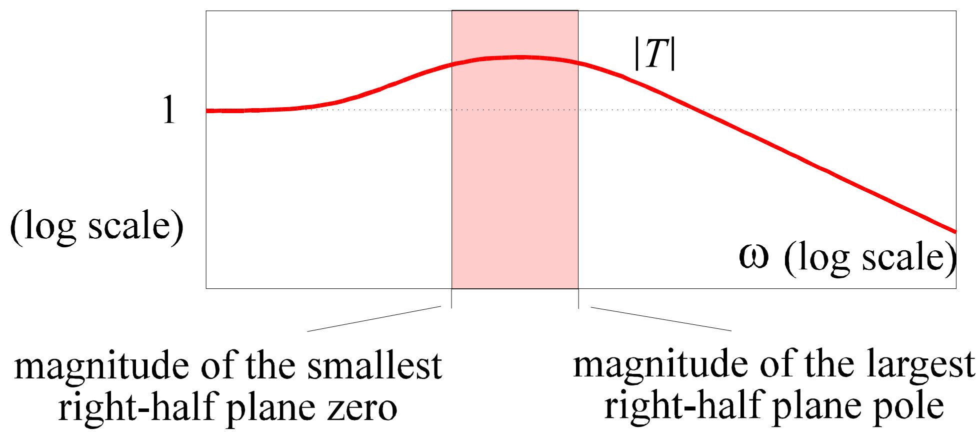
- Right-half plane zeros further impair the achievable reduction of $|T|$ (in particular, nearly-cancelling right-half plane pole-zero pairs)



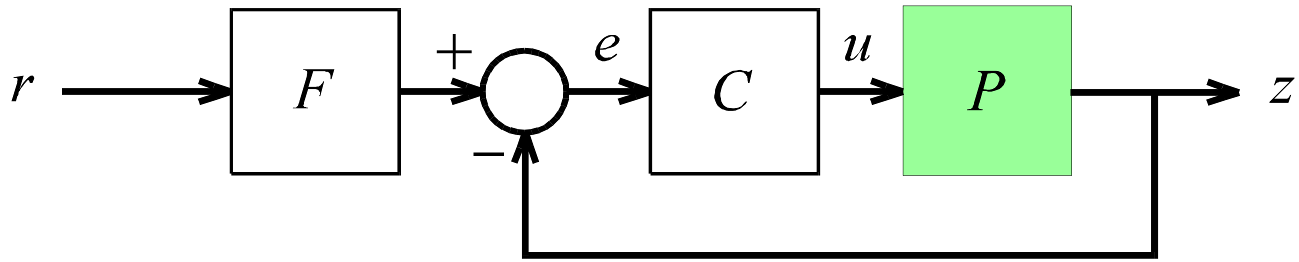
Freudenberg-Looze equality-8

Consequences for S and T





Two-degree-of-freedom systems-1



Let $P = \frac{N}{D}, \quad C = \frac{Y}{X}$

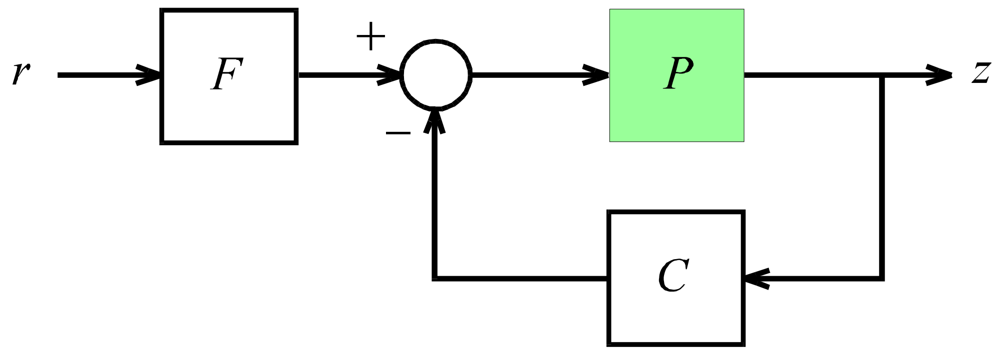
Then the closed-loop transfer function is

$$H = \frac{PC}{1+PC}F = \frac{NY}{D_{\text{cl}}}F \qquad D_{\text{cl}} = DX + NY$$

with D_{cl} the closed-loop characteristic polynomial

Two-degree-of-freedom systems-2

Other two-degree-of-freedom configuration:



$$H = \frac{NX}{D_{\text{cl}}} F$$

has zeros at the roots of N and X

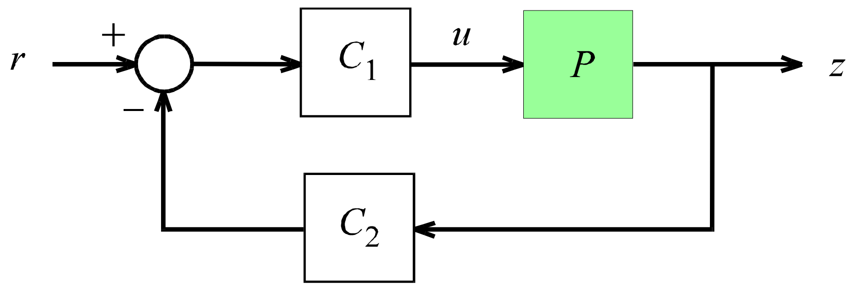
$$H = \frac{NY}{D_{\text{cl}}} F$$

has zeros at the roots of N and Y

Can the zeros of H be made independent of the feedback compensator?

Two-degree-of-freedom systems-3

Further two-degree-of-freedom configuration



$$P = \frac{N}{D}, \quad C_1 = \frac{Y_1}{X_1} \quad C_2 = \frac{Y_2}{X_2}$$

Need $X_1 X_2 = X, \quad Y_1 Y_2 = Y$ to achieve the same loop gain as in the two previous cases

Have

$$H = \frac{PC_1}{1+PC} = \frac{NX_2 Y_1}{D_{\mathrm{cl}}}$$

Two-degree-of-freedom systems-4

$$H = \frac{NX_2Y_1}{D_{\mathrm{cl}}}$$

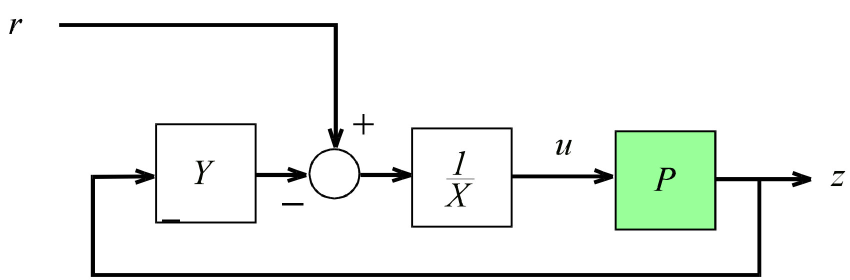
H is independent of the compensator if we let

$$Y_1 = X_2 = 1 \quad \Rightarrow \quad Y_2 = Y, \quad X_1 = X$$

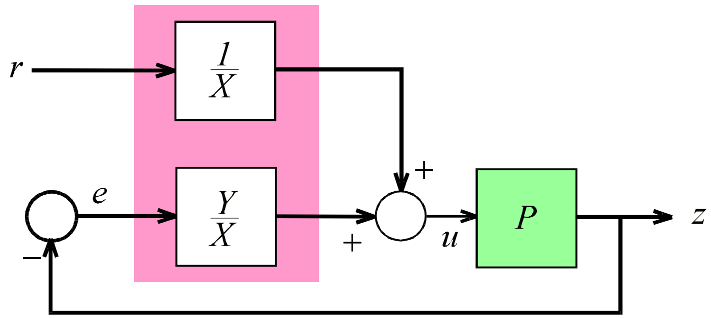
so that

$$C_1 = \frac{1}{X}, \quad C_2 = Y, \quad \boxed{H = \frac{N}{D_{\mathrm{cl}}}}$$

Two-degree-of-freedom systems-5



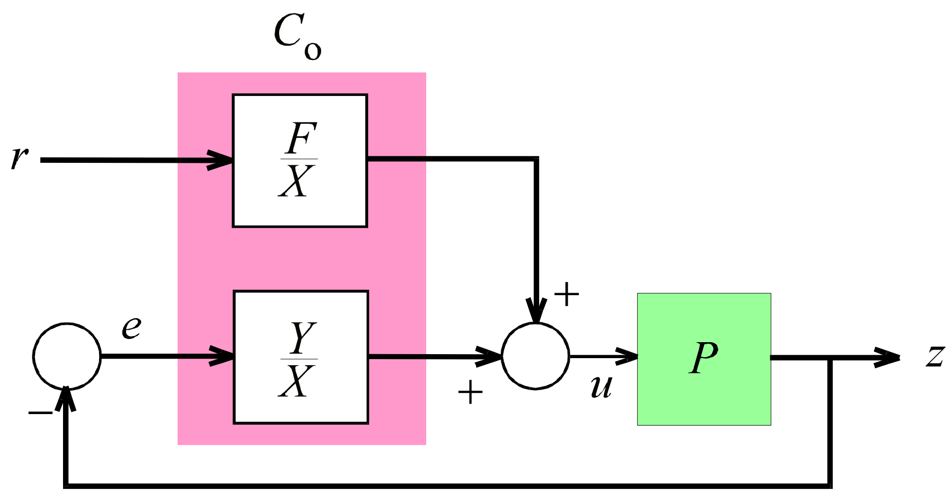
Resulting feedback system



Equivalent configuration

Two-degree-of-freedom systems-6

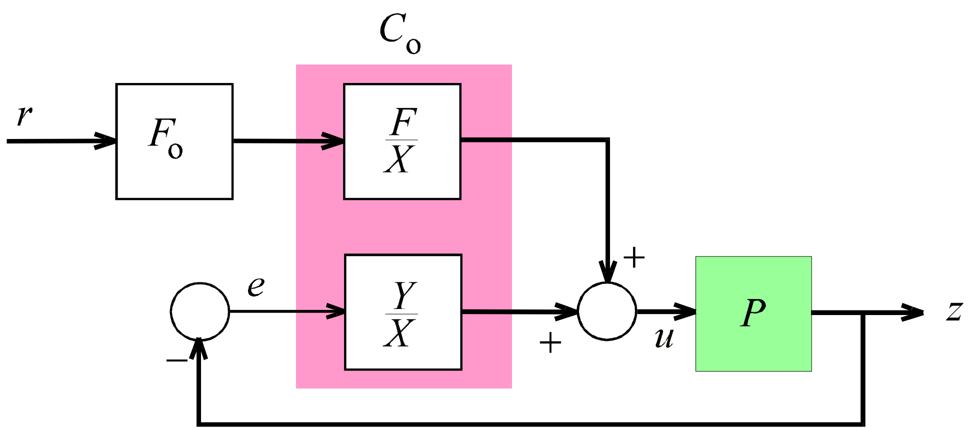
Extension



May choose F polynomial so that we obtain a "1½-degree-of-freedom" system

Two-degree-of-freedom systems-7

Further extension:



F polynomial, F_o rational:

"2½-degree-of-freedom" system

$$H = \frac{NF}{D_{cl}} F_o$$

$F = 1$, F_o rational:

2-degree-of-freedom system