Exercises joint distributions

8. a. Calculate $F_{X,Y}$ with the joint density function of example 1.3.2 for the following probabilities: $P(\frac{1}{2} < X \leq 2 \text{ and } \frac{1}{2} < Y \leq 3)$, $P(X < \frac{1}{2} \text{ and } Y > \frac{1}{2})$ and $P(X = \frac{1}{2} \text{ and } Y = \frac{1}{2})$.

   b. Calculate the probabilities again, now with the help of the joint probability density $f_{X,Y}$.

   c. Calculate for examples 1.3.2 and 1.3.9 the probability $P(X > Y)$.

9. The random variables X and Y contain a joint probability density given by

$$f_{X,Y}(x,y) = \begin{cases} ce^{-2(x+y)}, & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   a. Determine c (with the use of property 1.3.8).

   b. Determine the marginal probability densities f_X and f_Y.

   c. Calculate $P(X > 1 \text{ and } Y > 1)$, $P(X > 1 \text{ of } Y > 1)$, $P(X < Y)$ and $P(X = Y)$.

   d. Determine the joint distribution function of X and Y.

10. The random variables X and Y contain a joint probability density given by

$$f(x, y) = \begin{cases} 8xy, & \text{if } 0 < x < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Sketch the overlapping area of $f(x, y) > 0$ and determine the marginal probability densities f_X and f_Y.

b. Calculate $\rho(X, Y)$.

11. $f_{X,Y}(x, y) = \begin{cases} 16(x - x^3)y^3, & \text{if } 0 < x < 1 \text{ if } 0 < y < 1 \\ 0, & \text{else.} \end{cases}$

a. Determine the marginal probability densities f_X and f_Y.

b. Check that the marginal probability densities in problems 3 and 4 are the same, while the joint probability densities are different.

12. a. Show that for discrete random variables the statement "$P(X = x$ and $Y = y) = P(X = x) \cdot P(Y = y)$ for all x and y' is equivalent with definition 7.3.10 (prove both ways: prove the statement follows from the definition, because of a good choice of B and C ´and prove the other way around).

b. Prove that "$F_{X,Y}(x, y) = F_X(x)F_Y(y)$ for all x and y" follows from the definition 1.3.10 of independence (because of a good choice of B and C´and prove that properties 1.3.11a and 1.311b apply for continuous random variables).

13. We shoot at a practice target with a center 0 and a radius of 1. Every shot is a hit and the point where the target is hit is a pair of continuous random variables (X, Y) which are uniformly distributed on the disc $\{(x, y)|x^2 + y^2 \leq 1\}$, or so to say:

$$f_{X,Y}(x, y) = \begin{cases} c & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{else.} \end{cases}$$

a. Calculate c and sketch the graph of $f_{X,Y}$.

b. Give a graphical interpretation of the calculation of $f_X\left(\frac{1}{2}\right)$. Determine the marginal probability density of X (and of Y).

c. Are X and Y independent?

d. Calculate $f_X(x|Y = y)$

e. Calculate $E(X|Y = 1/2)$

14. X is uniformly distributed on [0,4]. Y is exponentially distributed with parameter $\lambda = 3$. X and Y are independent.

a. Give the joint probability density of X and Y.

b. Calculate $P(0 \leq X \leq 1$ and $Y \geq 1)$.

c. Calculate $P(Y > ln(X+1))$. sketch $\mathbb{R}^2$ first in the range $A = \{(x, y)|y > ln(x+1)\}$

15. X and Y are independent and exponentially distributed with parameters $\lambda = 2$ resp. $\lambda = 1$.

a. Determine the probability density of $Z = X + Y$.

b. Calculate $E(Z)$ and $\text{var}(Z)$.

c. Calculate $P(X > Y)$.

d. Calculate $\rho(X, X+Y)$ and $\rho(Y, X+Y)$.

16. $X_1, X_2, \ldots, X_n$ are independent and exponentially distributed with parameter λ.

a. Give the simutaneous density function and joint probability density of $X_1, X_2, \ldots, X_n$.

b. Determine the probability density of $W = \max(X_1, X_2, \ldots, X_n)$. (Express $F_W(w)$ in $F_{X_1, X_2, \ldots, X_n}$).