Solutions Trial test Calc. I – Functions, Limits, Continuity and Derivatives – April 2018

1. **a**. $y = f(x) = \frac{x+1}{1-x} \Leftrightarrow y(1-x) = x+1 \Leftrightarrow y - xy = x+1 \Leftrightarrow y-1 = x(y+1) \Leftrightarrow x = \frac{y-1}{y+1}$

Interchanging x and y we find: $y = f^{-1}(x) = \frac{x-1}{x+1}$

b. the domain $D_f = \{x \in \mathbb{R} | \, x \neq 1\}$ (denominator $1-x \neq 0$) and the range $R_f = D_{f^{-1}} = \{x \in \mathbb{R} | \, x \neq -1\}$

c. Vertical asymptote $x = 1$, since at $x = 1$ substitution leads to the form $\frac{2}{0}$.

Horizontal asymptote $y = -1$, since $\lim\limits_{x\to\infty} \frac{x+1}{1-x} = \lim\limits_{x\to\infty} \frac{1+\frac{1}{x}}{\frac{1}{x}-1} = \frac{1+0}{0-1} = -1$ and $\lim\limits_{x\to-\infty} \frac{x+1}{1-x} = -1$ as well.

d. $f \circ g(x) = f(2x - x^2) = \frac{2x-x^2+1}{1-2x+x^2}$, so $\lim\limits_{x\to-\infty} f \circ g(x) = \lim\limits_{x\to-\infty} \frac{2x-x^2+1}{1-2x+x^2} = \lim\limits_{x\to-\infty} \frac{\frac{2}{x}-1+\frac{1}{x^2}}{\frac{1}{x^2}-\frac{2}{x}+1} = \frac{0-1+0}{0-0+1} = -1$

2. **a.** $\lim\limits_{x\to4} \frac{x^2-4x}{x^2-3x-4} = \lim\limits_{x\to4} \frac{x(x-4)}{(x-4)(x+1)} = \lim\limits_{x\to4} \frac{x}{x+1} = \frac{4}{5}$

b. $\lim\limits_{x\to1} \frac{x-1}{2x-\sqrt{3+x^2}} = \lim\limits_{x\to1} \frac{(x-1)}{(2x-\sqrt{3+x^2})} \times \frac{(2x+\sqrt{3+x^2})}{(2x+\sqrt{3+x^2})} = \lim\limits_{x\to1} \frac{(x-1)(2x+\sqrt{3+x^2})}{4x^2-(3+x^2)} = \lim\limits_{x\to1} \frac{(x-1)(2x+\sqrt{3+x^2})}{43(x-1)(x+1)}$

$= \lim\limits_{x\to1} \frac{(2x+\sqrt{3+x^2})}{3(x+1)} = \frac{2+\sqrt{3+1}}{3(1+1)} = \frac{2}{3}$

c. $\lim\limits_{x\downarrow2} \frac{x^2-9x+14}{2x^2-4x} = \lim\limits_{x\downarrow2} \frac{(x-2)(x-7)}{2x(x-2)} = \lim\limits_{x\downarrow2} \frac{x-7}{2x} = \frac{2-7}{4} = -\frac{5}{4}$

d. $\lim\limits_{x\to1} \frac{e^x-1}{x^2} = \frac{e-1}{1} = e-1 \quad (\approx 1.72)$

e. $\lim\limits_{x\to\infty} \frac{1-e^x}{1+2e^x} = \lim\limits_{x\to\infty} \frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+2} = \frac{0-1}{0+2} = -\frac{1}{2}$

f. $\lim\limits_{x\to0} \frac{(x+3)^{-1} - 3^{-1}}{x} = \lim\limits_{x\to0} \frac{\frac{1}{x+3}-\frac{1}{3}}{x} \times \frac{3(x+3)}{3(x+3)} = \lim\limits_{x\to0} \frac{3-(3+x)}{3x(x+3)} = \lim\limits_{x\to0} \frac{-x}{3x(x+3)} = \lim\limits_{x\to0} \frac{-1}{3(x+3)} = -\frac{1}{9}$

3. **a.** $f'(x) = \lim\limits_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim\limits_{h\to0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim\limits_{h\to0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim\limits_{h\to0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

b. $m = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \Longrightarrow \textit{equation of the tangent}: y - 2 = \frac{1}{4}(x-4) \quad \Leftrightarrow \quad y = \frac{1}{4}x + 1$

c. f is continuous on its domain $[0, \infty)$, as all root functions are (at $x = 0$ f *is* continuous from the right $\lim\limits_{x\downarrow0} f(x) = f(0) = 0$). f is differentiable on $(0, \infty)$ (see a.), but f is not differentiable at $x = 0$, e.g. $\lim\limits_{x\downarrow0} f'(x) = \infty$.

4. **a.** Domain of g *is* $\mathbb{R}\backslash\{0\}$

b. **1.** $\lim\limits_{x\uparrow0} g(x) = \lim\limits_{x\uparrow0} \frac{1}{x} = +\infty$ **2.** $\lim\limits_{x\downarrow1} g(x) = \lim\limits_{x\downarrow1}(x+2) = 3$ **3.** $\lim\limits_{x\uparrow1} g(x) = \lim\limits_{x\uparrow1} \frac{1}{x} = 1$

4. $\lim\limits_{x\to1} g(x)$ does not exist , since the limit from the left (2.) is not equal to the limit from the right (3.)

5. $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \frac{1}{x} = 0$

c. g is **discontinuous at $x = 0$**: $x = 0$ is a VA, so an **asymptotic discontinuity**;
and g is discontinuous at **$x = 1$** since $\lim\limits_{x\to1} g(x)$ d.n.e.: b.2 and b.3 show it is a **jump discontinuity.**

d. g is not differentiable at $x = 2$, since $g'(x) = 1$ if $1 < x < 2$ and $g'(x) = 2x$, if $x > 2$. So $\lim\limits_{x\uparrow2} g'(x) = \lim\limits_{x\uparrow2} 1 = 1 \neq \lim\limits_{x\downarrow2} g'(x) = \lim\limits_{x\downarrow2} 2x = 4$

Or: $g'(2) = \lim\limits_{h\to0} \frac{g(2+h)-g(2)}{h}$ does not exist because:

$$\lim_{h\uparrow0} \frac{g(2+h)-g(2)}{h} = \lim_{h\uparrow0} \frac{(2+h+2)-4}{h} = \lim_{h\uparrow0} \frac{h}{h} = 1 \neq \lim_{h\downarrow0} \frac{g(2+h)-g(2)}{h} = \lim_{h\downarrow0} \frac{(2+h)^2-4}{h} = \lim_{h\downarrow0} \frac{4h+h^2}{h} = \lim_{h\downarrow0} \frac{4+h}{1} = 4$$