Trial partial test Probability and Statistics I – Chapters 1-5

This exam consists of 5 exercises and a formula sheet. A simple calculator is allowed

1. Suppose that in a country 22% of the breadwinners (of families) earn at least 60.000 euros a year. A fraction of 70% of these breadwinners own a car.
47% of the remaining breadwinners (< 60000) own a car.

a. Determine the probability that a random breadwinner owns a car.
In your solution you should define proper events, express the given probabilities in these events and then use rules of probability to determine the requested probability.

b. Determine the probability that a breadwinner who owns a car earns at least 60.000 euros a year.

2. Suppose Peter and Paul randomly divide 3 Mars and 3 Snicker candy bars, in such a way that they both get 3 candy bars. Determine the probability that Peter gets 3 candy bars of the same type.

3. Barney is given the opportunity by his sponsor to earn € 20000. To get this money, he has to hit bullseye at least once in five throws. Dart statisticians have discovered that Barney hits bullseye in 20 out of 100 throws.
What is the probability that Barney doesn't get his bonus? Clearly state the assumptions you make.

4. A student designed a new app that shows the places for going out in the region. Before investing in the application, he wants to poll the potential interest for the application. Because students will be his focus group, he takes a random sample of n students and asks them whether they are interested in the app if it would cost 1 euro. Let p denote the probability that a student is interested, and X the number of people that are interested out of a sample of n.

a. What is a suitable distribution for X? Under which assumptions?

b. $P(X > 0)$, if $n = 10$ and $p = 0.22$

c. $P(X < 2)$, if $n = 250$ and $p = 0.01$

5. The joint probability function $P(X = x \text{ and } Y = y)$ of X and Y is given in the accompanying table.

x \ y	0	1	2
0	0	$\frac{1}{7}$	$\frac{1}{7}$
1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
2	$\frac{1}{7}$	$\frac{1}{7}$	0

a. Determine $E(X)$ and $var(X)$.

b. Determine the distribution of $Z = X \cdot Y$ (in a table) and determine $E(Z)$.

c. Determine the value of $\rho(X, Y)$ and explain what this value means.

d. Determine $\rho(2X, -Y)$ (use $\rho(X, Y) = -0.4$ if you could not solve c.)

e. Determine $E(X|Y = 0)$.

Grade = $1 + \frac{points}{25} \times 9$, rounded to 1 decimal.

1		2	3	4			5					Total
a	b			a	b	c	a	b	c	d	e	
3	2	3	2	1	2	2	2	2	3	1	2	25

Formulas:

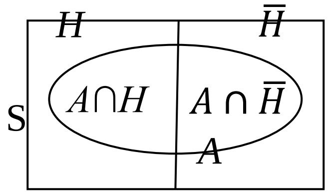
Distribution	$\boldsymbol{E(X)}$	$\boldsymbol{var(X)}$
Geometric	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$n \cdot \frac{R}{N}$	$n \cdot \frac{R}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$
Poisson $P(X = x) = \frac{e^{-\mu}\mu^x}{x!}, x = 0, 1, 2, \ldots$	μ	μ

$$var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum_{i \neq j}\sum cov(X_i, X_j)$$

Solutions

Exercise 1

a. Define events as: A = "Cost winner owns car" and
H= "Cost winner earns at least 60.000 a year"
Given: $P(H)=0.22$, $P(A\mid H)=0.70$, $P(A\mid \overline{H})=0.47$
Probability: (also see Venn-diagram):
$P(A) = P(AH) + P(A\overline{H}) = P(A|H)P(H) + P(A|\overline{H})P(\overline{H})$
Fill in (Using complement rule $P(\overline{H}) = 1 - P(H)$ = 0.78)
$\text{P(A)} = 0.70\times 0.22 + 0.47\times 0.78 = 0.5206$



b. $P(H|A) = \frac{P(H\cap A)}{P(A)} = \frac{P(A|H)P(H)}{P(A)} = \frac{0.70\times 0.22}{0.5206} \approx 29.6\%$

Exercise 2

If we take 3 candy bars at random out of $3 + 3 = 6$, we can get $\binom{6}{3} = 20$ combinations, all with the same probability. Two of these 20 combinations (Snicker+Snicker+Snicker and Mars+Mars+Mars) lead to the asked for probability: $\frac{2}{20} = 10\%$.
(Another way of reasoning: $2 \times P(3\ \text{ times Mars for Paul}) = 2 \times \left[\frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}\right] = \frac{1}{10}$)

Exercise 3

We assume that the trials to hit bull's eye are independent with probability $p = 0.2$. In that case, X (the number of trials needed) is geometrically distributed with $p = 0.2$. $P(X > 5) = (1-p)^5 = 0.8^5 \approx$ 32.8%. (If one only regards the first 5 trials,
then X = *number of bull's eye hits* $\sim B(5{,}0.2)$ *with* $P(X = 0) = 0.8^5$)

Exercise 4

a. X is $B(n,p)$-distributed if we assume that the n students are interested with probability p, each independent. (n times a Bernoulli experiment with probability p).
(*Further explanation: The Poisson approximation only may be applied for specific values of n and p. If one considers the situation as sampling without replacement, the hypergeometric distribution would be suitable. However, n is unknown. Because n is probably (very) large, the binomial distribution is appropriate.)*

b. n = 10 and p = 0.2: $P(X > 0) = 1 - P(X = 0) = 1 - 0.78^{10} = 91.7\%$
(*The probability* $P(X = 0)$ *also can be found in the B(10, 0.2)-table, if available.*)

c. For n = 250 and p = 0.01 (np = 2.5 < 5) we can use the Poisson approximation, with: $\mu = np = 2.5$ (< 5). So: $P(X < 2) = P(X = 0) + P(X = 1) = e^{-2.5} + 2.5e^{-2.5} = 3.5e^{-3.5} \approx 28{,}7\%$

Exercise 5

a. $P(X = 0) = P(X = 2) = \frac{2}{7}$ and $P(X = 1) = \frac{3}{7}$, so $E(X) = 1$ (symmetry).
$E(X^2) = \sum_x x^2 \cdot P(X = x) = 0 + 1 \cdot \frac{3}{7} + 2^2 \cdot \frac{2}{7} = \frac{11}{7} \Longrightarrow\ var(X) = E(X^2) - (EX)^2 = \frac{11}{7} - 1^2 = \frac{4}{7}$

b. $Z = XY$ can attain the values 0, 1 and 2:
$E(Z) = E(XY) = \frac{5}{7}$ (see the table)

z	0	1	2	Total
$P(Z = z)$	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	1
$zP(Z = z)$	0	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{5}{7} = E(Z)$

c. $\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$,
where $\sigma_X = \sigma_Y = \sqrt{var(X)} = \sqrt{\frac{4}{7}}$ and $cov(X,Y) = E(XY) - E(X)E(Y) = \frac{5}{7} - 1 \cdot 1 = -\frac{2}{7}$.
$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-2/7}{4/7} = -\frac{1}{2}$, meaning that there is a moderate, negative correlation of X and Y.

d. $\rho(2X,-Y) = -\rho(X,Y) = \frac{1}{2}$. (since $\rho(aX + b, cY + d) = -\rho(X,Y)$ if $a \cdot c < 0$)

e. $P(X = 1|Y = 0) = \frac{P(X=1 \text{ and } Y=0)}{P(Y=0)} = \frac{1/7}{2/7} = \frac{1}{2} = P(X = 2|Y = 0)$.
Hence $E(X|Y = 0) = 1.5$ (using symmetry).