Trial Exam Calculus I

The use of a calculator (without memory for formulas) is allowed. Motivate your solutions!

Exercise 1. Find the following limits, if they exist. Show your solution (including ∞ and -∞), if necessary in at most four steps. If a limit does not exist, state why not.

a. $\lim\limits_{x\to 0} \dfrac{x^2-8x}{x^2-9x+8}$ **b.** $\lim\limits_{x\to 8} \dfrac{x^2-8x}{x^2-9x+8}$ **c.** $\lim\limits_{x\to 2} \dfrac{4-x^2}{\sqrt{x+1}-\sqrt{3}}$

d. $\lim\limits_{t\to\infty} \dfrac{1}{\sqrt{2^t}}$ **e.** $\lim\limits_{x\to\infty} \dfrac{x^2+1}{e^{2x}}$ **f.** $\lim\limits_{x\to\infty} \dfrac{e^x+7e^{-x}}{5-2e^x}$

Exercise 2. Find the derivatives of the following functions:

a. $y = 3^x + \frac{3}{x}$ **b.** $f(x) = \sqrt{x^2-2x+8}$ **c.** $g(x) = \frac{\cos(2x)}{\sin(x)}$ **d.** $h(x) = x\ln(x) - x$

Exercise 3. Given is the function $f(x) = x^2 - 3x$

a. State the derivative of $f(x)$ (using the rules of differentiation).
b. Give the equation of the tangent line at the point (2, -2) of the graph of f.
c. Proof that the answer of part a. is correct, using the definition of derivative.

Exercise 4. The function g is defined as follows on the interval [-4, 1]: $g(x) = x^3e^x$
Find the absolute maximum and minimum of g on the interval [-4, 1]

Exercise 5. Consider the function $h(x) = \frac{2x^2-4x}{x^2-6x+8}$

a. Give the domain of h.
b. Find the equations of the horizontal and vertical asymptotes of h (if they exist).
c. At what values of x is $f(x)$ discontinuous? Investigate the nature of each discontinuity.
d. Give the interval(s) at which h is differentiable.

Exercise 6. Show that the line $y = 2x - 1$ is a slant asymptote of $s(x) = \frac{2x^2+x-5}{x+1}$

Exercise 7 Investigate the function $f(x) = 36x^2 - 2x^4$ as follows:

a. Find the y-intercept and the x-intercept(s) of f.
b. Does the function $f(x)$ have special properties: is it even, odd or periodic?
c. Compute $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ *(If the limit is not a number, reason if it is ∞ or -∞)*
d. Determine the derivative of f and all local and absolute minima and maxima.
e. Determine the second derivative and the inflection points.
f. Sketch the graph of f, using the results of **a.-e**.

Grading:	1	2	3			4	5				6	7						Total
	a-f	a-d	a	b	c		a	b	c	d		a	b	c	d	e	f	
	6×2	4×2	1	3	3	5	2	4	2	2	3	3	2	2	5	4	2	63

Solutions Trial Exam Calculus I

Exercise 1

a. $\lim\limits_{x\to 0} \frac{x^2-8x}{x^2-9x+8} = \frac{0}{8} = 0$ (direct substitution)

b. $\lim\limits_{x\to 8} \frac{x^2-8x}{x^2-9x+8} = \lim\limits_{x\to 8} \frac{x(x-8)}{(x-1)(x-8)} = \lim\limits_{x\to 8} \frac{x}{x-1} = \frac{8}{7}$

c. $\lim\limits_{x\to 2} \frac{4-x^2}{\sqrt{x+1}-\sqrt{3}} = \lim\limits_{x\to 2} \frac{(4-x^2)(\sqrt{x+1}+\sqrt{3})}{(\sqrt{x+1}-\sqrt{3})(\sqrt{x+1}+\sqrt{3})} = \lim\limits_{x\to 2} \frac{(2-x)(2+x)(\sqrt{x-1}+\sqrt{3})}{(x+1)-3} = \lim\limits_{x\to 2}$ -(2 + x)($\sqrt{x-1}+\sqrt{3}$) = -8$\sqrt{3}$

d. $\lim\limits_{t\to\infty} \frac{1}{\sqrt{2^t}} = \lim\limits_{t\to\infty} \frac{\sqrt{1}}{\sqrt{2^t}} = \lim\limits_{t\to\infty} \sqrt{\frac{1}{2^t}} = \sqrt{\lim\limits_{t\to\infty} \frac{1}{2^t}} = \sqrt{0} = 0$ (or use $\lim\limits_{t\to\infty}\left(\sqrt{2}\right)^t = \lim\limits_{t\to\infty}\left(\sqrt{2}\right)^t = +\infty$, since $\sqrt{2} > 1$)

e. $\lim\limits_{x\to\infty} \frac{x^2+1}{e^{2x}} =^H \lim\limits_{x\to\infty} \frac{2x}{2e^{2x}} = \lim\limits_{x\to\infty} \frac{x}{e^{2x}} =^H \lim\limits_{x\to\infty} \frac{1}{2e^{2x}} = 0$

f. $\lim\limits_{x\to\infty} \frac{e^x+7e^{-x}}{5-2e^x} = \lim\limits_{x\to\infty} \frac{1+\frac{7}{e^{2x}}}{\frac{5}{e^x}-2} = \frac{1+0}{0-2} = -\frac{1}{2}$

Exercise 2

a. $y = 3^x + \frac{3}{x} \qquad \Longrightarrow \quad y' = 3^x \ln(3) - \frac{3}{x^2}$

b. $f(x) = \sqrt{x^2-2x+8} \qquad \Longrightarrow f'(x) = \frac{1}{2}(x^2-2x+8)^{-\frac{1}{2}} \cdot (2x-2) = \frac{x-1}{\sqrt{x^2-2x+8}}$

c. $g(x) = \frac{\cos(2x)}{\sin(x)} \qquad \Longrightarrow g'(x) = \frac{-2\sin(2x)\cos(x)-\cos(2x)\sin(x)}{(\sin(x))^2}$

d. $h(x) = x\ln(x) - x \qquad \Longrightarrow h'(x) = \left(1\cdot\ln(x) + x\cdot\frac{1}{x}\right) - 1 = \ln(x) + 1 - 1 = \ln(x)$

Exercise 3

a. $f'(x) = 2x - 3$

b. The equation of the tangent line at (2, -2) has form $y = mx + b$, where the slope $m = f'(2) = 1$ and b can be found by substituting (2, -2) in $y = x + b$: $-2 = 2 + b$, so $b = -4$
The equation of the tangent line in (2, -2) is $y = x - 4$

c. $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{[(x+h)^2-3(x+h)]-(x^2-3x)}{h} = \lim\limits_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim\limits_{h\to 0} (2x+h-3) = 2x-3$

Exercise 4

We can apply the **closed interval method**, since g is a continuous function on [-4, 1]:

1. The critical points: $g'(x) = 3x^2\cdot e^x + x^3\cdot e^x = x^2(3+x)e^x$
   $g'(x) = 0 \Leftrightarrow x = 0$ or $x = -3$: $\boldsymbol{f(0) = 0}$ and $\boldsymbol{f(-3) = -27e^{-3}}$ **≈ -1.34**
2. The end points of the interval: $\boldsymbol{f(1) = e \approx 2.72}$ and $\boldsymbol{f(-4) = -64e^{-4}}$ **≈ -1.17**
3. The absolute maximum of g on [-3, 1] is $\boldsymbol{f(1) = e}$ and the absolute minimum is $\boldsymbol{f(-3) = -27e^{-3}}$

Exercise 5

a. h is not defined if $x^2 - 6x + 8 = (x-2)(x-4) = 0$, so if $x = 2$ or $x = 4$
The domain is $\mathbb{R}\backslash\{\mathbf{2,4}\}$ **(**or $(-\infty,2)\cup(2,4)\cup(4,\infty)$)

b. $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{2-\frac{4}{x}}{1-\frac{6}{x}+\frac{1}{x^2}} = \frac{2-0}{1-0+0} = 2$ and similarly $\lim_{x\to-\infty} h(x) = 1$: a **horizontal asymptote *y*= 2**

At $x = 4$ $h(x)$ has the form $\frac{16}{0}$, so we have a **vertical asymptote $x = 4$**

At $x = 2$ $h(x)$ has the form $\frac{0}{0}$: $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{2x(x-2)}{(x-2)(x-4)} = \lim_{x\to 2} \frac{2x}{x-4} = -2$, so no VA at $x = 2$.

c. A rational function is continuous and differentiable on its domain, so we have 2 discontinuities:

At $\boldsymbol{x = 4}$ $h(x)$ has an **asymptotic discontinuity**, since $x = 4$ is a vertical **asymptote** **(**see b.)

At $\boldsymbol{x = 2}$ $h(x)$ has a **removable discontinuity**, since $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{2x(x-2)}{(x-2)(x-4)} = \lim_{x\to 2} \frac{2x}{x-4} = -2$

d. $h(x)$ is continuous and differentiable on its domain and discontinuous at $x = 2$ and $x = 4$: h is not defined at these values so not continuous nor differentiable. h is differentiable at $(-\infty, 2), (2,4)$ and $(4, \infty)$.

Exercise 6 the line $y = 2x - 1$ is a slant asymptote if $\lim_{x\to\infty} [s(x) - (2x - 1)] = 0$ (*or use long division)*

$\lim_{x\to\infty} \left[\frac{2x^2+x-5}{x+1} - (2x-1)\cdot\frac{x+1}{x+1}\right] = \lim_{x\to\infty} \left[\frac{2x^2+x-5-(2x^2+x-1)}{x+1}\right] = \lim_{x\to\infty} \left[\frac{-4}{x+1}\right] = 0$: $y = 2x-1$ is a slant as.

Exercise 7 $f(x) = 36x^2 - 2x^4$

a. ***y*-intercept**: $\boldsymbol{f(0) = 0}$ and ***x*-intercepts**: solve $f(x) = 2x^2(18 - x^2) = 2x^2(\sqrt{18} - x)(\sqrt{18} + x) = 0$

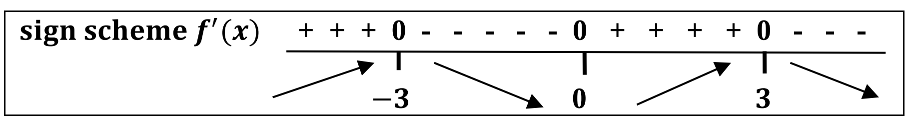
$\Leftrightarrow \boldsymbol{x = 0}$ **or** $\boldsymbol{x = \sqrt{18}}$ **or** $\boldsymbol{x = -\sqrt{18}}$ $\quad (\sqrt{18} = 3\sqrt{2} \approx 4.2)$

b. $f(-x) = 36(-x)^2 - (-x)^4 = 36x^2 - 2x^4 = f(x)$: $\boldsymbol{f}$ **is even** *(the graph can be reflected about the y-axis)*

c. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2x^2(\sqrt{18} - x)(\sqrt{18} + x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x^2(18 - x^2) = -\infty$

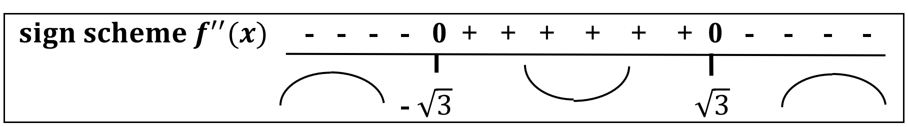
d. $f'(x) = 72x - 8x^3 = 8x(9 - x^2) = 8x(3 - x)(3 + x)$. So $f'(x) = 0$ if $x = 0$ or $x = 3$ or $x = -3$.

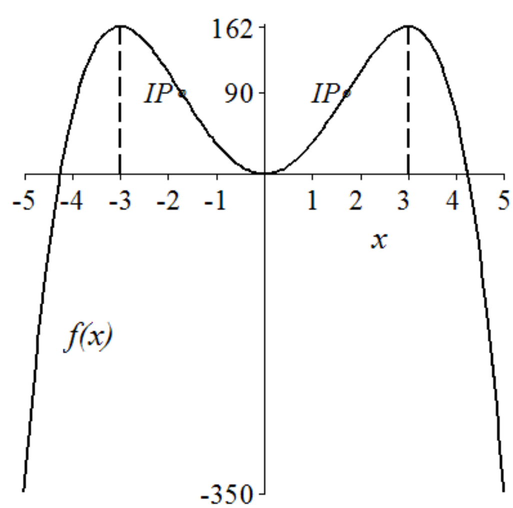
The sign scheme of the derivative: (*check e.g. that* $f'(4)$= -224, $f'(-4)$= 224, $f'(1) = 64$ *and* $f'(-1)$ = -64)



Considering the sign changes of $f'(x)$, $\boldsymbol{f(0) = 0}$ **is a local minimum**, but because of the limits in c. not absolute.
Likewise $\boldsymbol{f(-3) = 162}$ **and** $\boldsymbol{f(3) = 162}$ **are local <u>and</u> absolute maxima.**

e. $f''(x) = 72 - 24x^2 = 24(3 - x^2) = 24(\sqrt{3} - x)(\sqrt{3} + x)$. $f''(x) = 0$ if $x = \sqrt{3}$ or $x = -\sqrt{3}$





Considering the sign changes of $f''(x)$ at $-\sqrt{3}$ and $\sqrt{3}$,
we have two inflection points with $y = f(\pm\sqrt{3}) = 90$:
$(\boldsymbol{-\sqrt{3}, 90})$ and $(\boldsymbol{\sqrt{3}, 90})$.

f. The intercepts, extreme values, IP and some additional points:

x	0	± 1	$\pm\sqrt{3} \approx \pm 1.7$	± 2	± 3	$\pm\sqrt{18} \approx \pm 4.2$	± 5
$f(x)$	0	34	90	112	162	0	-350