Test Probability and Statistics I (Ch. 1-5) April 22, 2018, 14.30-16.00

This exam consists of 5 exercises and formulas (PTO). A simple scientific calculator is allowed.

1. A company participates in a bid-procedure to obtain a large project. The company's management thinks that the success probability is 60%. After submitting the bid the committee that assesses all bids can ask for additional information. From past experience it is known that 75% of the winning bids were asked for additional information whereas only 40% of the other bids were asked for additional information. Now, if we are told that the mentioned company is asked for additional information, what is the probability that this company obtains the project?
(Define relevant events, state the given (conditional) probabilities in terms of these events and compute the requested probability using probability rules).

2. John claims to be clairvoyant (i.e., he can supernaturally see things that other people cannot see). To test his claim 10 boxes are presented to him: in each of the boxes there is a (well closed) bottle with either water or vinegar: they told him that five of the boxes contain water and five vinegar. The organiser of the test asked John to select the 5 boxes containing water.
X is the number of the 5 chosen boxes that really contain water.
John will be considered clairvoyant if at least 4 of the five chosen boxes contain water. If we assume that John is not clairvoyant at all (so he chooses the 5 boxes at random), then determine $P(X \geq 4)$.

3. Alex is playing a game of Ludo with his friends. In order to start his game (a pawn on the game board), he first has to throw 6 face up with a (fair) dice.
X is the number of throws he needs to succeed.

a. Compute $P(X = 10)$ and $E(X)$.
b. Determine $P(X > 10)$.
c. Determine $P(X > 10 | X > 6)$.
d. What is the probability that Alex throws his third "6" in his tenth throw?

4. In the table the probability function $P(X = x \text{ and } Y = y)$ of X and Y is given:

x \ y	0	1	2
0	0.1	0.2	0.2
1	0.2	0.1	0
2	0.2	0	0

a. Give the distribution of X, $E(X)$ and $var(X)$.
b. Determine the covariance of X and Y.
c. Determine the correlation coefficient of X and Y, and give the proper interpretation of this value.
d. Are X and Y independent? Motivate your answer.
e. Give the conditional distribution of Y, given $X = 0$, and determine $E(Y|X = 0)$.

5. A car dealer sells 3 models (say: small, medium, big) of a certain brand. Let N denote the total number of new cars he sells per month. It is known that N has a Poisson distribution, with an average of 7 sold cars per month. The sales price X of a randomly selected sold car is 10000 Euros with probability 25% (small car), 15000 Euros with a probability of 50% (medium car), and 20000 Euros with a probability of 25% (big car). Let S denote "the total revenue of sold cars in a month".

a. Give $E(X)$ and $E(S|N = 12)$.
b. Express $E(S|N)$ in N and use this to calculate $E(S)$.

Grading: grade = $1 + \frac{\# points}{30} \times 9$, rounded at 1 decimal.

1	2	3				4					5		Tot
		a	b	c	d	a	b	c	d	e	a	b	
4	3	2	2	2	2	3	2	2	2	2	2	2	30

Formulas

Distribution	$\boldsymbol{E(X)}$	$\boldsymbol{var(X)}$
Geometric	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$n \cdot \frac{R}{N}$	$n \cdot \frac{R}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$
Poisson $P(X = x) = \frac{\mu^x e^{-\mu}}{x!}, x = 0, 1, 2, \dots$	μ	μ

$$var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} var(X_i) + \sum_{i \neq j}\sum cov(X_i, X_j)$$

Solutions

Exercise 1

Define O = "The company Obtains the project" and A = "additional information is requested".

Then $P(O) = 0.60$, $P(A|O) = 0.75$ en $P(A|\overline{O}) = 0.40$.

To be computed: $P(O|A)$. We will apply Bayes` rule:

$$P(O|A) = \frac{P(O \cap A)}{P(A)} = \frac{P(A|O)P(O)}{P(A|O)P(O) + P(A|\overline{O})P(\overline{O})} = \frac{0.75 \cdot 0.60}{0.75 \cdot 0.60 + 0.40 \cdot 0.40} = \frac{0.45}{0.61} = 0.738$$

Exercise 2

From the description we conclude that John will draw five times from 10 bottles, where 5 of them contain water and 5 vinegar. If the draws are random (not clairvoyant at all), we can use the hypergeometric distribution: $P(X \geq 4) = P(X = 4) + P(X = 5) = \frac{\binom{5}{4}\binom{5}{1}}{\binom{10}{5}} + \frac{\binom{5}{5}\binom{5}{0}}{\binom{10}{5}} = \frac{25+1}{252} \approx 10.3\%$

Exercise 3

a. X has a geometric distribution with $par.\, p = \frac{1}{6}$, so $P(X = 10) = \left(\frac{5}{6}\right)^9 \cdot \frac{1}{6} \approx 3.2\%$ and $E(X) = \frac{1}{p} = 6$.

b. $P(X > 10) = \left(\frac{5}{6}\right)^{10} \approx 16.2\%$.

c. Use the definition: $P(X > 10|X > 6) = \frac{P(X>10 \text{ and } X>6)}{P(X>6)} = \frac{P(X>10)}{P(X>6)} = \frac{\left(\frac{5}{6}\right)^{10}}{\left(\frac{5}{6}\right)^{6}} = \left(\frac{5}{6}\right)^4 \approx 48.2\%$

d. $P(\text{third "6" in his tenth throw}) = \binom{9}{2}\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^3 \approx 4.65\%$

($\frac{1}{6}$ probability that the 10th throw is "6": the other 9 throws should consist of 2 sixes and 7 other outcomes, each order having probability $\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^2$ and there are $\binom{9}{2}$ of these orders.)

Exercise 4

y / x	0	1	2	$P(X = x)$
0	0.1	0.2	0.2	**0.5**
1	0.2	0.1	0	**0.3**
2	0.2	0	0	**0.2**
$P(Y = y)$	0.5	0.3	0.2	**total = 1**

a. The distribution of X: see the table, first and last column.
$E(X) = \sum_x x \cdot P(X = x) = 1 \cdot 0.3 + 2 \cdot 0.2 = 0.7$
$var(X) = E(X^2) - (EX)^2$
$= (0^2 \times 0.5 + 1^2 \times 0.3 + 2^2 \times 0.2) - 0.7^2 = 0.61$

b. The covariance: $cov(X, Y) = E(XY) - EX \times EY$
Y has the same distribution as X: $E(X) = E(Y) = 0.7$ and $var(X) = var(Y) = 0.61$
$E(XY) = \sum\sum x \cdot y \cdot P(X = x \text{ and } Y = y) = 1 \times 1 \times 0.1 = 0.1$
hence $cov(X, Y) = 0.1 - 0.7 \times 0.7 = -0.39$

c. $\rho(X, Y) = \frac{cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{-0.39}{0.61} \approx -0.639$. X and Y are moderately negatively correlated.

d. X and Y are dependent, because (for example): $\rho \neq 0$
(*or, using the definition, because* $0 = P(X = 2 \text{ and } Y = 2) \neq P(X = 2) \cdot P(Y = 2) = 0.2^2$
or, using exercise e: $P(Y = 0|X = 0) = 0.2 \neq 0.5 = P(Y = 0)$)

e. $P(Y = 0|X = 0) = \frac{P(Y=0 \text{ and } X=0)}{P(X=0)} = \frac{0.1}{0.5} = 0.2$
and $P(Y = 1|X = 0) = P(Y = 2|X = 0) = \frac{0.2}{0.5} = 0.4$.
Hence $E(Y|X = 0) = \sum_y y \cdot P(Y = y|X = 0) = 0 \cdot 0.2 + 1 \cdot 0.4 + 2 \cdot 0.4 = 1.2$.

Exercise 5

a. $E(X) = \sum_x xP(X = x) = 15000$ Euro, using symmetry.
Suppose $N = 12$ cars are sold and $X_1, \ldots, X_{12}$ are the sales prices: then $S = X_1 + \cdots + X_{12}$.
$E(S|N = 12) = E(X_1 + \cdots + X_{12}) = E(X_1) + \cdots + E(X_{12}) = 12 \cdot 15000 = 180\,000$ Euro.

b. $E(S|N = n) = n \cdot 15000$, so $E(S|N) = 15000N$.
Hence $E(S) = E[E(S|N)] = E(15000N) = 15000 \cdot E(N) = 15000 \cdot 7 = 105\,000$ Euro.