Solutions test Calculus I – Functions, Limits, Continuity and Derivatives – April 2018

1. **a**. $f(-x) = 2 \cdot (-x) - (-x)^2 = -2x - x^2 \neq f(x)$ or $f(-x)$, f is odd nor even.

b. x-intercepts: $f(x) = 0 \Leftrightarrow x(2-x) = 0$. Hence $x = 0$ and $x = 2$ are the x-intercepts. y-intercept: $y = f(0) = 0$

c. Verification: $-(x-2)^2 + 4 = -(x^2 - 4x + 4) + 4 = -x^2 + 4x \neq f(x)$: error in exercise

Transformations of $y = x^2$: 1. Shift $y = x^2$ two units to the right

2. Reflect $y = (x-2)^2$ about the x-axis (or: "stretch" vertically by a factor -1)

3. Shift $y = -(x-2)^2$ four units upward.

d. Domain $D_f = \mathbb{R}$, range $R_f = (-\infty, 4]$ using $f(x) = -(x-2)^2 + 4$:the graph of f is a parabola, that opens downward with a top $y = 4$ at $x = 2$ (note that $y = -(x-2)^2 + 4 \leq 4$)

e. If $g(x) = \frac{1}{1-x}$, find $g \circ f(x) = g(2x - x^2) = \frac{1}{1-2x+x^2}$

$D_f = \mathbb{R}$ (no restrictions), but the denominator of $g \circ f$ is $1 - 2x + x^2 = (x-1)^2 \neq 0$, so $x \neq 1$: $D_{g \circ f} = \mathbb{R} \backslash \{1\}$ *(other notations:* $\{x \in \mathbb{R} | x \neq 1\}$ *or* $(-\infty, 1) \cup (1, \infty)$)

f. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[2(x+h) - (x+h)^2] - (2x - x^2)}{h} = \lim_{h \to 0} \frac{2h - 2xh - h^2}{h}$

$= \lim_{h \to 0} (2 - 2x - h) = 2 - 2x$

g. Use the result in f. to find the slope and the equation of the tangent line of the graph of $f(0) = 0$: the slope and equation of the tangent line of the graph at $(0, 0)$ are $f`(0) = 2$ and $y = 2x$, respectively.

2. **a.** $\lim_{x \to 5} \frac{x^2 - 25}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{(x-5)(x+5)}{(x-5)^2} = \lim_{x \to 5} \frac{x+5}{x-5}$ d.n.e. (Vertical asymptote at $x = 5$)

b. $\lim_{x \downarrow 5} \frac{x^2 - 25}{x^2 - 10x + 25} = \lim_{x \downarrow 5} \frac{(x-5)(x+5)}{(x-5)^2} = \lim_{x \downarrow 5} \frac{x+5}{x-5} = -\infty$ (if $x \downarrow 5$, $x - 5$ is close to 0 but negative)

c. $\lim_{x \to -5} \frac{x^2 - 25}{x^2 - 10x + 25} = \frac{0}{100} = 0$ (direct substitution)

d. $\lim_{x \to 4} \frac{x^2 + 4x - 32}{2 - \sqrt{x}} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} = \lim_{x \to 4} \frac{(x-4)(x+8)(2+\sqrt{x})}{4 - x} = \lim_{x \to 4} \frac{(x+8)(2+\sqrt{x})}{-1} = \frac{(4+8)(2+\sqrt{4})}{-1} = -48$

e. $\lim_{x \to \infty} \frac{e^{2x} - e^x}{1 + 2e^{2x}} = \lim_{x \to \infty} \frac{1 - e^{-x}}{e^{-2x} + 2} = \frac{1}{2}$ (since e.g. $\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$)

f. $\lim_{x \to \infty} \frac{\sin(x)}{x^2} = 0$, using the Squeeze Theorem $-\frac{1}{x^2} \leq \frac{\sin(x)}{x^2} \leq +\frac{1}{x^2}$ and $\lim_{x \to \pm\infty} \frac{1}{x^2} = 0$

g. $\lim_{x \to 0} \frac{\cos(x)}{x^2} = +\infty$ (shape $\frac{1}{0}$, where x^2 is always positive for x close to 0: VA $x = 0$)

h. $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$

3. **a.** 1. $\lim_{x \uparrow 0} g(x) = \lim_{x \uparrow 0} 2x = 0$ 2. $\lim_{x \downarrow 0} g(x) = \lim_{x \downarrow 0} x^2 = 0$ and hence 3. $\lim_{x \to 0} g(x) = 0$

4. $\lim_{x \downarrow 1} g(x) = \lim_{x \downarrow 1} (2x - 1) = 1$ 5. $\lim_{x \uparrow 1} g(x) = \lim_{x \uparrow 1} x^2 = 1$

b. At $\underline{x = 0}$ $f(x)$ is not defined, so discontinuous, but $\lim_{x \to 0} g(x) = 0$ (see a.3.), so it is removable discontinuity.

$\underline{x = 1}$: according to a.4. and 5, $\lim_{x \to 1} g(x) = 1 = g(1) = 1^2$: f is continuous at $x = 1$.

c. At $x = 0$ $f(x)$ is not defined (and not continuous), so not differentiable.

d. At $x = 1$: $\lim_{h \uparrow 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \uparrow 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \uparrow 0} \frac{2h + h^2}{h} = 2$ and $\lim_{h \downarrow 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \downarrow 0} \frac{[2(1+h) - 1] - 1}{h} = 2$. Hence $g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = 2$, so g is differentiable at $x = 1$.