Intermediate test Applied Statistics 29th of April 2018 (Ch 2-4)

A formula sheet and probability tables are provided. Write name and math number on your sheets

1. We determined the sodium content of twenty-five boxes organic cornflakes.
The ordered measurements (in milligram) are as follows:

128.24 128.33 128.71 128.77 128.77 129.00 129.29 129.39 129.53 129.54 129.64 129.65 129.73
129.74 129.78 130.12 130.14 130.42 130.69 130.72 130.80 130.91 130.92 131.15 133.15

Using certain software, we obtain the (classical) numerical summary of the data. The results are presented in the following table:

Sample size	Mean	Standard deviation	Variance
25	129.885	1.076	1.157

a. Determine a 95%-confidence interval for the mean sodium content. First state the model assumptions, necessary to compute the interval.
b. Is the following interpretation of the interval in a. correct: "About 95% of the future observations will be included in the interval, determined in a."? Why (not)?
c. Determine a 95%-confidence interval for the variance of the sodium content.
d. Can you use these data to support the claim that the mean sodium content deviates from 130 milligram? Use the testing procedure with $\alpha = 1\%$.

2. We are counting the number X of trials until a success occurs: X has a geometric distribution with unknown parameter p (see formula sheet). We observe the realization $x_1, x_2, \ldots, x_n$ of a random sample of X.

a. Determine the maximum likelihood estimator of p to show that
$\bar{X} = \frac{X_1+X_2+\cdots+X_n}{n}$ is the mle of $E(X)$.
b. Show that $\bar{X}$ is an unbiased and consistent estimator of $E(X)$
c. Apply Neyman-Pearson`s lemma to construct the MP-test for $H_0: p = 0.1$ against $H_1: p = 0.2$: express the test statistic as a function of $\bar{X}$ and determine the values of $\bar{X}$ for which the null hypothesis is rejected (determination of critical value(s) is not necessary).

3. A marketing consultant is designing an advertisement campaign for girl clothes in the age of 10-12 year. An important issue is to know who, in the end, decides about the purchase: the mother or the daughter. The consultant referred to a survey of 400 of these purchases, where in 243 times the decision was taken by the mother. Can we state, at a 5% level of significance, that in the majority of the purchases the mother decides?

a. Give (only) the hypotheses of this binomial test (with test statistic).
b. Compute (approximate) p-value for the observed number and explain what this probability implies.
c. Show that the rejection region of the test is $X \geq 217$.
d. Determine the probability of a type II error if in reality 60% of all purchases are decided by the mother. Give the power of the test as well, at this value of p.

Solutions

Exercise 1

a. 95%-CI(μ) = $\left(\overline{X} - c\frac{S}{\sqrt{n}}, \overline{X} + c\frac{S}{\sqrt{n}}\right)$, where $n = 25$, $\overline{x} = 129.885$, $s \approx 1.076$ and $c = 2.064$ from the t_{25-1}-table, such that $P(T_{24} > c) = 0.025$

95%-CI(μ) = (129.885– 2.064× $\frac{1.076}{\sqrt{25}}$, 129.885– 2.064× $\frac{1.076}{\sqrt{25}}$) ≈ (129.44, 130.33).

b. No, this interval is not for new observations (that would be a prediction interval), it is an interval for the **mean** sodium content.

c. $95\% - CI(\sigma^2) = \left(\frac{(n-1)S^2}{c_2}, \frac{(n-1)S^2}{c_1}\right)$,where $s^2 = 1.157$ and

$P(\chi^2_{25-1} \leq c_1) = P(\chi^2_{25-1} \geq c_2) = \frac{\alpha}{2} = 0.025$, so $c_1 = 12.40$ and $c_2 = 39.36$,

$95\% - CI(\sigma^2) = \left(\frac{24\cdot 1.157}{39.36}, \frac{24\cdot 1.157}{12.40}\right) \approx (0.71, 2.24)$

d. *We apply the one sample t-test (since we test on μ for a normal model with unknown μ and σ^2):*

1. Model: the observed sodium contents constitute a realization of a random sample $X_1, \dots, X_{25}$, drawn from a normal distribution with unknown mean sodium content μ and unknown σ^2.
2. Test $H_0: \mu = 130$ against $H_1: \mu \neq 130$ with α = 0.01
3. $T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{\overline{X} - 130}{S/\sqrt{25}}$
4. T is t_{24}-distributed if H_0 is true.
5. Observed value: $t = \frac{129.885 - 130}{1.076/\sqrt{25}} \approx -0.534$
6. It is a two-sided test: if $T \leq -c$ or $T \geq c$, then H_0 should be rejected
   For α = 0.05, we have c = 2.797, such that $P(T_{24} \geq c) = 0.005$.
7. t = -0.534 lies in between $-c$ and c, so we cannot reject H_0.
8. The mean sodium content does not deviate structurally from 130, at a significance level of 5%.

Exercise 2

a. $L(p) = \prod_{i=1}^n P(X = x_i) = \prod_{i=1}^n (1-p)^{x_i - 1} p = (1-p)^{\sum x_i - n} p^n \quad (\sum x_i = \sum_{i=1}^n x_i)$

$\ln L(p) = \left(\sum x_i - n\right) \ln(1-p) + n \ln(p)$

$\frac{d}{dp} \ln L(p) = 0$ if $-\frac{\sum x_i - n}{1-p} + \frac{n}{p} = 0$, or:

$\frac{-(\sum x_i - n)p + n(1-p)}{p(1-p)} = \frac{n - p\sum x_i}{p(1-p)} = 0$ if $p = \frac{n}{\sum x_i} = 1/\overline{x}$.

For this value of p $\ln L(p)$ attains its maximum, since $\frac{d}{dp} \ln L(p) > 0$ for $p < 1/\overline{x}$ and $\frac{d}{dp} \ln(L) < 0$ for $p > 1/\overline{x}$.

Conclusion: $\hat{p} = 1/\overline{X}$ is the maximum likelihood estimator (mle) of p.

Hence $\overline{X}$ is the mle of $E(X) = \frac{1}{p}$.

b. The MP-test is the test that rejects $H_0: p = 0.1$ in favour of $H_1: p = 0.2$ for small values of

$$r(x_1, \dots, x_n) = \frac{\prod_{i=1}^{n} P(X = x_i | p = 0.1)}{\prod_{i=1}^{n} P(X = x_i | p = 0.2)} = \frac{0.9^{\sum x_i - n} 0.1^n}{0.8^{\sum x_i - n} 0.2^n} = \left(\frac{1}{2}\right)^n \left(\frac{0.9}{0.8}\right)^{n(\bar{x}-1)}$$

r is an increasing function of $\bar{x}$, so the MP-test **rejects for small values of $\bar{x}$**: **reject H_0 if $\bar{X} \leq c$.** (c can be determined if the value of α_0 is given.)

Exercise 3

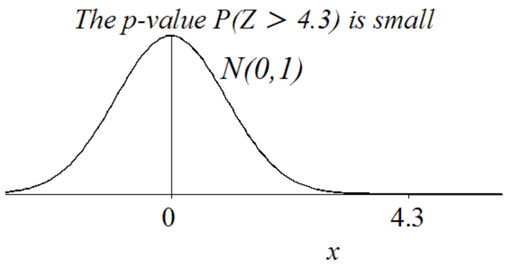
a. (step 2.) Test $H_0: p = \frac{1}{2}$ against $H_1: p > \frac{1}{2}$ with $\alpha = 5\%$.

b. If H_0 is true, we have: $X \sim B\left(400, \frac{1}{2}\right)$, so approximately according to the CLT:

$$X \sim N\big(np_0, np_0(1-p_0)\big) = N(200,100).$$

The p-value is $P(X \geq 243 | H_0)$

$$\approx P\left(Z \geq \frac{242.5 - 200}{10}\right) = 1 - \Phi(4.24)$$
$$< 0.0001$$



Since the p-value < 0.0001, hence and much smaller than α, (see graph) so we reject H_0. The evidence for the alternative is strong: we would even reject if $\alpha = 0.01\%$.

c. We will reject H_0 if $X \geq c$: $P(X \geq c | H_0) \leq \alpha_0 = 0.05$.

We will apply normal approximation with continuity correction:

$$P(X \geq c | H_0) \overset{\text{c.c.}}{=} P(X \geq c - 0.5 | H_0) \overset{\text{CLT}}{\approx} P\left(Z \geq \frac{c - 0.5 - 200}{\sqrt{100}}\right) \leq 0.05$$

and from $\Phi\left(\frac{c-0.5-200}{\sqrt{100}}\right) \geq 0.05$ it follows: $\frac{c-0.5-200}{\sqrt{100}} \geq 1.645$,

so $c \geq 200.5 + 10 \cdot 1.645 = 216.95$: $\boldsymbol{c = 217}$

d. The probability of a type II for $p = 0.60$ (computed with continuity correction):

$$P(X < 217 | p = 0.60) = P\left(Z \leq \frac{216.5 - 400 \times 0.60}{\sqrt{400 \times 0.6 \times 0.4}}\right) = P(Z \leq -2.40)$$
$$= 1 - \Phi(2.40) = 0.82\%$$

The power of the test for $p = 0.60$ is:

$P(X \geq 217 | p = 0.60) = 1 - 0.0082 = 99.18\%$.