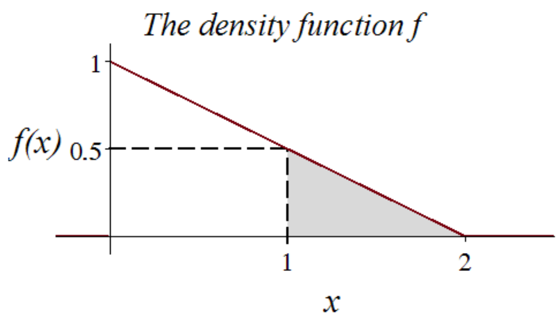
Solutions of exercises chapter 6



1. a. $P(X > 1) = \int_1^\infty f(x)dx = \int_1^2 \left(1 - \frac{1}{2}x\right) dx$

$= x - \frac{1}{4}x^2 \Big|_{x=1}^{x=2} = 1 - \frac{3}{4} = \frac{1}{4},$

(or determine the shaded area of the triangle: $\frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$)

b. $E(X) = \int_0^2 x \cdot \left(1 - \frac{1}{2}x\right) dx = \frac{1}{2}x^2 - \frac{1}{6}x^3 \Big|_{x=0}^{x=2} = \frac{2}{3}$

$E(X^2) = \int_0^2 x^2 \cdot \left(1 - \frac{1}{2}x\right) dx = \frac{1}{3}x^3 - \frac{1}{8}x^4 \Big|_{x=0}^{x=2} = \frac{2}{3}$

$var(X) = E(X^2) - (EX)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$

c. $F(x) = P(X \leq x)$, so $F(x) = 0$, if $x < 0$ and $F(x) = 1$, if $x > 2$.

if $0 \leq x \leq 2$, then $F(x) = \int_0^x \left(1 - \frac{1}{2}u\right) du = u - \frac{1}{4}u^2 \Big|_{u=0}^{u=x} = x - \frac{1}{4}x^2$

So $P(X > 1) = 1 - F(1) = 1 - \frac{3}{4} = \frac{1}{4}$

2. a. $f(x) = \lambda e^{-\lambda x}$, if $x \geq 0$ and $f(x) = 0$, if $x < 0$.

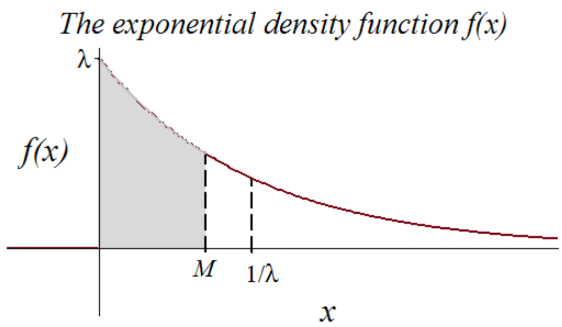
$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\big|_0^\infty = -0 - (-1) = 1$$ (graph: see below)

b. Use integration by parts

(see the appendix Mathematical Techniques: $f(x) = x$ and $g(x) = \lambda e^{-\lambda x}$):

$$E(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx = x \cdot -e^{-\lambda x}\big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x}\big|_0^\infty = +\frac{1}{\lambda}$$

$P(X > EX) = \int_{1/\lambda}^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\big|_{\frac{1}{\lambda}}^\infty = e^{-\lambda \cdot \frac{1}{\lambda}} = e^{-1} \approx 36.8\% \; (< \frac{1}{2})$



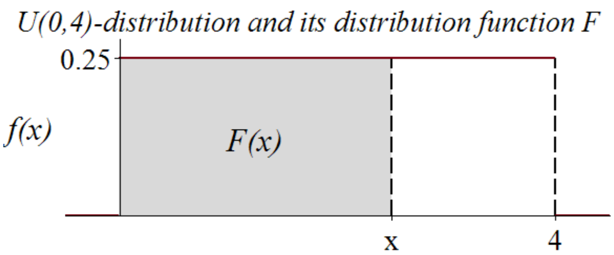
c. $P(X > M) = \int_M^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\big|_M^\infty = e^{-\lambda M} = \frac{1}{2}$, so $M = \frac{\ln(2)}{\lambda}$

d. The mode = 0 (see graph)

3. a. - $f(x) = \frac{1}{4}$, for $0 < x < 4$:

$\int_{-\infty}^\infty f(x)\, dx = 4 \cdot \frac{1}{4} = 1$

- $E(X) = 2 =$ median
  (because of symmetry of f)



So $P(X > EX) = P(X \le M) = \frac{1}{2}$

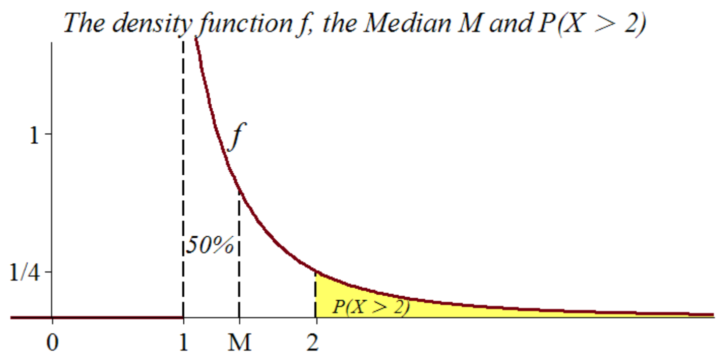
- The mode can be each value of the interval $(0,4)$.

b. $F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{x}{4} & \text{if } 0 < x < 4 \\ 1 & \text{if } x \ge 4 \end{cases}$

4. a. $f(x) = \frac{c}{x^3}$ if $x > 1$, so $\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} \frac{c}{x^3}\,dx = c \cdot -\frac{1}{2}x^{-2}\Big|_1^{x\to\infty} = 0 + c \cdot \frac{1}{2} = 1$

So $c = 2$.

$P(X > 2) = \int_2^{\infty} \frac{2}{x^3}\,dx = -x^{-2}|_{x=2}^{x\to\infty} = 0 - (-2^{-2}) = \frac{1}{4}$



b. $E(X) = \int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} x \cdot \frac{2}{x^3}\,dx = -2x^{-1}|_1^{x\to\infty} = 0 + 2 = 2$

The median M: $P(X \ge M) = \int_M^{\infty} \frac{2}{x^3}\,dx = -x^{-2}|_{x=M}^{x\to\infty} = 0 + \frac{1}{M^2} = \frac{1}{2}$, so $M = \sqrt{2}$

c. $F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \displaystyle\int_1^x \frac{2}{u^3}\,du = -u^{-2}|_{u=1}^{u=x} = 1 - \frac{1}{x^2} & \text{, if } x \ge 1 \end{cases}$

Check: $(x > 1)$: $f(x) = \frac{d}{dx}F(x) = -\left(-\frac{2}{x^3}\right) = \frac{2}{x^3}$ (correct).

5. a. 1. $F_Y(y) = P(5 - 2X \le y) = P(-2X \le y - 5) = P\left(X \ge \frac{y-5}{-2}\right) = 1 - F_X\left(-\frac{y-5}{2}\right)$

2. $f_Y(y) = \frac{d}{dy}F_y(y) = \frac{1}{2} f_X\left(\frac{y-5}{-2}\right)$

3. $f_X\left(\frac{y-5}{-2}\right) = 1$, if $0 < \frac{y-5}{-2} < 1$, so if $3 < y < 5 \rightarrow f_Y(y) = \frac{1}{2} \cdot 1$, for $3 < y < 5$

So $Y \sim U(3,5)$

b. Choose $Y = a + (b-a) \cdot X$

c. $Y = 2X$, if $X \sim Exp(\lambda = 3)$:

1. $F_Y(y) = P(2X \le y) = P\left(X \le \frac{1}{2}y\right) = F_X\left(\frac{1}{2}y\right)$

2. $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{1}{2}y\right) = \frac{1}{2} \cdot f_X\left(\frac{1}{2}y\right)$

3. Since $f_X(x) = 3e^{-3x}$ for $x \ge 0$, we have for $\frac{1}{2}y \ge 0$, or $y \ge 0$:

$f_Y(y) = \frac{1}{2} \cdot f_X\left(\frac{1}{2}y\right) = \frac{1}{2} \cdot 3e^{-3\cdot\frac{1}{2}y} = \frac{3}{2}e^{-\frac{3}{2}y}$. So $Y \sim Exp\left(\frac{3}{2}\right)$

$Z = X^2$, if $X \sim Exp(\lambda = 3)$:

1. $F_Z(z) = P(X^2 \le z) = P(-\sqrt{z} \le X \le \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z})$, if $z > 0$
2. $f_Z(z) = \frac{d}{dy}\left[F_X(\sqrt{z}) - F_X(-\sqrt{z})\right] = \frac{1}{2\sqrt{z}} \cdot f_X(\sqrt{z}) + \frac{1}{2\sqrt{z}} \cdot f_X(-\sqrt{z})$. $(z > 0)$
3. Since $f_X(x) = 3e^{-3x}$ for $x \ge 0$, we have $f_X(\sqrt{z}) = 3e^{-3\sqrt{z}}$ and $f_X(-\sqrt{z}) = 0$, so
   $f_Z(z) = \frac{1}{2\sqrt{z}} \cdot 3e^{-3\sqrt{z}} + \frac{1}{2\sqrt{z}} \cdot 0 = \frac{1}{2\sqrt{z}} \cdot 3e^{-3\sqrt{z}}$, for $z > 0$.

(This is not a regular distribution, but one can verify that it defines a density function: $f_Z(z) \ge 0$ *and*

$\int_{-\infty}^{\infty} f_Z(z)dz = \int_0^\infty \frac{1}{2\sqrt{z}} \cdot 3e^{-3\sqrt{z}}\, dz$, using a substitution: $x = \sqrt{z}$, so $dx = \frac{1}{2\sqrt{z}} dz$.

$= \int_0^\infty 3e^{-3x}\, dx$

$= -e^{-3x}\big|_{x=0}^{x\to\infty} = 0 + 1 = 1)$

6. a. 1. $F_Y(y) = P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right)$

$= 1 - F_X\left(\frac{1}{y}\right)$, $y > 0$

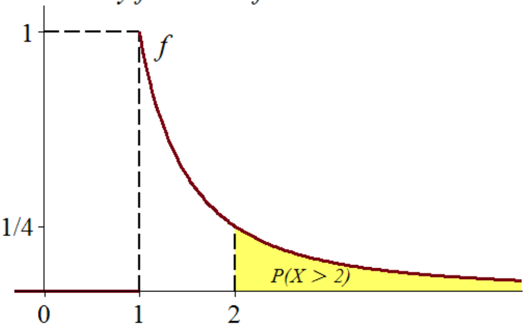
(And $F_Y(y) = P\left(\frac{1}{X} < y\right) = 0$ if $y < 0$)

2. $f_Y(y) = \frac{d}{dy} F_Y(y) = -\frac{1}{y^2} \cdot -f_X\left(\frac{1}{y}\right)$

3. $f_X\left(\frac{1}{y}\right) = 1$, if $\frac{1}{y} > 0$, so if $y > 1$

So $f_Y(y) = \frac{1}{y^2} \cdot 1 = \frac{1}{y^2}$ if $y > 1$

The density function of Y: EY does not exist



b. $P(Y > 2) = \int_2^\infty f_Y(y)\, dy = \int_2^\infty \frac{1}{y^2} dy = -y^{-1}\big|_2^\infty = \frac{1}{2}$

$P(Y > 2) = P\left(\frac{1}{X} > 2\right) = P\left(X < \frac{1}{2}\right) = \frac{1}{2}$

c. $E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_1^\infty y \cdot \frac{1}{y^2} dy = \ln(y)\,\big|_1^\infty = \infty$, so $E(Y)$ **does not exist**.

Likewise $E(Y) = E\left(\frac{1}{X}\right) = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)dx = \int_0^1 \frac{1}{x} dy = \ln(x)\,\big|_0^1$ does not exist.

7. a. if $y > 0$ we have: $F_Y(y) = P\left(\sqrt{|X|} \le y\right) = P(|X| \le y^2) = P(-y^2 \le X \le y^2)$

$= F_X(y^2) - F_X(-y^2)$

Since $F_X(x) = 1 - e^{-x}$, if $x \ge 0$ and $F_X(x) = 0$, if $x < 0$,

$F_Y(y) = \left(1 - e^{-y^2}\right) - 0 = 1 - e^{-y^2}$, for $y > 0$.

b. $f_Y(y) = \frac{d}{dy} F_y(y)$ *(see a.)* $= 2ye^{-y^2}$, for $y > 0$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)dy = \int_0^\infty y \cdot 2ye^{-y^2} dy$ *(apply integration by parts)*

$= \left[y \cdot e^{-y^2}\right]_0^{y\to\infty} - \int_0^\infty 1 \cdot -1e^{-y^2} dy = 0 + \int_0^\infty e^{-y^2} dy$,

Here we can apply the standard normal distribution, so $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$,

so $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$ and $\int_0^\infty e^{-\frac{1}{2}x^2} dx = \frac{1}{2}\sqrt{2\pi}$,

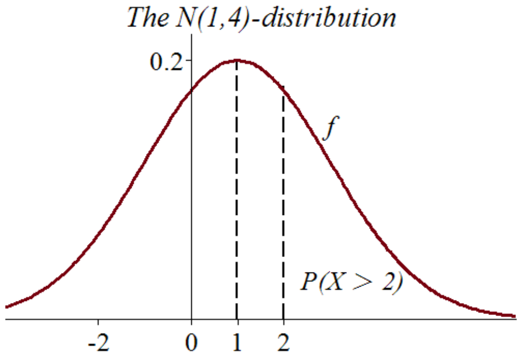
Use the substitution $x = \sqrt{2} \cdot y$, where $dx = \sqrt{2} \cdot dy$ to find

$\int_0^\infty e^{-\frac{1}{2}x^2} dx = \int_0^\infty e^{-y^2}\sqrt{2}dy = \frac{1}{2}\sqrt{2\pi}$, so $E(Y) = \int_0^\infty e^{-y^2}\sqrt{2}dy = \frac{\frac{1}{2}\sqrt{2\pi}}{\sqrt{2}} = \frac{1}{2}\sqrt{\pi}$.

(note that the a.-part of this exercise and the first two lines of b. are applications of Probability Theory, whereas the second part of b. is merely application of calculus techniques. Tests will focus on the application of Probability Theory, but sometimes some Calculus is inevitable.

8. a. $P(X > 2) = P\left(\frac{X-1}{2} > \frac{2-1}{2}\right)$

$= P\left(Z > \frac{1}{2}\right) = 1 - \Phi(0.5) = 0.3085.$



$P(|X| > 2) = P(X > 2) + P(X < -2)$

$= P\left(Z > \frac{1}{2}\right) + P\left(Z < \frac{-2-1}{2}\right)$

$= 1 - \Phi(0.5) + \Phi(-1.5)$

$= 0.3085 + (1 - 0.9332) = 0.3753$

$P(|X - 1| < 2) = P(-2 < X - 1 < +2)$

$= P\left(-\frac{2}{2} < \frac{X-1}{2} < \frac{2}{2}\right)$

$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 2 \cdot 0.8417 - 1 = 68.34\%.$

b. $P(X \le c) = P\left(\frac{X-1}{2} \le \frac{c-1}{2}\right) = \Phi\left(\frac{c-1}{2}\right) = 90\%$, so $\frac{c-1}{2} = 1.28.$

$c = 1 + 2 \cdot 1.28 = 3.56$ is the 90^{th} percentile of X.

c. $P(X \le c) = P\left(\frac{X-1}{2} \le \frac{c-1}{2}\right) = \Phi\left(\frac{c-1}{2}\right) = 10\%$, so $\frac{c-1}{2} = -1.28.$

$c = 1 - 2 \cdot 1.28 = -1.56$ is the 10^{th} percentile of X.

9. For instance: $P(-2\sigma < X - \mu < 2 \cdot \sigma) = P\left(-2 < \frac{X-\mu}{\sigma} < 2\right) = P(-2 < Z < 2)$

$= \Phi(2) - \Phi(-2)$

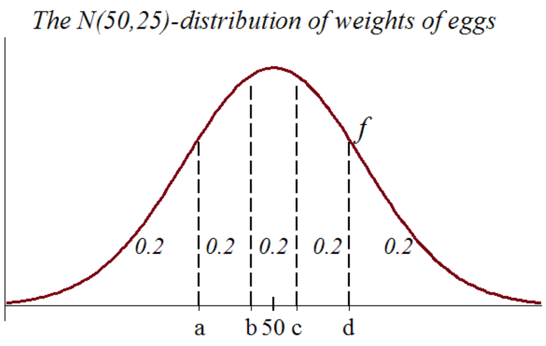
$= 2\Phi(2) - 1$

$= 2 \cdot 0.9772 - 1$

$= 0.9544 \approx 95.4\%$

And: $P(-\sigma < X - \mu < \sigma) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 2 \cdot 0.8413 - 1 \approx 68.3\%$

$P(-3\sigma < X - \mu < 3\sigma) = \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 2 \cdot 0.9987 - 1 = 99.68\%$



10. The interval bounds a, b, c and d are such that b and c are symmetric about 50, as are a and d.

$P(X \le c) = 0.60$, so $P\left(Z \le \frac{c-50}{5}\right) =$

$\Phi\left(\frac{c-50}{5}\right) = 0.60$

In the $N(0,1)$-table we find that $\Phi(z) = 0.60$, if $z \approx 0.25$,

so $\frac{c-50}{5} \approx 0.25$, or $c \approx 50 + 5 \cdot 0.25 = 51.25$ gram and $b \approx 50 - 1.25 = 48.75$ gram.

(*since the probability 0.60 is not found exact exactly in the N(0,1)-table, we could approximate the z-value more precise by linear interpolation:* $z = 0.254$ *and* $c \approx 51.27$ *gram.*)

Similarly:

$P(X \le d) = \Phi\left(\frac{d-50}{5}\right) = 0.80$, so $d \approx 50 + 5 \cdot 0.84 = 54.20$ and $a \approx 45.80$ gram.

11. a. Since $E(X) = \mu$, we have:

$E(X-\mu)^3 = E(X^3 - 3 \cdot X^2 \cdot \mu + 3 \cdot X \cdot \mu^2 - \mu^3)$

$= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 = E(X^3) - 3\mu E(X^2) + 2\mu^3$

b. $E(X) = \frac{1}{2}$, $E(X^2) = \int_0^1 x^2 \cdot dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}$ and $E(X^3) = \int_0^1 x^3 \cdot dx = \frac{1}{4}x^4|_0^1 = \frac{1}{4}$.

And applying a.: $E(X-\mu)^3 = \frac{1}{4} - 3 \cdot \frac{1}{2} \cdot \frac{1}{3} + 2 \cdot \left(\frac{1}{2}\right)^3 = 0$

(of directly: $E(X-\mu)^3 = \int_0^1 \left(x-\frac{1}{2}\right)^3 dx = \frac{1}{4}\left(x-\frac{1}{2}\right)^4 |_0^1 = \frac{1}{4}\left(\frac{1}{16}-\frac{1}{16}\right) = 0$).

c. $E(X) = \frac{1}{\lambda} = 1$ and since $var(X) = E(X^2) - (EX)^2 = 1,$ we have $E(X^2) = 1 + 1 = 2$.

$E(X^3) = \int_0^\infty x^3 \cdot e^{-x} dx = x^3 \cdot -e^{-x}|_0^1 + 3\int_0^\infty x^2 \cdot e^{-x} dx = 3E(X^2) = 6.$

$E(X-\mu)^3 = 6 - 3 \cdot 1 \cdot 2 + 2 \cdot 1 = 2$

d. Correct: the uniform distribution is symmetric: $E(X-\mu)^3 = 0$.

and the exponential distribution is skewed to the right: $E(X-\mu)^3 = +2$.